Fair mixing: the case of dichotomous preferences

Haris Aziz*, Anna Bogomolnaia**, and Hervé Moulin**

November 2017

* Data61, CSIRO and University of New South Wales
** University of Glasgow and HSE St Petersburg

Abstract

Agents vote to choose a fair mixture of public outcomes; each agent likes or dislikes each outcome. We discuss three outstanding voting rules.

The *Conditional Utilitarian* rule, a variant of the random dictator, is Strategyproof and guarantees to any group of like-minded agents an influence proportional to its size. It is easier to compute and more efficient than the familiar *Random Priority* rule. Its worst case (resp. average) inefficiency is provably (resp.in numerical experiments) low if the number of agents is low.

The efficient *Egalitarian* rule protects similarly individual agents but not coalitions. It is *Excludable Strategyproof*: I do not want to lie if I cannot consume outcomes I claim to dislike.

The efficient *Nash Max Product* rule offers the strongest welfare guarantees to coalitions, who can force any outcome with a probability proportional to their size. But it fails even the excludable form of Strategyproofness.

1 Introduction

Interactive democracy aka *Liquid Democracy* (see e.g., (Behrens, 2017; Brill, 2017)) is a new approach to voting well suited for low stakes/ high frequency decisions, and easily implemented on the internet (Grandi, 2017). An especially successful instance is budgetary participation (Cabannes, 2004) where the stakeholders (citizens, employees of a firm, club members) vote to decide which subset of public projects the community, firm, or club should implement.

We discuss a stylized version of this process in the probabilistic voting model (Fishburn, 1984; Gibbard, 1977). The guiding principle of our analysis is that the selection of a single (deterministic) public outcome is prima facie unfair: fairness requires compromise, we must select a *mixture* of several mutually exclusive outcomes. The mixture may come from actual randomization, or the allocation of time-shares, or the distribution of a fixed amount of some resource (e.g., money) over these outcomes. Some typical examples follow.

In reference (Cabannes, 2004), the city authority must divide funds or staff between several projects (library, sport center, concert hall) taking into account the citizens' wishes. The scheduling of one or several weekly club meetings (gym classes, chess club, study group) must accomodate the time constraints reported by the club members. Or the local public TV, after polling its audience, must divide broadcasting time between different languages, or different types of program (news, sports, movies). In the *fair knapsack* problem, the server schedules repeatedly jobs of different reported or observed sizes under a capacity constraint, and must pick a (random) serving protocol.

In all these examples, fairness requires to give some share of the public resources to everyone: each club member should have access to some meetings; everyone should enjoy at least some TV programs, etc.. This contrasts with traditional high stakes/low frequency voting contexts, where the first best is to select a single (deterministic) outcome, and randomization over outcomes is only second best.[1]

We run into the familiar conflict between protecting minorities and submitting to the will of the majority (Young, 1950; Gordon, 1994; Porta, de Silanes, Schleifer, and Vishny, 2000). On the one hand, the larger the support for a public outcome, the bigger should be its share in the final compromise: numbers matter. On the other hand, we must protect minorities with their idiosyncratic preferences for outcomes disliked by the majority. So the club meetings will be more frequent when many members can attend, but nobody will be entirely excluded; the knapsack server will favor short jobs because this increases the number of satisfied customers, but it cannot ignore long jobs entirely; and so on.

We analyze this tradeoff when preferences can be represented in a very simple Facebook-style *dichotomous* form: each agent likes or dislikes each outcome, and her utility is simply the total share of her *likes*. Agents in the knapsack problem care only about their expected service time, and in the club example, about the number of meetings they can attend. Though less natural in the public TV and the library funding examples, where they rule out any complementarities between outcomes, dichotomous preferences are still of practical interest because they are easy to elicit.

We discuss the fairness and incentive compatibility properties of three mostly well known social choice rules.

Our results. The *Fair Share guarantee* principle is central to the fair division literature since the earliest *cake division* papers (Steinhaus, 1948). In our model this is the *Individual Fair Share* (IFS) axiom: each one of the n agents "owns" a $1/n$-th share of decision power, so she can ensure an outcome she likes at least $1/n$-th of the time (or with probability at least $1/n$). To capture the more subtle ideas that minorities should be protected, and numbers should matter as

[1] It is used to break ties, or to play the role of an absent deterministic Condorcet winner: for instance (Laffond, Laslier, and Le Breton, 1993; Aziz, Brandl, Brandt, and Brill, 2017a; Brandt, 2017) identifies a lottery that, in a certain sense, wins the majority tournament.

well, we strengthen IFS to *Unanimous Fair Share* (UFS), giving to any group of like-minded agents an influence proportional to its size: so if 10% of the agents have identical preferences they should like the outcome at least 10% of the time.

Our starting point is the impossibility result in (Bogomolnaia, Moulin, and Stong, 2005), where our model and the two fairness properties IFS and UFS appear first: *no mixing rule can be efficient, incentive compatible in the prior-free sense of Strategyproofness (SP), and meet Unanimous, or even Individual, Fair Share*. We introduce new fairness and incentives properties and offer instead possibility results. Three remarkable mixing rules (two of them well known) meet IFS and achieve, loosely speaking, two out of the three goals of efficiency, group fairness (in the sense of UFS or other more demanding properties), and incentive compatibility.

Start with the *Egalitarian* (EGAL) rule, adapting to our model a celebrated principle of distributive justice. Taking the probability that the selected outcome is liked by agent i as her canonical utility, the rule maximizes first the utility level we can guarantee to all agents; among the corresponding mixtures, it maximizes the utility we can guarantee to all agents but one; and so on. It is efficient and implements IFS, therefore it is not strategyproof, by the above mentioned result. However if public outcomes are non rival but *excludable*, we can force agents to consume only those outcomes they claim to like, so it becomes more costly to fake a dislike and the strategyproofness is correspondingly weakened. A meeting of the club is such an *excludable* public outcome: it is easy to exclude from the meeting those who reported they could not attend; broadcasting via cable TV is similarly excludable, not so via aerial broadcasting. The Egalitarian rule is efficient as well as *Excludable Strategyproof* (EXSP): misreporting one's preferences does not pay, provided an agent is excluded from consuming those public outcomes she reportedly dislikes (Theorem 1). Thus weakening SP to EXSP resolves the impossibility result.

But numbers do not matter to the egalitarian rule: it treats a unanimous group of agents exactly as if it contained a single agent, so the UFS property obviously fails. A related problem is that if I have a clone (another agent with preferences identical to mine), I can simply stay home and nothing will change. The *Strict Participation* (PART*) axiom takes care of this disenfranchisement problem by insisting that casting his vote is strictly benefitial to each voter. So the EGAL rule is only appealing if we focus on individual guarantees and are comfortable treating a homogenous group as a single person. This makes sense if the club must offer some important training to its members. But in the budgetary participation or the broadcasting examples, numbers should definitely matter.

The *Conditional Utilitarian* (CUT) rule is a simple variant of the classic "random dictator". Each agent identifies, among the outcomes he likes, those with the largest support from the other agents: then he spreads the probability (time share) of $1/n$ uniformly over the outcomes he likes. So the utilitarian concern is conditional upon guaranteeing one's full utility first: *charity begins at home*. The CUT rule is related to, but much simpler than, the *Random Priority* (RP) rule averaging outcomes of all deterministic priority rules. Both rules are

SP, meet PART* and guarantee UFS. Therefore they are inefficient. But CUT is much easier to compute and strictly more efficient than RP (Theorem 2). In numerical simulations (Section 9) and for relatively small values of n, its inefficiency is consistently low.

Our third rule is the familiar *Nash Max Product* (NMP) rule picking the mixture maximizing the product of individual utilities. It is efficient and offers much stronger welfare guarantees to groups than UFS. We introduce two requirements, each one a considerable strengthening of UFS, intuitively but not logically related. The *Core Fair Share* (CFS) property has an incentive flavor in the spirit of cumulative voting (Gordon, 1994; Sawyer and MacRae, 1962): any group of agents can pool their shares of decision power and object to the proposed mixture z by enforcing another mixture z' with a probability proportional to the group size. Core Fair Share rules out any such objection. Finally *Average Fair Share* (AFS) applies to any coalition with a common liked outcome: the *average* utility in such a group cannot be smaller than its relative size. In simple examples, AFS limits very effectively the set of acceptable efficient mixtures. Theorem 3 shows that the efficient NMP rule meets PART*, CFS and AFS but fails even EXSP.

The results suggest several challenging open questions about the impossibility frontiers of our model.

2 Related literature

Budgetary participation is an important new aspect of participative democracy, reviewed in (Cabannes, 2004). Our model casts this process as a probabilistic voting problem, introduced first by Gibbard (1977) as a way to design non dictatorial strategyproof decision rules. The literature he inspired viewed randomization as a way around the defects of deterministic rules, mostly to allow anonymous and neutral rules, or to circumvent the absence of Condorcet winners (see e.g, Fishburn, 1984; Laffond et al., 1993; Aziz et al., 2017a; Aziz and Stursberg, 2014; Brandl, Brandt, and Seedig, 2016). But recent work turns its attention to mixtures of outcomes with time-sharing or compromise in mind: see e.g., (Bogomolnaia et al., 2005; Aziz and Stursberg, 2014; Aziz, 2013; Aziz, 2017; Fain, Goel, and Munagala, 2016; Benade, Nath, Procaccia, and Shah, 2017).

Our works takes direct inspiration from Bogomolnaia, Moulin, and Stong (2002); Bogomolnaia et al. (2005) who introduced the model of randomised voting under dichotomous preferences. In the same mathematical model we present several new results about new normative requirements such as participation incentives, a decentralization axiom, weaker forms of strategyproofness, and stronger forms of fairness.

Two of our rules, EGAL and NMP, maximize respectively a familiar social welfare ordering and a classic collective utility function. The EGAL rule is the lead mechanism in the related assignment model with dichotomous preferences in Bogomolnaia and Moulin (2004). In probabilistic voting, the Egalitarian

Simultaneous Reservation rule of Aziz and Stursberg (2014) can be seen as an adaptation of the Egalitarian rule.

Recent literature emphasizes that the NMP rule is central to the competitive approach of the fair division of private commodities, whether divisible or indivisible (Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, 2016; Bogomolnaia, Moulin, Sandomirskyi, and Yanovskaya, 2017). We find here a new application of this rule in the public decision making context, closer in spirit to Nash's original bargaining model (Nash, 1950). Our results are related to those of Fain et al. (2016), who also propose the NMP rule for budgetary participation, reinterpret this rule as a Lindahl equilibrium, and discuss its computational complexity. They allow for more general preferences than ours (in particular, full-fledged vNM utilities), and show the Core Fair Share property (Corollary 1 Section 2.3) just like in statement *i*) of our Theorem 3. They do not discuss incentives properties or any alternative rule.

The rules CUT and RP are non welfarist, in that they do not maximize any social welfare ordering. The RP rule is well known (and was discussed in Bogomolnaia et al. (2005)), and CUT is a fairly simple twist on the random dictator first introduced by Duddy (2015) who noted that it is strategyproof but did not develop its normative appeal.

Fair Share is an early design constraint of decision mechanisms: see the mathematical literature on cake cutting (Steinhaus, 1948), and on fair division of microeconomic commodities (Moulin, 2003; Varian, 1974; Thomson, 2016).

The group version of Fair Share captures the ubiquitous "protection of minorities" principle that is formally related to cooperative stability in standard voting. It is also related to the proportional veto principle (Moulin, 1981, 1982) and motivates practical twists in the rules such as cumulative voting, especially concerned with the protection of ethnic minorities in political elections (Sawyer and MacRae, 1962), or minority stockholders in corporate governance (Young, 1950; Gordon, 1994; Porta et al., 2000). See also the same concerns for EU enlargement Hughes and Sasse (2003). Our fairness notions are closely related to proportional representation axioms in multi-winner voting as well (see e.g., Aziz, Brill, Conitzer, Elkind, Freeman, and Walsh (2017b)).

Strict Participation has been considered in the deterministic voting model, leading mostly to negative results. Our results complement those of Brandl, Brandt, and Hofbauer (2015), who undertook a formal study of participation incentives in probabilistic voting.

3 The Model

A generic agent is $i \in N$, and $n = |N|$. A *pure* public outcome is $a \in A$, and a *mixture* of public outcomes is an element z of the simplex $\Delta(A)$, interpreted as a lottery over A, a profile of time shares, or shares of other types of resources between the outcomes in A. Both N and A are finite.

A utility function $u_i = (u_{ia})_{a \in A}$ is an element of $\{0,1\}^A$. Agents who dislike all outcomes play no role in any or the rules we discuss, thus we exclude them at once: the domain of preferences is $\Omega = \{0,1\}^A \diagdown \{\mathbf{0}\}$, where $\mathbf{0} = 0^A$; and $u \in \Omega^N$ is a profile of utility functions. In the examples we always represent u as a $N \times A$ matrix filled with 0-s and 1-s, and we use the notation: $u_S = \sum_{i \in S} u_i$ and $u_{SB} = \sum_{i \in S} \sum_{a \in B} u_{ia}$ for $S \subseteq N$ and $B \subseteq A$.

A *problem* M is a triple $M = (N, A, u)$ where $u \in \Omega^N$.

Actual utilities (welfare) at z are written $U_i = u_i \cdot z$, and the corresponding utility profile is written $U = u \cdot z \in [0,1]^N$. The set of feasible utility profiles is $\Phi(M) = \{U = u \cdot z | z \in \Delta(A)\}$. Given $U \in \Phi(M)$ we set $\varphi^{-1}(U) = \{z \in \Delta(A) | U = u \cdot z\}$.

Definition 1

i) In problem $M = (N, A, u)$ a feasible utility profile $U \in \Phi(M)$ is efficient if there is no profile $U' \in \Phi(M)$ such that $U \leq U'$ and at least one inequality $U_i \leq U'_i$ is strict.

ii) A mixture $z \in \Delta(A)$ is efficient in M if the profile $u \cdot z$ is efficient.

iii) Fix $\varepsilon \in [0,1]$; the profile $U \in \Phi(M)$ is ε-efficient if there exists $U' \in \Phi(M)$ such that $U \leq \varepsilon U'$.

Definition 2

i) A rule F picks one $U \in \Phi(M)$ for each problem M; the mapping f picks the corresponding mixtures: $f(M) = \varphi^{-1}(F(M))$, so that $F(M) = u \cdot f(M)$. Moreover F and f are Anonymous (treat agents symmetrically) and Neutral (treat outcomes symmetrically).

ii) The rule F is efficient if it selects an efficient profile in every problem. For any n the rule is $\varepsilon(n)$-inefficient if a) there exists a problem M of size n and a profile $U \in \Phi(M)$ such that $F(M)$ is $\varepsilon(n)$-inefficient, and b) no smaller number $\varepsilon'(n)$ meets this property.

A rule is "welfarist" by design, in the sense that it does not distinguish between mixtures resulting in the same utility profile. For instance if two outcomes a, b are "clones" in problem M (liked by exactly the same agents), a rule is oblivious to shifting some weight from a to b.

The efficient pure outcomes in A are easy to recognize: a is efficient *if and only if* there is no b such that the set of agents liking b is strictly larger than the set liking a. We call such outcomes *undominated.* In the following example

$N \downarrow A \rightarrow$	a	b	c	d	e
1	0	0	0	1	1
2	0	0	1	1	0
3	1	1	0	0	0
4	1	0	1	0	0
5	0	1	0	1	1

(1)

outcome e is dominated by d, and the four other outcomes are undominated. However, convex combinations of undominated outcomes may well be inefficient. In the example, any mixture z such that z_b, z_c are both positive, say $z_b, z_c \geq \alpha > 0$, can be improved by redistributing the weight α to a and to d. That is, z is Pareto inferior to the mixture

$$z' = (z_a + \alpha, z_b - \alpha, z_c - \alpha, z_d + \alpha, z_e).$$

Of special interest are those problems where any mixture of undominated pure outcomes is efficient: in the probabilistic interpretation of our model this means that ex post efficiency implies ex ante efficiency. Indeed the four rules we discuss below mix only undominated outcomes, so in such problems their efficiency is guaranteed.

In our first (minor) result, the set of outcomes liked by an agent is called her *like-set*.

Lemma *All mixtures of undominated (pure) outcomes are efficient in problem M in two cases:*
i) If $|A| \leq 3$ *and/or* $|N| \leq 4$;
ii) If A can be ordered in such a way that the like-set of every agent is an interval.

Statement *i*) is proven by Duddy (2015); it implies that example (1) has the smallest sizes of A and N for which a combination of undominated outcomes is inefficient.

Proof of statement *ii*)
Fix a problem M as in statement *ii*). If some outcomes are "clones" (liked by exactly the same set of agents), a class of clones is an interval as well and it is clearly enough to prove the statement for the "decloned" problem where each interval of clones has shrunk to a single outcome. Thus we can assume that our problem has no clones.

Let A^* denote the subset of undominated pure outcomes. We fix a mixture z with support in A^* ($z \in \Delta(A^*)$) and assume some other mixture $y \in \Delta(A^*)$ makes everyone weakly better off than z: we will show $y = z$, which implies the statement.

We keep in mind that for any two a, b in A^* there is some agent i who likes a but not b, because a and b are not clones. Write the ordered set A^*as $\{1, \cdots, K\}$ and apply this remark to the first two agents: some agent i likes 1 but not 2, hence i likes only 1 and $u_i \cdot z \leq u_i \cdot y$ implies $z_1 \leq y_1$. Some agent j likes 2 but not 3, hence j likes $1, 2$ or just 2, so $u_j \cdot z \leq u_j \cdot y$ is either $z_{12} \leq y_{12}$ or $z_2 \leq y_2$ and either way we deduce $z_{12} \leq y_{12}$. Similarly there is some k who likes 3 but not 4, so $u_k \cdot z \leq u_k \cdot y$ means that at least one of z_3, z_{23}, and z_{123} increases weakly and inequality $z_{123} \leq y_{123}$ follows in each case. An obvious induction argument gives

$$z_{12\cdots k} \leq y_{12\cdots k} \text{ for all } k, 1 \leq k \leq K$$

The symmetric argument starting from outcome K gives

$$z_{k(k+1)\cdots K} \leq y_{k(k+1)\cdots K} \text{ for all } k, 1 \leq k \leq K$$

and the desired conclusion $y = z$ follows. ■

4 Excludable Strategyproofness; the Egalitarian rule

We start with the familiar prior-free incentive compatibility requirement that misreporting one's preferences is never profitable if no agent can coordinate this move with other agents.[2]

Notation: upon replacing in the profile u the coordinate u_i by another $u_i' \in \Omega$, the resulting profile is $(u|^i u_i')$.

Strategyproofness (SP)**:** $u_i \cdot f(M) \geq \max_{z' \in f(N,A,(u|^i u_i'))} u_i \cdot z'$ for all M, i and u_i'.

The simplest strategyproof rule adapts approval voting to our model: it selects only those outcomes liked by the largest number of agents. Write $\Phi^p(M)$ for the set of utility profiles implemented by pure outcomes in A: $\Phi^p(M) = \{U \in [0,1] | \exists a \in A \forall i \in N, U_i = u_{ia}\}$. With the notation $avg(Y)$ for the uniform average operation on a set Y of utility profiles, we define the

Utilitarian rule (UTIL): $F^{ut}(M) = avg\{\arg \max_{U \in \Phi^p(M)} U_N\}$.

Note that the rule deliberately treats a problem with two identical columns exactly as the reduced problem where only one column remains.

The careful reader can check that this defines a rule in the sense of Definition 1, one that is efficient and strategyproof. However UTIL ignores minority opinions entirely so it fails to address the normative concerns described in the Introduction.

If an agent gets a fair $1/n$-th share of total decision power, she will use it on an outcome she likes. We take the following lower bound on individual welfare as the first test that mixing is fair:

Individual Fair Share (IFS): $U = F(M) \Longrightarrow U_i \geq \frac{1}{n}$ for all M and all i.

The main result of Bogomolnaia et al. (2005) is that a rule cannot be together Efficient, Strategyproof, and meet the Individual Fair Share. Our first result is that this impossibility disappears if we weaken SP as explained below. To motivate this weakening, we adapt to our model the extremely familiar idea of equalizing individual utilities while respecting efficiency.

The lexicographic ordering in $[0,1]^{\{1,\cdots,n\}}$ maximizes the first coordinate, and when this is not decisive, the second one, and so on. For a utility profile $U \in [0,1]^N$ the vector $U^* \in [0,1]^{\{1,\cdots,n\}}$ is obtained by rearranging its coordinates increasingly. Then the leximin ordering $\succ_{leximin}$ compares U^1 and U^2 in $[0,1]^N$ exactly as the lexicographic ordering compares U^{1*} and U^{2*} in $[0,1]^{\{1,\cdots,n\}}$.

Egalitarian rule (EGAL): $F^{eg}(M) = \arg \max_{U \in \Phi(M)} \succ_{leximin}$.

[2]Recall from Propositions 2 and 3 in Bogomolnaia et al. (2005) that in our model group versions of SP are not compatible with efficiency, even in the ex post sense.

This maximization yields a unique and efficient utility profile (see e. g., Lemma 1.1 in (Moulin, 1988)). Anonymity and Neutrality are clear. To check Individual fair Share, pick for each agent i a pure outcome a_i she likes, and observe that the uniform average of the a_i-s ensures utility at least $1/n$ to each agent: therefore the egalitarian profile U^{eg} must have $U_1^{eg*} \geq 1/n$.

Here is the simplest problem where the rule EGAL is vulnerable to a misreport of preferences:

$$\text{true profile } u = \begin{array}{c|ccc} N\downarrow A\rightarrow & a & b & c \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array} \rightarrow \text{ misreport } \widetilde{u} = \begin{array}{c|ccc} N\downarrow A\rightarrow & a & b & c \\ 1 & 1 & \widetilde{0} & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}.$$

At the true profile u outcome a is dominated and EGAL mixes b and c, $z = (0, \frac{1}{2}, \frac{1}{2})$. After the misreport by agent 1, outcome a no longer appears dominated and EGAL mixes equally the three outcomes, $\widetilde{z} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Agent 1's utility raises from 1/2 at z to 2/3 at $\widetilde{u}$, *because he can enjoy outcome b despite pretending not to.* The latter is avoidable if the public outcome are excludable: based on reported preferences, the mechanism excludes agents from consuming outcomes they claim to dislike. Recall the discussion of this possibility in the examples of Section 1.

The following incentives property, where we use the notation $u_i \wedge u'_i$ for the coordinate-wise minimum of the two utility functions, captures the resulting weaker incentive compatibility requirement:

Excludable Strategyproofness (EXSP)

$$u_i \cdot f(M) \geq \max_{z' \in f(N,A,(u|^i u'_i))} (u_i \wedge u'_i) \cdot z' \quad \text{for all } M,\, i \text{ and } u'_i.$$

To make this definition more explicit, we identify the true utility u_i by its *like-set* $L_i = \{a \in A | u_{ia} = 1\}$, and partition it as $L_i = L_i^0 \cup L_i^-$. Agent i's misreports is $L'_i = L_i^0 \cup L_i^+$ where $L_i^+ \subseteq A \diagdown L_i$: she pretends to like L_i^+ and to dislike L_i^-. The like-set of $u_i \wedge u'_i$ is L_i^0 therefore EXSP reads:

$$z_{L_i^0 \cup L_i^-} \geq z'_{L_i^0} \quad \text{for all } z \in f(M),\, z' \in f(N,A,(u|^i u'_i)).$$

It is useful to decompose EXSP in two statements. In the first one agent i misreports only by inflating her like-set ($L_i = L_i^0$):

$$\mathbf{SP}^+\text{: } u_i \cdot f(M) \geq \max_{z' \in f(N,A,(u|^i u'_i))} u_i \cdot z' \quad \text{for all } M,\, i \text{ and } u'_i \text{ s. t. } u_i \leq u'_i.$$

and in the second one, only by decreasing this set ($L_i^+ = \varnothing$):

$$\mathbf{SP}^-\text{: } u_i \cdot f(M) \geq \max_{z' \in f(N,A,(u|^i u'_i))} u'_i \cdot z' \quad \text{for all } M,\, i \text{ and } u'_i \text{ s. t. } u'_i \leq u_i.$$

That EXSP equals the combination of SP^+ and SP^- is clear by applying first SP^- from u_i to $u_i \wedge u'_i$, then SP^+ from $u_i \wedge u'_i$ to u'_i. Similarly SP is the combination of SP^+ and SP^*:

$$\mathbf{SP}^*\text{: } u_i \cdot f(M) \geq \max_{z' \in f(N,A,(u|^i u'_i))} u_i \cdot z' \text{ for all } M,\, i \text{ and } u'_i \text{ s. t. } u'_i \leq u_i.$$

and the above example shows that EGAL violates SP^*.

Theorem 1 *The Egalitarian rule is Efficient, Excludable Strategyproof, and guarantees Individual Fair Shares.*

Proof

Preliminary notation and remarks: If $M \subseteq N$ and $U \in [0,1]^M$ then U^* is the set of *distinct* coordinates U^{*k} of U arranged increasingly; so U^* may be of lower dimension than M.

Fix a problem $\mathcal{M} = (N, A, u)$. For any $M \subseteq N$ and convex compact $C \subseteq \Delta(A)$ the projection on M of the set of feasible utility profiles $\Phi(C) = \{U = u \cdot z | z \in C\}$ is convex and compact, so it admits a unique leximin optimal element that we write $F^{eg}(M, C, u) \in [0,1]^M$. This extends the domain of the mapping F^{eg}, and note that we abuse notation by keeping u instead of its restriction to $M \times A$.

Recall the algorithm defining $U = F^{eg}(M, C, u)$. Start with

$$U^{*1} = \max_{z \in C} \min_{j \in M} \{u_j \cdot z\}..$$

Write N^1 for the set of agents achieving this minimum, $P^1 = N \diagdown N^1$, and $C^1 = \{z \in C | u_j \cdot z = U^{*1} \text{ for all } j \in N^1\}$. We stop if $N^1 = N$, otherwise we set $U^{*2} = \max_{z \in C^1} \min_{j \in P^1} \{u_j \cdot z\}$. We let N^2 be the set of agents achieving U^{*2}, $P^2 = N \diagdown (N^1 \cup N^2)$, and C^2 the subset of C^1 achieving U^{*2} in N^2; we stop if $P^2 = \varnothing$, otherwise we set $U^{*3} = \max_{z \in C^2} \min_{j \in P^2} \{u_j \cdot z\}$, and so on. We end up with a partition $N = \cup_{k=1}^K N^k$ such that U_i equals U^{*k} whenever $i \in N^k$.

Turning to the proof of statement i), we saw that it is enough to show separately SP^- and SP^+. Fix an arbitrary $\mathcal{M} = (N, A, u)$. An agent who likes all outcomes, $u_i = \mathbf{1}$, cannot benefit from any misreport; pick now $i \in N$ such that $u_{ia} = 0$ for at least one a, and a profile $\widetilde{u}$ identical to u for all $j \in N \diagdown i$ and such that $u_i \lneqq \widetilde{u}_i$ (so at least one 0 in u_i is changed to a 1). Let $U = F^{eg}(N, A, u)$ and $\widetilde{U} = F^{eg}(N, A, \widetilde{u})$ be implemented respectively by some lotteries z and $\widetilde{z}$. We prove successively:

$$\widetilde{U}_i \geq U_i \tag{2}$$

$$U_i = u_i \cdot z \geq u_i \cdot \widetilde{z} \tag{3}$$

The first inequality implies SP^- (when i with true $\widetilde{U}_i$ reports U_i), the second gives SP^+ (when i with true U_i reports $\widetilde{U}_i$).

We clearly have $\widetilde{U} \succeq_{leximin} U$, in particular $\widetilde{U}^{*1} \geq U^{*1}$: this proves (2) if $U_i = U^{*1}$. Assume for the rest of the proof $U_i = U^{*\ell}$ where $\ell \geq 2$. We check first $\widetilde{U}^{*1} = U^{*1}$. If $\widetilde{U}^{*1} > U^{*1}$ we pick $\varepsilon \in]0,1]$, and note that the mixture $z' = \varepsilon \widetilde{z} + (1 - \varepsilon) z$ ensures $u_j \cdot z' > U^{*1}$ for all $j \in N \diagdown i$; for ε small enough we also have $u_i \cdot z' > U^{*1}$ because $u_i \cdot z > U^{*1}$. This contradicts the definition of U^{*1}.

Set $N^1 = \{j|U_j = U^{*1}\}$ and $\widetilde{N}^1 = \{j|\widetilde{U}_j = U^{*1}\}$. We use a similar argument to show next $N^1 \subseteq \widetilde{N}^1$. If $j \in N^1$ and $u_j \cdot \widetilde{z} > U^{*1}$, then for any $\varepsilon \in]0,1]$ the mixture $z' = \varepsilon\widetilde{z} + (1-\varepsilon)z$ gives $u_k \cdot z' \geq U^{*1}$ for all $k \in N \diagdown \{j,i\}$ and $u_j \cdot z' > U^{*1}$; for ε small enough we also have $u_i \cdot z' > U^{*1}$ (because $U_i = U^{*\ell} > U^{*1}$) and then z' guarantees exactly U^{*1} to a smaller set of agents than z, and strictly more to all others. This implies that $u \cdot z'$ leximin-dominates $u \cdot z$, contradiction.

Similarly the strict inclusion $N^1 \subsetneq \widetilde{N}^1$ would imply that the vector $\widetilde{U}$ is strictly leximin-dominated by U, which we saw is not true.

So far we have shown that the maxi-minimization of feasible utilities – the first step in the algorithm defining the leximin solution– gives at u and $\widetilde{u}$ identical values U^{*1} and $\widetilde{U}^{*1}$, and identical sets N^1 and $\widetilde{N}^1$. Now the second step of the algorithms, delivering U^{*2}, $\widetilde{U}^{*2}$, and N^2, $\widetilde{N}^2$, is the same maxi-minimization problem applied in both cases to $C^1 = \{z \in \Delta(A)|u_j \cdot z = U^{*1} \text{ for all } j \in N^1\}$ and $P^1 = N \diagdown N^1$. Mimicking the above proof we deduce that, if $U_i = U^{*2}$ then $\widetilde{U}_i \geq U_i$, and if $U_i = U^{*\ell}$ for some $\ell \geq 3$, then $U^{*2} = \widetilde{U}^{*2}$, $N^2 = \widetilde{N}^2$. The induction argument establishing $U^{*k} = \widetilde{U}^{*k}$, $N^k = \widetilde{N}^k$ up to $k = \ell - 1$, and finally (2) is now clear.

To prove (3) we compare the profiles $u \cdot z$ and $u \cdot \widetilde{z}$. We just saw that they coincide on $N \diagdown P^{\ell-1} = \cup_{k=1}^{\ell-1} N^k$, and that if a mixture guarantees utility U^{*k} to all agents in N^k for $k = 1, \cdots, \ell-1$, it cannot guarantee (at u) more than $U^{*\ell}$ to all agents in $P^{\ell-1}$: z and $\widetilde{z}$ are two such lotteries, so if $u_i \cdot \widetilde{z} > u_i \cdot z = U^{*\ell}$, there is some $j \in P^{\ell-1}$ for whom $u_j \cdot \widetilde{z} < U^{*\ell}$. But $\widetilde{U}^{*\ell} \geq U^{*\ell}$ (because $\widetilde{U}$ weakly leximin-dominates U) and $\widetilde{u}_j \cdot \widetilde{z} = u_j \cdot \widetilde{z} \geq \widetilde{U}^{*\ell}$, thus we reach a contradiction. ∎

5 Strict Participation and Unanimous Fair Share

A striking feature of the Egalitarian rule is *Clone Invariance*: if at least one voter who shares my preferences does vote, adding my own vote will not change the resulting mixture. This holds because, fixing an agent i, the leximin ordering compares two utility profiles U and U' in the same way as $\widetilde{U}$ and $\widetilde{U'}$, where from V to $\widetilde{V}$ we add an $(n+1)$-th coordinate repeating V_i. Thus the rule is oblivious to the size of support for a particular preference, an unpalatable feature in all the examples discussed in the introduction.

We now define two requirements capturing, each in a different way, the concern that numbers should matter. The first one is an incentive property.

Given a problem M and agent i, define $M(-i) = (N \diagdown i, A, u_{-i})$ and $U_i(-i) = \max_{z \in f(M(-i))} u_i \cdot z$.

Participation (PART)**:** $F_i(M) \geq U_i(-i)$ for all M and i.

The violation of Participation is commonly called the No Show Paradox (Fishburn and Brams, 1983): a voter is better off abstaining to go to the polls. In the context of budgetary participation, we want more: everyone should have a strict incentive to show up, lest many agents, feeling disenfranchised, will stay home or put a blank ballot, and the result of the vote will not give an accurate picture of the opinion profile.

Strict Participation (PART*):

$$F_i(M) \geq U_i(-i) \text{ and } \{U_i(-i) < 1 \Longrightarrow F_i(M) > U_i(-i)\} \text{ for all } M \text{ and } i.$$

A consequence of PART* is *Clone Responsiveness*: I am strictly better off if one or more agents with preferences identical to mine cast their vote. Thus the Egalitarian rule violates PART*, although it satisfies PART.[3]

The second axiom, in the spirit of cumulative voting, allows groups of agents with identical preferences to pool their respective shares of decision power. This leads to the following strengthening of IFS, where we set again $U = F(M)$:

Unanimity Fair Share (UFS) :

$$\text{for all } S \subseteq N\text{: } \{u_i = u_j \text{ for all } i, j \in S\} \Longrightarrow U_i \geq \frac{|S|}{n} \text{ for all } i \in S.$$

In the statement of UFS the unanimous group S can be a minority or a majority. However unanimous preferences are much more likely in small than large groups, so this property will be more relevant in practice to minorities.

All three rules discussed in the next two sections meet Strict Participation and Unanimous Fair Share. Thus they cannot be both efficient and strategyproof. We start with two strategyproof rules.

6 Incentive compatibility and Fairness; the Conditional Utilitarian rule

We introduce two rules adapting to our model the familiar random dictator mechanism (see Gibbard (1977)). The difficulty is the treatment of indifferences: if I can dictate the outcome for a $1/n$-th share of the time, how should I choose in my like-set ?

The first rule, introduced by Duddy (2015), applies a simple utilitarian test: I focus on the outcomes liked by the largest number of other agents. Consider the set $\Phi^p(M; i) = \{U \in \Phi^p(M) | U_i = 1\}$ of all the utility profiles corresponding to the like-set of agent i. Each agent spreads her share $\frac{1}{n}$ equally between the profiles in $\Phi^p(M; i)$ with maximal support:

Conditional Utilitarian (CUT) rule: $F^{cut}(M) = \dfrac{1}{n} \sum_{i \in N} avg\{U | U \in \arg \max_{U' \in \Phi^p(M;i)} U'_N.\}$

[3] Define $U^* = \arg\max_{U \in \Phi(M)} \succ_{leximin}$; $\overline{U}_{-1} = \arg\max_{U \in \Phi(M(-1))} \succ_{leximin}$ and $\overline{U}_1 = U_1(-1)$. If $\overline{U}_1 > U^*_1$ we have successively $\overline{U} \succeq_{leximin} (\overline{U}_1, U^*)$ then $(\overline{U}_1, U^*) \succ_{leximin} U^*$, contradiction.

Remark 1. Our definition of the domain Ω allows for agents who like all outcomes, $u_i = 1^A$. The presence of such agents is of no consequence for the rules UTIL, EGAL, RP and NMP, but it does impact the mixture selected by the CUT rule, as such agents put their weight on the utilitarian outcomes (those with largest support). Suppose we choose to exclude those agents in the definition of the CUT rule: this will not affect the incentives and fairness properties of the rule identified below, nor its Decentralization property in Section 8.

The next rule uses a familiar hierarchical rule to resolve indifferences, that plays a critical role in probabilistic voting ((Aziz, Brandt, and Brill, 2013)), as well as for assigning indivisible private goods ((Abdulkadiroğlu and Sönmez, 1998), (Bogomolnaia and Moulin, 2001)). Let $\Theta(N)$ be the set of strict orderings σ of N. For any $\sigma \in \Theta(N)$ the *σ-Priority rule* F^σ guarantees full utility to agent $\sigma(1)$; next to agent $\sigma(2)$ as well if 1 and 2 like a common outcome, else $\sigma(2)$ is deemed irrelevant; next to agent $\sigma(3)$ if she likes an outcome that all relevant agents before her like, else she is irrelevant; and so on.

Random Priority rule (RP)

$$F^{rp}(M) = \frac{1}{n!} \sum_{\sigma \in \Theta(N)} F^\sigma(M) \text{ where } F^\sigma(M) = \arg \max_{U \in \Phi(M)} \succ^\sigma_{lexico} .$$

If mixtures in $\Delta(A)$ represent lotteries, the RP rule picks an ordering σ with uniform probability and computes U^σ. But in other interpretations, time shares or the distribution of other resources, this simple implementation is not available. We retain nevertheless the intuitive probabilistic terminology.

After checking that both rules are incentive compatible and fair, we compare them from the efficiency angle, and recap our discussion in Theorem 2 below.

Clearly each σ-Priority rule F^σ is strategyproof, and SP is preserved by convex combinations, thus RP is strategyproof as well. Checking PART* is equally easy: to each ordering $\widetilde{\sigma}$ of $N \diagdown i$ we associate the n orderings σ of N where i can have any rank from first to last: agent i is weakly better off at $F^\sigma(M)$ than at $F^{\widetilde{\sigma}}(M(-i))$, strictly so if he is gets utility 0 in $F^{\widetilde{\sigma}}(M(-i))$; thus the only case where i does not strictly benefit by showing up is when he gets utility 1 in $F^{rp}(M(-i))$. For UFS, it is enough to observe that a member of coalition S is first in σ with probability $\frac{|S|}{n}$.

Check now that CUT meets the same three properties. UFS is clear. We decompose SP into the combination of SP$^+$ and SP*. Start with SP*: with the notations just before Theorem 1, assume agent i's like-set is $L_i = L_i^0 \cup L_i^-$, and he reports L_i^0 instead. Fix another agent j and check that the potential change in the way j uses her $1/n$-th share does not hurt i. If j was putting no weight in L_i^-, then she still loads exactly the same set. If j was only loading (a subset of) L_i^-, she was helping i as much as possible and cannot do more after the change. If j was loading some outcomes in L_i^-, some in $B \subseteq L_i^0$ and some in $C \subseteq A \diagdown L_i$, where $B \cup C \neq \varnothing$, then she redistributes all her weight on L_i^- uniformly in $B \cup C$ and this clearly cannot benefit agent i. Turning to SP$^-$ we assume i likes L_i and reports $L'_i = L_i \cup L_i^+$ instead. If j was putting some load in L_i^+, she now loads only L_i^+, so she does not help i at all. If she was not loading L_i^+ at all, she may do so now, in which case some weight will be taken uniformly from the set she was loading before: this cannot benefit i strictly.

Check PART*. Fix a problem M, an agent i, and for every $j \in N \diagdown i$ let B_j be the set of outcomes agent j loads in problem $M(-i)$. Set $N^+ = \{j \in N \diagdown i | B_j \cap L_i \neq \varnothing\}$ and $N^- = N \diagdown (N^+ \cup i)$. Before participating agent i's utility was

$$\frac{1}{n-1} \sum_{j \in N^+} \lambda_j \text{ where } \lambda_j = \frac{|B_j \cap L_i|}{|B_j|}.$$

After i shows up every j in N^+ loads only $B_j \cap L_i$, and agents in N^- may give some of their load to L_i therefore i's utility is at least $\frac{1}{n}(1 + |N^+|)$. The inequality

$$\frac{1}{n-1} \sum_{j \in N^+} \lambda_j \leq \frac{|N^+|}{n-1} \leq \frac{1}{n}(1 + |N^+|).$$

proves PART. And both inequalities are equalities if and only if each $\lambda_j = 1$ and $|N^+| = n - 1 \Leftrightarrow N^+ = N \diagdown i$; the latter implies that i's utility is already 1 in $M(-i)$.

Example (1) above shows that both RP and CUT are inefficient. Under the CUT rule agents $1, 2$ and 5 load only d, while agent 3 spreads his load between a and b, and agent 4 between a and c, resulting in the mixture $z^{cut} = (\frac{1}{5}, \frac{1}{10}, \frac{1}{10}, \frac{3}{5}, 0)$. Under RP we get $z^{rp} = (\frac{1}{5}, \frac{1}{6}, \frac{1}{6}, \frac{7}{15}, 0)$; for instance b is selected in two cases only: if 3 is first, and 5 comes before 4 (proba. $\frac{1}{10}$), or 5 is first and 3 is first among $1, 2, 3$ (proba. $\frac{1}{15}$). As noted at the end of Section 3, shifting the weight of b and c to a and d is a Pareto improvement. Clearly, then, z^{rp} is more inefficient than z^{cut}.

In our next example, with $n = 6$ and $|A| = 5$,

$N \downarrow A \rightarrow$	a	b	c	d	e
1	1	0	0	1	0
2	1	0	0	0	1
3	0	1	0	1	0
4	0	1	0	0	1
5	0	0	1	1	0
6	0	0	1	0	1

(4)

the CUT rule selects the efficient mixture $z^{cut} = (0, 0, 0, \frac{1}{2}, \frac{1}{2})$ and $U_i^{cut} = 0.5$ for all i, while RP picks $z^{rp} = (\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, \frac{1}{3})$ and $U_i^{rp} = 0.44$ for all i: thus z^{cut} is strictly Pareto superior to z^{rp}.

The reverse situation cannot happen: the RP mixture never Pareto dominates the CUT one. This follows because in all problems, total utility under RP is at most that under CUT: $U_N^{rp} \leq U_N^{cut}$. Indeed U^{cut} is the uniform average of profiles $U(i)$ maximizing total utility in $\Phi^p(M; i)$, and for each ordering σ where i is first, the corresponding profile U^σ is in $\Phi^p(M; i)$ as well.[4]

[4] A consequence of this remark is that CUT and RP pick the same utility profile at problem M if and only if all undominated outcomes of M are liked by the same number of agents.

We prove finally that *the CUT rule is efficient more often than RP*: whenever RP picks an efficient mixture, so does CUT. Observe first that both rules only give weight to undominated pure outcomes. In the case of RP every such outcome a has a positive weight, because it is selected whenever the set of agents who like a has the highest priority. Thus the support of the RP mixture is exactly the set of all undominated columns. Therefore RP selects an efficient mixture if and only if all mixtures with support in this set are efficient as well. The claim follows because the CUT rule is also a combination of undominated columns.

The next result adds to the discussion above some worst case computations reinforcing the strong efficiency advantage of CUT over RP.

Theorem 2

i) Both rules CUT and RP are strategyproof and meet Strict Participation and Unanimity Fair Share.

ii) Total utility at the CUT mixture is never below that at the RP mixture, and the former may Pareto dominate the latter. If RP picks an efficient mixture at some problem M, so does CUT.

iii) The CUT rule is $\varepsilon^{cut}(n)$-efficient with $\varepsilon^{cut}(n) = O(n^{-\frac{1}{3}})$ and for all $n \geq 5$ we have

$$\varepsilon^{cut}(n) \geq \frac{1}{n} + (1 - \frac{1}{n^{\frac{1}{3}}})\frac{3}{n^{\frac{1}{3}}} \tag{5}$$

The RP rule is $\varepsilon^{rp}(n)$-efficient with $\varepsilon^{rp}(n) \leq O(\frac{\ln(n)}{n})$.

iv) The CUT mixture is computed in time polynomial in $n + |A|$; computing the RP rule is #P-complete in $n + |A|$.

Recall from the Lemma in Section 3 that both CUT and RP are efficient if $n \leq 4$. For small values of n, the lower bound (5) implies a high guaranteed efficiency of CUT, a lower bound on $\varepsilon^{cut}(n)$, and the computations in Step 2 of the proof below yield a much smaller worst case efficiency of RP, an upper bound on $\varepsilon^{rp}(n)$:

n	6	8	12	32	64	1024	16384
$\varepsilon^{cut} \geq$	91%	87%	82%	68%	58%	27%	11%
$\varepsilon^{rp} \leq$	83%	72%	64%	40%	24%	3%	0.12%

Together statements *ii*) to *iv*) make a very strong case that in our model the CUT rule is a much more efficient interpretation of the random dictator idea than RP.

Proof of statement *iii*) (Statement *iv*) is explained in Section 9)

Step 1 Worst case inefficiency of CUT

Step 1.a: We construct a problem with large n where the CUT profile is $O(n^{-\frac{1}{3}})$ inefficient. We fix N of size n, a partition $N = N_1 \cup N_2$, and an integer p such that

$$p < n_1, n_2 \text{ and } n_1 \text{ divides } (p-1)n_2 \text{ where } n_i = |N_i|, i = 1, 2.$$

Problem M has $2n_2 + 1$ outcomes labeled as $A = \{a\} \cup B \cup C$, where $B = \{b_j, j \in N_2\}$ and $C = \{c_j, j \in N_2\}$. Setting $(p-1)n_2 = qn_1$, each agent $i \in N_1$

likes a, exactly q outcomes in B, and none in C; and each $j \in N_2$ dislikes a, likes only outcome b_j in B, and exactly $p-1$ outcomes in C. Moreover the problem is symmetric in N_1 and in N_2, which can be achieved by arranging cyclically the like-sets of the N_1 agents in B and the like-sets of the N_2 agents in C. Here is an example with $n_1 = n_2 = 5, p = 4$ and $q = 3$, and the top five agents form N_1:

a	b_1	b_2	b_3	b_4	b_5	c_1	c_2	c_3	c_4	c_5
1	1	0	0	1	1	0	0	0	0	0
1	1	1	0	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0
1	0	1	1	1	0	0	0	0	0	0
1	0	0	1	1	1	0	0	0	0	0
0	1	0	0	0	0	1	0	0	1	1
0	0	1	0	0	0	1	1	0	0	1
0	0	0	1	0	0	1	1	1	0	0
0	0	0	0	1	0	0	1	1	1	0
0	0	0	0	0	1	0	0	1	1	1

Note that each outcome b_j is liked by exactly p agents, all but one of them in N_1, and each c_j is liked by exactly $p-1$ agents, all in N_2.

Under the CUT rule, each agent $i \in N_1$ loads only a because $n_1 > p$, so $z_a = \frac{n_1}{n}$, and each $j \in N_2$ loads only b_j so $z_{b_j} = \frac{1}{n}$; there is no weight on C. Total utility in each group is

$$U_{N_1} = \frac{(n_1)^2}{n} + \frac{n_2}{n}(p-1) \ ; \ U_{N_2} = \frac{n_2}{n}$$

and by the symmetries these are equally shared in N_1 and N_2 respectively.

Now consider the mixture z': $z'_a = \frac{2}{3}$, $z'_{c_j} = \frac{1}{3n_2}$ for all $j \in N_2$, and zero weight on B, resulting in the total utilities

$$U'_{N_1} = \frac{2}{3}n_1 \ ; \ U'_{N_2} = \frac{1}{3}(p-1)$$

again equally shared in each N_i.

For n large enough we can pick n_1 and p such that $n_1 \simeq n^{\frac{2}{3}}$ and $p-1 \simeq n^{\frac{1}{3}}$ (if n is a cube these values are exact and $q = n^{\frac{2}{3}} - n^{\frac{1}{3}}$) so that $\frac{n_2}{n} \simeq 1$. This yields the ratios

$$\frac{U'_{N_1}}{U_{N_1}} \simeq \frac{\frac{2}{3}n^{\frac{2}{3}}}{2n^{\frac{1}{3}}} = \frac{1}{3}n^{\frac{1}{3}} = \frac{U'_{N_2}}{U_{N_2}}$$

and completes the proof of Step 1.a.

Step 1.b. For an arbitrary problem M we give an upper-bound of the inefficiency of the CUT mixture.

We fix a problem M and partition the agents according to their scores $\max_{U \in \Phi^p(M;i)} U_N$, i.e., the utilitarian score of the outcomes on which they spread their weight under the CUT rule. Let $p_1 > p_2 > \cdots > p_K > 0$ be the sequence of such scores and N_k the subset of agents who load outcomes with score p_k. Note that $n_1 \geq p_1$. Set A_k to be the set of outcomes loaded by at least one agent in N_k: they all have the same score p_k so the A_k-s are pairwise disjoint. Note also that agents in N_k do not like any outcome in A_ℓ for $\ell < k$.

Consider finally the outcomes b in $B = A \diagdown (\cup_1^K A_k)$, if any. Their utilitarian score u_{N_b} is at most $p_1 - 1$. We partition B by gathering in B_k all the outcomes with a score in $[p_{k+1}, p_k[$, with the convention $p_{K+1} = 0$. Therefore the agents in N_k do not like any outcome in B_ℓ for $\ell < k$.

We prove first that for any feasible profile $U \in \Phi(M)$, we can find convex weights $\pi_1, \cdots, \pi_K$ such that

$$U_{N_k} \leq \pi_k p_k \text{ for } k = 1, \cdots, K \tag{6}$$

Pick $z \in \Delta(A)$ implementing U and write for simplicity $z_{A_k} = x_k$ and $z_{B_k} = y_k$. The total contribution[5] $U_{NA_k} = x_k p_k$ of A_k to U_N is shared between the agents of $\cup_1^k N_\ell$ only, so there are some convex weights $\gamma_\ell^k, 1 \leq \ell \leq k$, such that

$$U_{N_\ell A_k} = \gamma_\ell^k x_k p_k \text{ for all } 1 \leq \ell \leq k \leq K.$$

Similarly the contribution U_{NB_k} of B_k is shared in $\cup_1^k N_\ell$ and $U_{NB_k} \leq y_k p_k$. So we can find convex weights $\delta_\ell^k, 1 \leq \ell \leq k$, such that

$$U_{N_\ell B_k} \leq \delta_\ell^k y_k p_k \text{ for all } 1 \leq \ell \leq k \leq K.$$

Combining the above equality and inequality we have for all k

$$U_{N_k} = \sum_{\ell=k}^{K} (U_{N_k A_\ell} + U_{N_k B_\ell}) \leq \sum_{\ell=k}^{K} (\gamma_k^\ell x_\ell + \delta_k^\ell y_\ell) p_\ell \leq p_k \sum_{\ell=k}^{K} (\gamma_k^\ell x_\ell + \delta_k^\ell y_\ell)$$

so the weights $\pi_k = \sum_{\ell=k}^{K} (\gamma_k^\ell x_\ell + \delta_k^\ell y_\ell)$ are indeed convex and satisfy (6).

Next we evaluate the blocks of the profile U^{cut} in the same fashion. Agents in N_k load exclusively A_k therefore if z implement U^{cut} we have $z_{A_k} = \frac{n_k}{n}$ and $U_{NA_k}^{cut} = \frac{n_k}{n} p_k$. We can find as above convex weights $\theta_\ell^k, 1 \leq \ell \leq k$, such that

$$U_{N_\ell A_k}^{cut} = \theta_\ell^k \frac{n_k}{n} p_k \text{ for all } 1 \leq \ell \leq k \leq K$$

and then as above we get

$$U_{N_k}^{cut} = \sum_{\ell=k}^{K} \theta_k^\ell \frac{n_\ell}{n} p_\ell.$$

Assume now the profile U^{cut} is ε-efficient: $U^{cut} \leq \varepsilon U$ for some feasible U. From (6) we find convex weights π such that $U_{N_k}^{cut} \leq \varepsilon \pi_k p_k$ for all k, which implies

$$\varepsilon \geq \sum_{k=1}^{K} \frac{1}{p_k} U_{N_k}^{cut} = \sum_{\ell=1}^{K} \frac{n_\ell}{n} \sum_{k=1}^{\ell} \theta_k^\ell \frac{p_\ell}{p_k}.$$

[5]Recall our notation $u_{SB} = \sum_{i \in S} \sum_{a \in B} u_{ia}$.

The key inequality is $U^{cut}_{N_k A_k} \geq \frac{n_k}{n}$ because agent $i \in N_k$ loads only A_k containing his like-set: this implies $\theta^k_k \geq \frac{1}{p_k}$. Moreover in the sum $\sum_{k=1}^{\ell} \theta^\ell_k \frac{p_\ell}{p_k}$ the terms $\frac{p_\ell}{p_k}$ increase in k. Combining these two observations we have for any $\ell \geq 2$:

$$\sum_{k=1}^{\ell} \theta^\ell_k \frac{p_\ell}{p_k} \geq (\sum_{k=1}^{\ell-1} \theta^\ell_k) \frac{p_\ell}{p_1} + \theta^\ell_\ell \geq (1 - \frac{1}{p_\ell}) \frac{p_\ell}{p_1} + \frac{1}{p_\ell} = \frac{p_\ell - 1}{p_1} + \frac{1}{p_\ell}.$$

We invoke now the inequality $\frac{\alpha-1}{p_1} + \frac{1}{\alpha} \geq \frac{2}{\sqrt{p_1}} - \frac{1}{p_1}$, for any $\alpha > 0$, that we apply to each $\alpha = p_\ell, \ell \geq 2$, and combine with the two inequalities above as well as $\theta^1_1 = 1$:

$$\varepsilon \geq \frac{n_1}{n} + (1 - \frac{n_1}{n})(\frac{2}{\sqrt{p_1}} - \frac{1}{p_1}).$$

Finally the term $\frac{2}{\sqrt{p_1}} - \frac{1}{p_1}$ decreases in p_1 and we know $p_1 \leq n_1$, so we get

$$\varepsilon \geq \frac{1}{n}(n_1 + (n - n_1)(\frac{2}{\sqrt{n_1}} - \frac{1}{n_1})).$$

It remains to compute the minimum of the above expression for fixed n and variable $n_1 \in [1, n]$. With the real variable x instead of n_1 the right hand term and its derivative are

$$\varphi(x) = \frac{1}{n}(1 + x - 2\sqrt{x}) + (\frac{2}{\sqrt{x}} - \frac{1}{x}) \Longrightarrow \varphi'(x) = (1 - \frac{1}{\sqrt{x}})(\frac{1}{n} - \frac{1}{x^{\frac{3}{2}}}).$$

therefore $x = n^{\frac{2}{3}}$ achieves the minimum and we compute

$$\varepsilon \geq \varphi(n^{\frac{2}{3}}) = \frac{1}{n} + (1 - \frac{1}{n^{\frac{1}{3}}}) \frac{3}{n^{\frac{1}{3}}}.$$

which is inequality (5).

Step 2: Lower bounding the worst case inefficiency of RP

Fix N and integers k, d, ℓ such that $n = kd$ and $2 \leq \ell < k$. Fix a partition $N^1 \cup \cdots \cup N^d$ of N where each subset contains k agents. This construction requires $n \geq 6$ and is not feasible for all n.

We consider the problem with $A = D \cup C$ where $D = \{1, \cdots, d\}$ and each $\delta \in D$ is liked exactly by the k agents in N^δ; also $|C| = \binom{n}{\ell}$ and each outcome in C is liked exactly by a different subset of ℓ agents.

The symmetric (egalitarian) and efficient outcome is the uniform distribution in D and yields the utility profile $U^*_i = \frac{1}{d}$ for all i. We compute now the symmetric profile U implemented by RP.

Fix an ordering $\sigma \in \Theta(N)$ and let L be the set of its ℓ highest priority agents. In the resulting profile U^σ, the first ℓ agents have full utility (because there is $a \in C$ where they all do). Two cases arise. In the favourable case L is contained in some set N^δ: then δ is the only efficient pure outcome liked by all agents in L, thus it must be chosen by the σ-priority rule and $U^\sigma_N = k$. In the unfavourable case L straddles two or more sets N^δ and there is only one outcome (in C) that everyone in L like, so that $U_N^\sigma = \ell$. Therefore

$$U_N = \frac{d\binom{k}{\ell}}{\binom{n}{\ell}} \cdot k + (1 - \frac{d\binom{k}{\ell}}{\binom{n}{\ell}}k) \cdot \ell = (k - \ell)\frac{n}{k}\frac{\binom{k}{\ell}}{\binom{n}{\ell}} + \ell.$$

$$\Longrightarrow \varepsilon(n) \leq \frac{U_N}{U_N^*} = (1 - \frac{\ell}{k})\frac{(k-1)\cdots(k-\ell+1)}{(n-1)\cdots(n-\ell+1)} + \frac{\ell}{k} \qquad (7)$$

For the asymptotic statement we use the inequality $\frac{\binom{k}{\ell}}{\binom{n}{\ell}} \leq (\frac{k}{n})^\ell$ and compute

$$\Rightarrow \frac{U_i}{U_i'} = \frac{U_N}{U_N'} \leq (\frac{k}{n})^{\ell-1} + \frac{\ell}{k}.$$

Then we choose $k \simeq \frac{n}{e}$ and $\ell \simeq \ln(n)$ so that $(\frac{k}{n})^{\ell-1} + \frac{\ell}{k} \simeq e\frac{\ln(n)}{n}$. The systematic inequality $\varepsilon^{rp}(n) \leq 6\frac{\ln(n)}{n}$ is obtained by numerical estimations of (7), omitted for brevity.■

Remark 2 The proof of Step 2 improves upon, with a similar proof technique, Example 1 in (Bogomolnaia et al., 2002) establishing that RP is $\frac{2}{\sqrt{n}}$*inefficient.*

7 Efficiency and Fairness; the Nash Max Product rule

Our last rule of interest is a familiar compromise between the Utilitarian and Egalitarian rules:

Nash Max Product rule (NMP): $F^{nsh}(M) = \arg\max_{U \in \Phi(M)} \sum_{i \in N} \ln U_i.$

This rule is well defined because it solves a strictly convex program, and obviously efficient.

Recall that Unanimity Fair Share offers welfare guarantees only to coalitions of agents with identical preferences (clones). The first of our two new "Fair Share" axioms applies, much more generally, to any group who can find at least one outcome that everyone likes:

Average Fair Share (AFS)

for all $S \subseteq N$: $\{\exists a \in A : u_{ia} = 1 \text{ for all } i \in S\} \Longrightarrow \frac{1}{|S|}U_S \geq \frac{|S|}{n}$.

The next property conveys the idea that, as each agent is endowed with $1/n$-th of total decision power, any coalition of size s can cumulate these shares and impose that a mixture of their choice be chosen with probability at least $\frac{s}{n}$:

Core Fair Share (CFS)

for all $S \subseteq N : \nexists z \in \Delta(A)$ s. t. $\forall i \in S,\ U_i \leq \frac{|S|}{n}(u_i \cdot z)$ and $\exists i, U_i < \frac{|S|}{n}(u_i \cdot z)$..

This is a familiar core stability property.

That UFS follows from either AFS or CFS is clear because we only consider anonymous rules. Applying CFS to $S = N$ implies that the rule is efficient, therefore neither the CUT or the RP rule meets CFS. In the example (1) it happens that the AFS property selects uniquely the Nash mixture,[6] therefore CUT and RP fail AFS as well.

We illustrate the bite of AFS in the following example

	a	b	c	d
1	1	0	0	0
2	1	1	1	0
3	0	0	1	1
4	0	1	0	1
5	0	0	0	1

For a mixture $z = (\alpha, \beta, \gamma, \delta)$ we apply AFS to $S = \{1, 2\}$ and $T = \{3, 4, 5\}$

$$2\alpha + \beta + \gamma \geq \frac{4}{5}\ ;\ \beta + \gamma + 3\delta \geq \frac{9}{5}.$$

Adding both inequalities gives $\delta \geq \frac{3}{5}$, then the first one implies $\alpha \geq \frac{2}{5}$ so that $z = (\frac{2}{5}, 0, 0, \frac{3}{5})$: this is precisely the mixture selected by the NMP rule. Clearly z meets CFS as well.

By contrast Core Fair Share selects many more outcomes than the Nash mixture, for instance $z' = (\frac{1}{5}, \frac{3}{20}, \frac{3}{20}, \frac{1}{2})$. So in this example AFS is more demanding than CFS, but in general the two axioms are not logically related. For instance in the problem

1	1	0	0
2	1	1	0
3	0	1	0
4	0	0	1

the mixture $z = (\frac{7}{20}, \frac{7}{20}, \frac{3}{10})$ gives the profile $U = (\frac{7}{20}, \frac{7}{10}, \frac{7}{20}, \frac{3}{10})$. It passes the AFS test but fails CFS because coalition $\{1, 2, 3\}$ achieves utilities $(\frac{3}{8}, \frac{3}{4}, \frac{3}{8})$ by implementing $\frac{3}{4}$ of the mixture $z' = (\frac{1}{2}, \frac{1}{2}, 0)$, and they all improve strictly.

Theorem 3

i) The NMP rule is efficient and meets Strict Participation, Average Fair Share, and Core Fair Share.

ii) The NMP rule is not Excludable Strategyproof.

Proof

Step 1: We prove AFS and CFS.

[6] We leave the proof to the reader and give a similar example in the next paragraph.

The separation inequality capturing the optimality of the Nash utility profile $U^* = F^{nsh}(M)$ at problem M writes as follows:

$$\sum_{i\in N}\frac{U_i}{U_i^*} \leq \sum_{i\in N}\frac{U_i^*}{U_i^*} = n \text{ for all } U \in \Phi(M) \tag{8}$$

Fix $S \subseteq N$ and combine (8) with Cauchy's inequality as follows

$$nU_S^* \geq (\sum_{i\in S}\frac{U_i}{U_i^*}).(\sum_{i\in S}U_i^*)\geq(\sum_{i\in S}\sqrt{U_i})^2 \Longrightarrow$$

$$U_S^* \geq \frac{1}{n}\max_{U\in\Phi(M)}(\sum_{i\in S}\sqrt{U_i})^2 \tag{9}$$

The AFS property follows, because if there is some $a \in A$ such that $u_{ia} = 1$ for all $i \in S$, the maximum on the right hand side is $|S|^2$. To check CFS we assume there is a mixture z such that $U_i^* \leq \frac{|S|}{n}(u_i \cdot z)$ for all $i \in S$ and use again (8) to compute:

$$n \geq \sum_{i\in S}\frac{u_i \cdot z}{U_i^*} \geq \frac{n}{|S|}\sum_{i\in S}\frac{U_i^*}{U_i^*} = n$$

therefore none of the inequalities $U_i^* \leq \frac{|S|}{n}(u_i \cdot z)$ can be strict.

Step 2: We check PART*.

In a preliminary result we fix $S \subset \mathbb{R}_+^N$ convex and compact, and write $S(-1)$ for its projection on $\mathbb{R}_+^{N\diagdown 1}$. Define

$$U^* = \arg\max_{U\in S}\sum_{i\in N}\ln(U_i)$$

$$\overline{U}_{-1} = \arg\max_{U_{-1}\in S(-1)}\sum_{i\in N\diagdown 1}\ln(U_i) \text{ and } \overline{U}_1 = \max_{(U_1,\overline{U}_{-1})\in S}U_1.$$

Inequality $U_1^* < \overline{U}_1$ brings a contradiction as follows

$$\sum_{i\in N}\ln(\overline{U}_i) \geq \ln(\overline{U}_1) + \sum_{i\in N\diagdown\{1\}}\ln(U_i^*) > \sum_{i\in N}\ln(U_i^*).$$

Assume next $U_1^* = \overline{U}_1$. The right hand inequality above becomes an equality, so we get $\sum_{i\in N}\ln(\overline{U}_i) = \sum_{i\in N}\ln(U_i^*)$ and finally $\overline{U} = U^*$. Summing up, we have just proven:

$$U_1^* \geq \overline{U}_1\text{; and if } U_1^* = \overline{U}_1 \text{ then } U_{-1}^* = \overline{U}_{-1} \tag{10}$$

Applying (10) to $S = \Phi(M)$, $U^* = F^{nsh}(M)$, $\overline{U}_{-1} = F^{nsh}(M(-1))$ gives $\overline{U}_1 = U_1(-1)$ and $U_1^* \geq \overline{U}_1$, the first inequality in PART* (i.e., PART). To check the second we can assume that any two columns of u are different, for if two columns are identical one of them can be eliminated as a redundant outcome. Also recall that no row of u is null.

Because $U_i^* > 0$ for all i, the statement is true if $\overline{U}_1 = 0$. We assume now $0 < U_1^* = \overline{U}_1 < 1$ and derive a contradiction. Property (10) implies $U^* = \overline{U}$, therefore there is some $z \in \Delta(A)$ solving both problems: $z \in f^{nsh}(M) \cap f^{nsh}(M(-1))$.

As $0 < U_1^* < 1$ the mixture z cannot be deterministic, moreover there exists two outcomes a, b in the support $[z]$ of z such that $u_{1a} = 1, u_{1b} = 0$. Writing $N(x; y)$ for the set of agents in N who like x and dislike y, this means $1 \in N(a; b)$.

Note that $N(b; a)$ must contain at least one $i \in N \diagdown 1$: otherwise the column U_a dominates column U_b (outcome b is Pareto inferior to a) which contradicts the efficiency of z in M. We claim that $N(a; b)$ as well contains some $j \in N \diagdown 1$: suppose not, then the restriction of column U_b to $N \diagdown 1$ either dominates the corresponding restriction of U_a, or these two restricted columns are equal; the former case contradicts efficiency of z in $M(-1)$, the latter contradicts its efficiency in M.

We have shown that $N(a; b)$ and $N(b; a)$ both contains at least one outcome in $N \diagdown 1$. Recalling that z_a, z_b are both positive, we define $z(\varepsilon)$ by shifting the weight ε from a to b: this outcome is well defined for ε small enough and of arbitrary sign; such a shift does not affect agents outside $N(a; b) \cup N(b; a)$. From $z \in f^{nsh}(M(-1))$ we see that the strictly concave function

$$\varphi(\varepsilon) = \sum_{i \in (N(a;b) \cup N(b;a)) \diagdown 1} \ln(u_i \cdot z(\varepsilon)).$$

reaches its maximum at $\varepsilon = 0$. And $z \in f^{nsh}(M)$ implies that the function $\varphi(\varepsilon) + \ln(u_1 \cdot z(\varepsilon))$ is also maximal at $\varepsilon = 0$: this is a contradiction because $\ln(u_1 \cdot z(\varepsilon))$ decreases strictly in ε.■

The proof that the NMP rule fails EXSP is more involved and relegated to the Appendix. There we construct an example with $|A| = 4$ and $n = 860$ where it violates the SP^+ property. We also report a computer generated example with 36 agents proving the same point. This prompts the following open question: *what are the smallest sizes of* $|N|, |A|$ *ensuring that NMP violates EXSP ?*

Remark 3. Another version of the group fair share requirement is proposed by Bogomolnaia et al. (2002). The same concept was independently proposed by Duddy (2015) who referred to simply as *proportional share* (Duddy, 2015). For the sake of consistency with our other notions, we will refer to it as *Group Fair Share*(GFS).

Writing u^{*S} for the maximum of all utility functions in S ($u_a^{*S} = \max_{i \in S} u_{ia}$), this condition is

$$U^{*S} \geq \frac{|S|}{n} \text{ for all } S.$$

It is clearly stronger than UFS, but strictly weaker than CFS. Both CUT and RP satisfy GFS.

Remark 4. It has been mentioned as an open problem, in the more general voting model with vNM-preferences, whether there exists some rule that satisfies Very Strong Stochastic Dominance Participation and Stochastic Dominance Efficiency for weak orders (Brandl et al., 2015; Brandt, 2017). Because NMP satisfies both Strict Participation and Efficiency, we see that this question is resolved at least for the case of dichotomous preferences.

8 Decentralization

We introduce a *Decentralization* (DEC) property for polarized societies. Say the agents and the deterministic outcomes are color-coded with the same set of colors: we call a profile of preferences *polarized* if each agent only likes outcomes of his own color. The requirement is that if I am red, the number of green agents will matter to me but not their preferences inside green outcomes. This natural independence property adds to the appeal of the NMP rule, but also of the CUT and RP rules.

Consider a problem $M = (N, A, u)$ and two partitions $\Gamma = (N^k)_{k=1}^K$ and $\Lambda = (A^k)_{k=1}^K$ of N and A respectively. We call this problem *polarized along the partitions* Γ, Λ if $u_{ia} = 0$ whenever $i \in N^k, a \in A^k$, and $k \neq k'$. Then if u^k is the restriction of u to $N^k \times A^k$, problem M is captured by its K *subproblems* $M^k = (N^k, A^k, u^k)$. We write $\Pi(\Gamma, \Lambda)$ the set of polarized problems.

Decentralization (DEC): for any Γ, Λ and k

$$\{M, M' \in \Pi(\Gamma, \Lambda) \text{ and } u_{ia} = u'_{ia} \text{ if } i \in N^k, a \in A^k\} \Longrightarrow F_i(M) = F_i(M') \text{ for } i \in N^k.$$

Combined with the UFS property, this implies that in a polarized problem, each colored subset N^k chooses the distribution in $\Delta(A^k)$ as if other colors were not present, then the selected outcome in $f(M^k)$ is weighted down in proportion of the size of N^k.

Proposition *The Nash, Conditional Utilitarian, and Random Priority rules meet Decentralization. Moreover for any polarized problem* $M \in \Pi(\Gamma, \Lambda)$ *they satisfy*

$$F(M) = \sum_{k=1}^{K} \frac{|N^k|}{n} F(M^k) \tag{11}$$

where the profile $F(M^k)$ *is filled with zeros outside* M^k.

Check that the Utilitarian and Egalitarian rules violate DEC. Consider the two polarized problems along the partition $\{1\} \cup (2, 3\}$:

$$M : \begin{matrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{matrix} \qquad M' : \begin{matrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 \end{matrix}.$$

Both UTIL and EGAL choose $z = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ at M, but at M' they pick respectively $z' = (0, 1, 0)$ and $z'' = (\frac{1}{2}, \frac{1}{2}, 0)$, in contradiction of DEC.

Proof of Proposition

Step 1. NMP meets DEC

We prove it for two partitions $\Gamma = (N_1, N_2)$ and $\Lambda = (A_1, A_2)$ as the general case with arbitrary K is just as easy. We fix a problem $M \in \Pi(\Gamma, \Lambda)$ with the two subproblems $M^k = (N^k, A^k, u^k)$, $k = 1, 2$: there is no null row in u^k because there is none in the grand matrix u. Let $z^* \in f^N(M)$ be (one of) NMP's choice in M. By IFS both weights $z^*_{A_1} = \lambda_1$ and $z^*_{A_2} = \lambda_2$ are strictly positive (and sum to 1). By definition (12) the restriction $z^*(A^k)$ of z^* to A^k solves

$$\max_{\widetilde{z} \geq 0, \widetilde{z}_{A^k} = \lambda_k} \sum_{i \in N^k} \ln(u_i \cdot \widetilde{z}).$$

Changing the variables $\widetilde{z}$ to $z = \frac{1}{\lambda_k}\widetilde{z}$, we see that $\frac{1}{\lambda_k} z^*(A^k)$ solves

$$\max_{z \in \Delta(A^k)} \sum_{i \in N^k} \ln(u_i \cdot \lambda z) = n_k \ln \lambda_k + \max_{z \in \Delta(A^k)} \sum_{i \in N^k} \ln(u_i \cdot z)$$

therefore $z^*(A^k) = \lambda_k z^k$, where $z^k \in f^N(M^k)$, and

$$\max_{\widetilde{z} \geq 0, \widetilde{z}_{A^k} = \lambda_k} \sum_{i \in N^k} \ln(u_i \cdot \widetilde{z}) = n_k \ln \lambda_k + \sum_{i \in N^k} \ln F_i^N(M^k).$$

It is now clear that the optimal choice of λ_1, λ_2 is

$$(\lambda_1, \lambda_2) = \arg \max_{\lambda_1 + \lambda_2 = 1} \{n_1 \ln \lambda_1 + n_2 \ln \lambda_2\} = (\frac{n_1}{n}, \frac{n_2}{n}).$$

In particular λ depends only on the sizes of the partition sets, and $U_i^* = \frac{n_k}{n} F_i^N(M^k)$ for each $i \in N^k$, the desired property (11).

Step 2 CUT and RP meet DEC

Fix $M \in \Pi(\Gamma, \Lambda)$ some $k, 1 \leq k \leq K$ and some $i \in N^k$. Under the CUT rule only the agents in N^k will load the outcomes that i likes, and they will do it exactly as in the problem M^k, except that each $j \in N^k$ will spread a total weight of $\frac{1}{n}$ instead of $\frac{1}{n_k}$. This implies (11). The proof for RP is just as easy. ■

9 Computation

In this section we first discuss the computational aspects of the rules we have considered in the paper. We then report on some experiments where we examined the utilitarian performance of the four rules, which in turn gives a lower bound on their efficiency.

1. Computational complexity

The CUT rule is the easiest to compute of the four. In the like-set of each agent we simply need to identify those liked by the largest number of other agents.

For EGAL, the outcome can be computed in polynomial-time by solving at most $n+1$ linear programs each with $|A|$ variables. The algorithm was presented by Aziz and Stursberg (2014).

The RP outcome is #P-complete to compute even under dichotomous preferences (Aziz et al., 2013). Therefore unless P=NP, it is unlikely that there exists an efficient algorithm for computing the RP outcome. For RP, it is even open whether there exists an FPRAS (Fully Polynomial-time Approximation Scheme) for computing the outcome shares/probabilities.

As for NMP, in contrast to RP, an approximate solution can be computed relatively fast by using general optimisation packages and solvers. The problem is to maximize a convex objective $\sum_{i\in N}\log(u_i \cdot z)$ where z is a feasible mixture. Bogomolnaia et al. (2005) discussed some standard approaches to approximate the solution.

2. Experiments

We ran some experiments for small numbers of agents and outcomes. It is difficult to evaluate in a given problem the degree of inefficiency of a given mixture z as in Definition 1, *iii*). However the ratio of utilitarian welfare at z to the maximum utilitarian welfare gives a lower bound on ε, and it is much easier to compute.

For each combination of n and $|A|$ in $\{3, 5, 7, 10, 15, 20\}$ and for each rule, we examined under the impartial culture (1) the minimum of this ratio and (2) its average. The results are listed in Tables 1–8. For RP, we did not run the experiments for $n = 15$ and 20 because the computation becomes very slow. This illustrates the computational infeasibility of RP when we want the exact mixture, even for a relatively modest number of agents.

As the number of agents increase, the ratios start to get worse. But for a fixed number of agents, the ratios do not necessarily get worse as we increase the number of alternatives. We note that CUT seems to fare marginally but consistently better than NMP, RP, and EGAL in the utilitarian metric. This is especially so when we consider the average rather than the worst ratios.

We note that NMP rule's fairness constraints also lead to loss of utilitarian welfare. Fain et al. (2016) show that on certain real-world participatory budgeting datasets, core fair outcomes often coincide with welfare maximizing ones. Since the objective of EGAL is diametrically opposed to utilitarian objectives, it is not surprising that EGAL fares the worst in the utilitarian metric among the rules we consider. In particular its worst case ratios drop rapidly as we increase the number of agents and outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.8314	0.8155	0.8069	0.8005	0.781	0.7149
5	0.7777	0.7778	0.7322	0.7531	0.7072	0.7172
7	0.7678	0.80790	0.7373	0.695	0.7581	0.7109
10	0.7524	0.7334	0.808	0.7843	0.7857	0.7204
15	0.7862	0.8029	0.7561	0.7801	0.7747	0.7737
20	0.792	0.8234	0.7764	0.8155	0.7505	0.7896

Table 1: Minimum ratio of utilitarian welfare under the NMP rule to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.9451	0.9652	0.9722	0.9678	0.9759	0.9634
5	0.9171	0.9309	0.9421	0.9377	0.9335	0.9004
7	0.8926	0.9324	0.9171	0.9277	0.9121	0.8856
10	0.8921	0.9014	0.91	0.9094	0.9056	0.8873
15	0.893	0.9013	0.8911	0.9049	0.8984	0.8774
20	0.8948	0.9001	0.8909	0.9047	0.9049	0.8941

Table 2: Average ratio of utilitarian welfare the NMP rule to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.75	0.6397	0.5333	0.4815	0.4333	0.3743
5	0.625	0.3919	0.4244	0.4592	0.4956	0.403
7	0.5833	0.492	0.3632	0.5102	0.5599	0.5799
10	0.5834	0.375	0.4952	0.4253	0.5689	0.5696
15	0.5129	0.5525	0.57	0.4361	0.5198	0.5817
20	0.6001	0.625	0.5927	0.5525	0.6425	0.5656

Table 3: Minimum ratio of utilitarian welfare under EGAL to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.9325	0.9256	0.8838	0.8075	0.844	0.8408
5	0.8482	0.8484	0.781	0.8019	0.82	0.8175
7	0.8221	0.8131	0.7817	0.7978	0.7992	0.8118
10	0.8176	0.8049	0.7902	0.7639	0.8152	0.7803
15	0.8267	0.807	0.7805	0.7476	0.8259	0.8009
20	0.8414	0.8278	0.8121	0.7748	0.8265	0.8084

Table 4: Average ratio of utilitarian welfare under EGAL to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333
5	0.8	0.7333	0.8	0.8	0.8	0.8667
7	0.75	0.7619	0.8571	0.8214	0.8857	0.8571
10	0.8	0.8	0.8714	0.86	0.8667	0.8833
15	0.8	0.8444	0.8583	0.8417	0.8741	0.8815
20	0.8038	0.85	0.8773	0.9	0.8944	0.8727

Table 5: Minimum ratio of utilitarian welfare under CUT to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.9333	0.9717	0.9717	0.9867	0.9867	0.995
5	0.9372	0.9452	0.959	0.9748	0.969	0.9757
7	0.9139	0.9468	0.9549	0.9624	0.969	0.9778
10	0.9194	0.9383	0.9502	0.9586	0.9576	0.965
15	0.9263	0.9276	0.9483	0.9483	0.9567	0.9634
20	0.9195	0.9332	0.9486	0.955	0.9588	0.9631

Table 6: Average ratio of utilitarian welfare under CUT to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333
5	0.7778	0.7	0.7778	0.7778	0.7	0.8
7	0.7679	0.75	0.8036	0.75	0.7943	0.7778
10	0.7778	0.7737	0.7596	0.8116	0.7684	0.8031

Table 7: Minimum ratio of utilitarian welfare under RP to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

$N \downarrow A \rightarrow$	3	5	7	10	15	20
3	0.9483	0.9733	0.9883	0.99	0.9867	0.9933
5	0.8992	0.9302	0.9351	0.9471	0.9512	0.962
7	0.8851	0.8952	0.9143	0.9182	0.929	0.9305
10	0.8839	0.89	0.8911	0.8969	0.9	0.8997

Table 8: Average ratio of utilitarian welfare under RP to maximum utilitarian welfare for 100 profiles draws under impartial culture assumption for each combination of # agents and # outcomes.

10 Conclusion and Open Questions

1) We compared the relative merits of some well-known rules (EGAL, RP, NMP) and of an (essentially) new one (CUT), for the model of probabilistic/fractional voting under dichotomous preferences. We did so by taking a more nuanced and fine-grained approach to standard concepts such as strategyproofness, participation incentives, and welfare guarantees, of which we introduced new versions, both weaker and stronger than the existing ones. Some of the results are summarised in Table 9.

Properties	RP	CUT	UTIL	EGAL	NMP
EFF (Efficiency)	−	−	+	+	+
EXSP = $SP^- \wedge SP^+$ (Excludable SP)	+	+	+	+	−
SP = $SP^* \wedge SP^+$ (Strategyproofness)	+	+	+	−	−
IFS (Individual Fair Share)	+	+	−	+	+
GFS (Group Fair Share)	+	+	−	−	+
AFS (Avg. Fair Share)	−	−	−	−	+
CFS (Core Fair Share)	−	−	−	−	+
PART (Participation)	+	+	+	+	+
PART* (Strict participation)	+	+	−	−	+
DEC (Decentralisation)	+	+	−	−	+
Known Polynomial-time Algorithm	−	+	+	+	−

Table 9: Properties satisfied by rules under dichotomous preferences.

The two rules that are especially desirable in the instances where protection of minorities and participation concerns matter most are CUT and NMP. The *Conditional Utilitarian* rule is strongly incentive compatible, but in extreme cases it may be severely inefficient. The *Nash Max Product* rule is efficient and gives much better guarantees to groups of agents than CUT, but it fails even the weak form of strategyproofness where outcomes are excludable.

2) Our results also identify two especially interesting open questions. We know from Bogomolnaia et al. (2005) that Efficiency, Individual Fair Share and Strategyproofness are incompatible. If we are content to achieve only the excludable version of Strategyproofness, this incompatibility disappears, and the Egalitarian rule is an example. The unpalatable feature of this rule is that it pays no attention to clones (subgroups of agents with identical preferences) hence offers no protection to sizable minorities. But can a rule combine Efficiency, Excludable Strategyproofness and Strict Participation; or Efficiency, Excludable Strategyproofness and Unanimous Fair Share? Such a rule would be a serious new contender in our fair mixing model.

3) Bogomolnaia et al. (2005) defined, and Bogomolnaia et al. (2002) studied, a family of welfarist rules directly borrowed from classical social choice theory. Fix an increasing, strictly concave, and continuous function h on [0.1]. A rule in the sense of Definition 1 is obtained by maximizing the sum of individual

utilities weighted by h:

$$\textbf{h-rule:}\ f(M) = \arg\max_{U\in\Phi(M)} \sum_{i\in N} h(U_i) \tag{12}$$

This maximization has a unique solution in $\Phi(M)$. The NMP rule is of course a paramount example.

All h-rules are efficient, and by mimicking Step 2 in the proof of Theorem 3, we see that they satisfy PART* provided $h'(0) = \infty$. They satisfy (resp. fail) IFS if h is at least as concave as (resp. less concave than) the *log* function; but NMP is the only h-rule meeting UFS (these two facts are already proven in Bogomolnaia et al. (2002)). Finally all h-rules fail EXSP and only NMP meets DEC. Thus they don't add much to our axiomatic discussion.

However, once we observe that the EGAL and UTIL rules are the two end points of the family of h-rules[7] the following intriguing facts emerges: most h-rules meet PART* but neither EGAL nor UTIL does; EGAL and UTIL meet EXSP, but none of the h-rules does.

11 Acknowledgements

The authors thank Edward Lee for assistance with the code for some of the rules. The comments of seminar participants at Seoul National University,, Université Paris 1, and the Paris School of Economics are gratefully acknowledged.

References

Abdulkadiroğlu, A., Sönmez, T., 1998. Random serial dictatorship and the core from random endowments in house allocation problems. Econometrica 66 (3), 689–701.

Aziz, H., 2013. Maximal Recursive Rule: A New Social Decision Scheme. In: Proceedings of the 23nd International Joint Conference on Artificial Intelligence (IJCAI). AAAI Press, pp. 34–40.

Aziz, H., 2017. A probabilistic approach to voting, allocation, matching, and coalition formation.

Aziz, H., Brandl, F., Brandt, F., Brill, M., 2017a. On the tradeoff between efficiency and strategyproofnessWorking paper.

Aziz, H., Brandt, F., Brill, M., 2013. The computational complexity of random serial dictatorship. Economics Letters 121 (3), 341–345.

[7]When h converges pointwise to a linear function, e.g. $h(x) = x^q$ with $q \uparrow 1$, the h-rule converges pointwise to UTIL; when h becomes infinitely concave, e.g. $h(x) = -x^q$ with $q \downarrow -\infty$, it converges pointwise to EGAL.

Aziz, H., Brill, M., Conitzer, V., Elkind, E., Freeman, R., Walsh, T., 2017b. Justified representation in approval-based committee voting. Social Choice and Welfare 48 (2), 461–485.

Aziz, H., Stursberg, P., 2014. A generalization of probabilistic serial to randomized social choice. In: Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI). AAAI Press, pp. 559–565.

Behrens, J., 2017. The origins of liquid democracy. The Liquid Democracy Journal 5.

Benade, G., Nath, S., Procaccia, A. D., Shah, N., 2017. Preference elicitation for participatory budgeting. In: Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI). AAAI Press, pp. 376–382.

Bogomolnaia, A., Moulin, H., 2001. A new solution to the random assignment problem. Journal of Economic Theory 100 (2), 295–328.

Bogomolnaia, A., Moulin, H., 2004. Random matching under dichotomous preferences. Econometrica 72 (1), 257–279.

Bogomolnaia, A., Moulin, H., Sandomirskyi, F., Yanovskaya, E., 2017. Competitive division of a mixed manna. Econometrica.

Bogomolnaia, A., Moulin, H., Stong, R., 2002. Collective choice under dichotomous preferences.

Bogomolnaia, A., Moulin, H., Stong, R., 2005. Collective choice under dichotomous preferences. Journal of Economic Theory 122 (2), 165–184.

Brandl, F., Brandt, F., Hofbauer, J., 2015. Incentives for participation and abstention in probabilistic social choice. In: Proceedings of the 14th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS). IFAAMAS, pp. 1411–1419.

Brandl, F., Brandt, F., Seedig, H. G., 2016. Consistent probabilistic social choice. Econometrica 84 (5), 1839–1880.

Brandt, F., 2017. Rolling the dice: Recent results in probabilistic social choice. In: Endriss, U. (Ed.), Trends in Computational Social Choice. AI Access, Ch. 1, pp. 3–26.

Brill, M., 2017. Interactive democracy: New challenges for social choice theory. In: Laslier, J., Moulin, H., Sanver, R., Zwicker, W. (Eds.), The Future of Economic Design. Springer.

Cabannes, Y., 2004. Participatory budgeting: a significant contribution to participatory democracy. Environment and Urbanization 16 (1), 27–46.

Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A. D., Shah, N., Wang, J., 2016. The Unreasonable Fairness of Maximum Nash Welfare. In: Proceedings of the 17th ACM Conference on Economics and Computation (ACM-EC). pp. 305–322.

Duddy, C., 2015. Fair sharing under dichotomous preferences. Mathematical Social Sciences 73, 1–5.

Fain, B., Goel, A., Munagala, K., 2016. The core of the participatory budgeting problem. In: Web and Internet Economics - 12th International Conference, WINE 2016, Montreal, Canada, December 11-14, 2016, Proceedings. pp. 384–399.

Fishburn, P. C., 1984. Probabilistic social choice based on simple voting comparisons. Review of Economic Studies 51 (4), 683–692.

Fishburn, P. C., Brams, S. J., 1983. Paradoxes of preferential voting. Mathematics Magazine 56 (4), 207–214.

Gibbard, A., 1977. Manipulation of schemes that mix voting with chance. Econometrica 45 (3), 665–681.

Gordon, J., 1994. Institutions as relational investors: A new look at cumulative voting. Columbia Law Review 94 (1), 124–192.

Grandi, U., 2017. Agent-mediated social choice. In: Laslier, J., Moulin, H., Sanver, R., Zwicker, W. (Eds.), The Future of Economic Design. Springer.

Hughes, J., Sasse, G., 2003. Monitoring the monitors : Eu enlargement conditionality and minority protection in the ceecs. Journal on ethnopolitics and minority issues in Europe 1, 35.

Laffond, G., Laslier, J.-F., Le Breton, M., 1993. The bipartisan set of a tournament game. Games and Economic Behavior 5 (1), 182–201.

Moulin, H., 1981. The proportional veto principle. Review of Economic Studies 48 (3).

Moulin, H., 1982. Voting with proportional veto power. Econometrica 50 (1), 145–162.

Moulin, H., 1988. Axioms of Cooperative Decision Making. Cambridge University Press.

Moulin, H., 2003. Fair Division and Collective Welfare. The MIT Press.

Nash, J. F., 1950. The bargaining problem. Econometrica 18 (2), 155–162.

Porta, R. L., de Silanes, F. L., Schleifer, A., Vishny, R., 2000. Investor protection and corporate governance. Journal of Financial Economics 58 (1–2), 3–27.

Sawyer, J., MacRae, D., 1962. Game theory and cumulative voting in illinois: 1902-1954. The American Political Science Review 56 (4), 936–946.

Steinhaus, H., 1948. The problem of fair division. Econometrica 16, 101–104.

Thomson, W., 2016. Introduction to the theory of fair allocation. In: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A. D. (Eds.), Handbook of Computational Social Choice. Cambridge University Press, Ch. 11.

Varian, H. R., 1974. Equity, envy, and efficiency. Journal of Economic Theory 9, 63–91.

Young, G., 1950. The case for cumulative voting. Wisconsin Law Review 49–56.

12 Appendix

12.1 The NMP rule fails EXSP

12.1.1 A numerical example

Consider the following example with 36 agents and 4 outcomes.

No. of agents types	a	b	c	d
4	1	0	0	0
4	0	1	1	1
1	0	0	1	0
1	0	0	0	1
2	1	1	1	0
2	1	1	0	1
7	1	0	1	0
7	1	0	0	1
4	0	1	1	0
4	0	1	0	1

The outcome of NMP is (0.4163514575435199, 0.08787730532715962, 0.2479123840667547, 0.24785885306256383). If one agent of type one additionally liked b, the profile is as follows.

No. of agents types	a	b	c	d
3	1	0	0	0
1	1	1	0	0
4	0	1	1	1
1	0	0	1	0
1	0	0	0	1
2	1	1	1	0
2	1	1	0	1
7	1	0	1	0
7	1	0	0	1
4	0	1	1	0
4	0	1	0	1

In this case, The outcome of NMP is 0.4179621510380684, 0.1389580435629242, 0.22150747720884034, 0.22157232819017458). Note that the misreporting agent gets more utility (equivalently more probability for outcome a by additionally liking b).

12.1.2 A formal construction

We fix N and A and describe a profile $u \in \Omega$ by the vector $(n_S)_{S \in 2^A \setminus \varnothing}$ of non negative integers, where n_S is the number of agents i with like-set $L_i = S$. Because f^{nsh} is Anonymous, this is all we need to describe $f^{nsh}(u)$. If $z \in \Delta(A)$ has full support ($z_a > 0$ for all a), it is (uniquely) selected at u if and only if the gradient of $z \to \varphi(z) = \sum_N \ln(z_{L_i})$ is parallel at z to 1^A. Write $\Theta(a)$ for the set of subsets of A containing a, then we have

$$\frac{\partial \varphi}{\partial z_a}(z) = \sum_{i: a \in L_i} \frac{1}{z_{L_i}} = \sum_{S \in \Theta(a)} \frac{n_S}{z_S}$$

so if $\Theta(a - b)$ is the set of coalitions containing a and not b, and S^c is $A \setminus S$, we have

$$\frac{\partial \varphi}{\partial z_a}(z) = \frac{\partial \varphi}{\partial z_b}(z) \Longleftrightarrow \sum_{S \in \Theta(a-b)} \left(\frac{n_S}{z_S} - \frac{n_{S^c}}{z_{S^c}}\right) = 0 \tag{13}$$

Thus $f^{nsh}(u) = z$ holds iff the right hand equation above holds for $|A| - 1$ independent pairs a, b.

Constructing an example violating SP^+ We note first that SP^+ is equivalent to the following property: if at profile u a set of K agents have identical preferences S, and u' is obtained from u when they all report instead $S \cup T$, ceteris paribus, then the total weight of S decreases weakly from u to u'. Indeed by SP^+, when our K agents misreport their preferences one at a time, the weight of a must decrease weakly at each step.

We fix now $A = \{a, b, c_1, c_2\}$ and two mixtures $z, z' \in \Delta(A)$, both symmetric in c_1, c_2 and such that

$$0 < z_a < z'_a \ , \ 0 < z_b < z'_b \ , \ z_c > z'_c > 0 \qquad (14)$$

(where z_c stands for $z_{c_1} = z_{c_2}$). We show that, under some additional restrictions on z, z' we can choose $u \in \Omega$ and an integer K such that $f^{nsh}(u) = z$ and $f^{nsh}(u') = z'$, where u' is obtained from u when K agents of type a switch to ab. This contradicts SP^+.

The profile u is symmetric in c_1, c_2 and we write n_{ac} in lieu of n_{ac_1} etc..Then $\varphi(u) = z$ holds iff the two equations (13) applied respectively to a, c_1 and to b, c_1, are true: note that (13) for c_1, c_2 is obtained by the symmetry assumption. These two equations are

$$\left(\frac{n_a}{z_a} - \frac{n_{bcc}}{z_{bcc}}\right) + \left(\frac{n_{ab}}{z_{ab}} - \frac{n_{cc}}{z_{cc}}\right) + \left(\frac{n_{ac_2}}{z_{ac_2}} - \frac{n_{bc_1}}{z_{bc_1}}\right) + \left(\frac{n_{abc_2}}{z_{abc_2}} - \frac{n_{c_1}}{z_{c_1}}\right) = 0 \qquad (15)$$

$$\left(\frac{n_b}{z_b} - \frac{n_{acc}}{z_{acc}}\right) + \left(\frac{n_{ab}}{z_{ab}} - \frac{n_{cc}}{z_{cc}}\right) + \left(\frac{n_{bc_2}}{z_{bc_2}} - \frac{n_{ac_1}}{z_{ac_1}}\right) + \left(\frac{n_{abc_2}}{z_{abc_2}} - \frac{n_{c_1}}{z_{c_1}}\right) = 0 \qquad (16)$$

We choose u in such a way that all parenthesis above are null, that is we pick five positive parameters $\alpha, \beta, \gamma, \delta, \varepsilon$, such that

$$\frac{n_a}{z_a} = \frac{n_{bcc}}{z_{bcc}} = \alpha K \ ; \ \frac{n_b}{z_b} = \frac{n_{acc}}{z_{acc}} = \beta K \ ; \ \frac{n_c}{z_c} = \frac{n_{abc}}{z_{abc}} = \gamma K \qquad (17)$$

$$\frac{n_{ab}}{z_{ab}} = \frac{n_{cc}}{z_{cc}} = \delta K \ ; \ \frac{n_{ac}}{z_{ac}} = \frac{n_{bc}}{z_{bc}} = \varepsilon K \qquad (18)$$

Note that we must have $n_a \geq K \iff \alpha \geq \frac{1}{z_a}$ in order to construct u' by transforming K agents who only like $\{a\}$ to agents who like a and b. And if the coordinates of z and the numbers $\alpha, \cdots, \varepsilon$ are all rational, we can choose K large enough so that the above system delivers integers n_S for all S.

The profile u' has $n'_a = n_a - K$ and $n'_{ab} = n_{ab} + K$, and other terms n_S are as in u. The desired equality $f^{nsh}(u') = z'$ requires two equations like (15) and (16). For instance (15) becomes

$$\left(\frac{n_a}{z'_a} - \frac{n_{bcc}}{z'_{bcc}}\right) + \left(\frac{n_{ab}}{z'_{ab}} - \frac{n_{cc}}{z'_{cc}}\right) + \left(\frac{n_{ac}}{z'_{ac}} - \frac{n_{bc}}{z'_{bc}}\right) + \left(\frac{n_{abc}}{z'_{abc}} - \frac{n_c}{z'_c}\right) = K\left(\frac{1}{z'_a} - \frac{1}{z'_{ab}}\right).$$

Taking (17), (18) into account this becomes

$$\left(\frac{z_a}{z'_a} - \frac{z_{bcc}}{z'_{bcc}}\right)\alpha + \left(\frac{z_{ab}}{z'_{ab}} - \frac{z_{cc}}{z'_{cc}}\right)\delta + \left(\frac{z_{ac}}{z'_{ac}} - \frac{z_{bc}}{z'_{bc}}\right)\varepsilon + \left(\frac{z_{abc}}{z'_{abc}} - \frac{z_c}{z'_c}\right)\gamma = \frac{1}{z'_a} - \frac{1}{z'_{ab}}.$$

$$\iff \frac{z_a - z'_a}{z'_a \cdot (1 - z'_a)} \cdot \alpha + \frac{z_{ab} - z'_{ab}}{z'_{ab} \cdot (1 - z'_{ab})} \cdot \delta + \frac{z_{ac} - z'_{ac}}{z'_{ac} \cdot (1 - z'_{ac})} \cdot \varepsilon + \frac{z'_c - z_c}{z'_c \cdot (1 - z'_c)} \cdot \gamma = \frac{z'_b}{z'_a \cdot z'_{ab}}.$$

Now we use inequalities (14) to check that in the above sum, all numerators except $z_{ac} - z'_{ac}$ are negative. Therefore $z_{ac} - z'_{ac}$ is positive and we can rewrite this equation as

$$\frac{z_{ac}-z'_{ac}}{z'_{ac}\cdot(1-z'_{ac})}\cdot\varepsilon=\frac{z'_b}{z'_a\cdot z'_{ab}}+\frac{z_c-z'_c}{z'_c\cdot(1-z'_c)}\cdot\gamma+\frac{z'_a-z_a}{z'_a\cdot(1-z'_a)}\cdot\alpha+\frac{z'_{ab}-z_{ab}}{z'_{ab}\cdot(1-z'_{ab})}\cdot\delta \tag{19}$$

where all fractions are positive on both sides.

The second equation we need to ensure $f^{nsh}(u')=z'$ is the counterpart of (16) and reads

$$(\frac{n_b}{z'_b}-\frac{n_{acc}}{z'_{acc}})+(\frac{n_{ab}}{z'_{ab}}-\frac{n_{cc}}{z'_{cc}})+(\frac{n_{bc}}{z'_{bc}}-\frac{n_{ac}}{z'_{ac}})+(\frac{n_{abc}}{z'_{abc}}-\frac{n_c}{z'_c})+\frac{K}{z'_{ab}}=0.$$

A similar computation using (17), (18) to change the terms n_S into z_S and inequalities (14) to sign the fractions gives

$$\frac{1}{z'_{ab}}=\frac{z'_b-z_b}{z'_b\cdot(1-z'_b)}\cdot\beta+\frac{z'_{ab}-z_{ab}}{z'_{ab}\cdot(1-z'_{ab})}\cdot\delta+\frac{z_{ac}-z'_{ac}}{z'_{ac}\cdot(1-z'_{ac})}\cdot\varepsilon+\frac{z_c-z'_c}{z'_c\cdot(1-z'_c)}\cdot\gamma \tag{20}$$

where again all ratios are positive.

We must show that the non negative rational numbers $\alpha,\cdots,\varepsilon$ can be chosen solving system (19), (20) and $\alpha\geq\frac{1}{z_a}$. Note that (19) implies

$$\frac{z_{ac}-z'_{ac}}{z'_{ac}\cdot(1-z'_{ac})}\cdot\varepsilon>\frac{z'_b}{z'_a\cdot z'_{ab}}+\frac{z'_a-z_a}{z'_a\cdot(1-z'_a)}\cdot\frac{1}{z_a}$$

and (20) gives

$$\frac{z_{ac}-z'_{ac}}{z'_{ac}\cdot(1-z'_{ac})}\cdot\varepsilon<\frac{1}{z'_{ab}}$$

We can choose ε meeting these two inequalities if and only if

$$\frac{z'_b}{z'_a\cdot z'_{ab}}+\frac{z'_a-z_a}{z'_a\cdot(1-z'_a)}\cdot\frac{1}{z_a}<\frac{1}{z'_{ab}}\iff\frac{z'_a-z_a}{z_a\cdot(1-z'_a)}<\frac{z'_a-z'_b}{z'_{ab}}$$

$$\iff z_a>\frac{z'_a+z'_b}{2-z'_a+z'_b}. \tag{21}$$

and in this case we can also pick $\alpha\geq\frac{1}{z_a}$ as well as β,γ,δ solving (19), (20).

Summing up the requirements on z,z': we need inequalities (14), (21) as well as $z'_{ac}<z_{ac}\iff z'_a-z'_b<z_a-z_b$. Note that (21) and $z'_a>z_a$ together imply $z'_a>z'_b$. We can construct such a pair z,z' as follows.

Write r the RHS in (21), and check $r<z'_a$ as long as so $z'_a>z'_b$. Thus it is enough to pick z_a in the interval $]\max\{r,z'_a-z'_b\},z'_a[$, and then to pick z_b small enough that $z'_a-z'_b<z_a-z_b$.

For instance we can choose

$$z_a=\frac{9}{20},\ z_b=\frac{1}{20},\ z_c=z_{c'}=\frac{1}{4}\ \ ;\ \ z'_a=\frac{1}{2},\ z'_b=z'_c=z'_{c'}=\frac{1}{6}$$

the system (19), (20) is then

$$\frac{3}{20}\varepsilon = \frac{1}{2} + \frac{3}{5}\gamma + \frac{3}{4}\delta + \frac{1}{5}\alpha$$

$$\frac{3}{5}\gamma + \frac{3}{20}\varepsilon + \frac{21}{25}\beta + \frac{3}{4}\delta = \frac{3}{2}$$

where we recall the constraint $n_a \geq K \Leftrightarrow \frac{9}{20}\alpha \geq 1$.

A relatively simple solution of the system above is

$$\alpha = \frac{25}{11},\ \gamma = \frac{5}{11},\ \varepsilon = \frac{90}{11},\ \beta = \delta = 0$$

for which we derive the profile $u \in \otimes$ by system (17), (18). Here $K = 44$ is the smallest integer delivering integer coordinates, and we end up with 860 agents and the profile

$$n_a = 45,\ n_{bcc'} = 55\ ;\ n_c = n_{c'} = 5,\ n_{abc} = n_{abc'} = 15\ ;\ n_{ac} = n_{ac'} = 252;$$

$$n_{bc} = n_{bc'} = 108.$$

Example where the NMP rule violates (a slightly stronger version of) SP^0 We use the same technique. Set $A = \{a, a', a'', b, c\}$ and construct two profiles u, u' such that u is obtained from u' when K agents who all like $\{a, a', a'', b\}$ all declare $\{a, a', a''\}$ and end up better off even though they cannot consume b anymore (so u' is the true profile). This property implies a group version of SP^0.

At profile u the K agents in question declare $\{a, a', a''\}$ and $\varphi(u) = z$; at u' they switch to $\{a, a', a'', b\}$ (nothing else changes) and $\varphi(u') = z'$. The profiles are entirely symmetric in a, a', a''. We define

$$z_a = \frac{1}{6}, z_b = \frac{1}{32}, z_c = \frac{15}{32}\ ;\ z'_a = \frac{1}{16}, z'_b = \frac{1}{4}, z'_c = \frac{9}{16}$$

Note that $z'_{aaab} < z_{aaa}$ as desired but $z_{aab} < z'_{aab}$.

We have six types of preferences and ten homogenous coalitions of which the sizes meet the analog of system (11), (5) for some positive parameters γ, δ:

$$\frac{n_{aaa}}{z_{aaa}} = \frac{n_{bc}}{z_{bc}} = K\ ,\ \frac{n_{aaab}}{z_{aaab}} = \frac{n_c}{z_c} = \gamma K\ ,\ \frac{n_{ac}}{z_{ac}} = \frac{n_{aab}}{z_{aab}} = \delta K$$

(we have three coalitions who like one of the a-s and c, and another three who like two of the a-s and b). This implies $\varphi(u) = z$. To ensure $\varphi(u') = z'$ we need two instances of the first order condition (13), respectively for b, c and for a^*, b where a^* is an arbitrary selection from a, a', a''; symmetry implies (13) between two a-s.

Straightforward computations as above, omitted for brevity, show that (13) for b,c reduces to

$$\frac{z_{aaab} - z'_{aaab}}{z'_{aaab} \cdot z'_c}\gamma K + \frac{16}{7}K = 3\frac{z_{ac} - z'_{ac}}{z'_{ac} \cdot z'_{aab}}\delta K$$

and for a^*, b

$$\frac{z_{aaa} - z'_{aaa}}{z'_{aaa} \cdot z'_{bc}} K + \frac{z_{a^*c} - z'_{a^*c}}{z'_{a^*c} \cdot z'_{aab}} \delta K = \frac{16}{3} K$$

With our choice of z, z' these equations boil down to

$$\frac{32}{85}\gamma + \frac{16}{7} = 3(\frac{2}{45}\delta) \; ; \; \frac{160}{39} + \frac{2}{45}\delta = \frac{16}{3}$$

$$\Longrightarrow \gamma = \frac{340}{91} \; , \; \delta = \frac{360}{13}.$$

Finally we pick $K = 8 \times 19 \times 91 = 13832$ and get

$$\begin{aligned} n_{aaa} &= n_{bc} = K \; ; \; n_{aaab} = 17 \times 85 \times 19 = 27455 \; ; \; n_c = 15 \times 85 \times 19 = 24225 \\ n_{aab} &= 35 \times 90 \times 56 \simeq 176400, n_{ac} = 61 \times 90 \times 56 \simeq 307440. \end{aligned}$$