Modeling Dependence Structure and Forecasting Market Risk with Dynamic Asymmetric Copula[*]

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Abstract

We investigate the dynamic and asymmetric dependence structure between equity portfolios from the US and UK. We demonstrate the statistical significance of dynamic asymmetric copula models in modelling and forecasting market risk. First, we construct "high-minus-low" equity portfolios sorted on beta, coskewness, and cokurtosis. We find substantial evidence of dynamic and asymmetric dependence between characteristic-sorted portfolios. Second, we consider a dynamic asymmetric copula model by combining the generalized hyperbolic skewed t copula with the generalized autoregressive score (GAS) model to capture both the multivariate non-normality and the dynamic and asymmetric dependence between equity portfolios. We demonstrate its usefulness by evaluating the forecasting performance of Value-at-Risk and Expected Shortfall for the high-minus-low portfolios. From backtesting, we find consistent and robust evidence that our dynamic asymmetric copula model provides the most accurate forecasts, indicating the importance of incorporating the dynamic and asymmetric dependence structure in risk management.

Key words: asymmetry, tail dependence, dependence dynamics, dynamic skewed t copulas, VaR and ES forecasting.

JEL codes: C32, C53, G17, G32

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1. Introduction

The finance and econometrics literature provides a wealth of evidence that the conditional correlation or dependence structure between assets varies through time (see Longin and Solnik, 1995; Jondeau and Rockinger, 2006; Giacomini et al., 2009; Dias and Embrechts, 2010, etc). Moreover, asset returns also exhibit greater correlation, or more generally, greater dependence during market downturns than market upturns. One feature of the recent financial crisis is the extent to which assets that had previously behaved mostly independently suddenly moved together. This phenomenon is usually termed asymmetric dependence, see for instance Longin and Solnik (2001), Ang and Bekaert (2002), Ang and Chen (2002), Poon et al. (2004), Patton (2006), Okimoto (2008) and Christoffersen and Langlois (2013). The presence of asymmetric correlations or dependence is empirically important, as it can cause serious problems in hedging effectiveness and portfolio diversification (see Hong et al., 2007). In the foreign exchange markets, Patton (2006) suggests that this asymmetry is possibly caused by the asymmetric responses of central banks to exchange rate movements. In the equity markets, although there have been many studies of asymmetric dependence, there is no consensus on the underlying economic cause. One possible cause is that risk-averse investors treat downside losses and upside gains distinctively, which is consistent with "Prospect Theory" (see Kahneman and Tversky, 1979).

Clearly, the key to portfolio risk management is therefore to recognize how quickly and dramatically the dependence structure changes. An increasingly popular method for constructing high dimensional dependence is based on copulas. Copulas are functions that connect multivariate distributions to their one-dimensional margins (Sklar, 1959). The copula approach is particularly useful in portfolio risk measurement for the following reasons. First, copulas can describe the dependence between assets under extreme circumstances, as they use a quantile scale. Second, they utilize a flexible bottom-up approach that can combine a variety of marginal models with a variety of possible dependence specifications (McNeil et al., 2005). Ideally, an appropriate copula for financial modeling should be capable of accommodating both positive and negative dependence, capturing both symmetric and asymmetric dependence, and allowing for possible tail dependence. The skewed t copula of Demarta and McNeil (2005) can be

viewed as a flexible extension that contains all these desirable properties.

Further, the time-variation of dependence motivates the consideration of dynamic copula models which allow the correlation parameter to change dynamically. One such model is proposed by Patton (2006) who extended Sklar's theorem for conditional distributions and proposed an observation driven conditional copula model. This model defined the time-varying dependence parameter of a copula as a parametric function of transformations of the lagged data and an autoregressive term. Another example is the dynamic conditional correlation (DCC) model proposed by Engle (2002). Christoffersen et al. (2012) and Christoffersen and Langlois (2013) develop a dynamic asymmetric copula (DAC) model based on the DCC model to capture long-run and short-run dependence, multivariate nonnormality, and dependence asymmetries.

Creal et al. (2013) propose a class of Generalized Autoregressive Score (GAS) models, which use the scaled score of a likelihood function to update the parameters over time. The GAS model is a consistent and unified framework, which encompasses many successful econometric models including the GARCH, the autoregressive conditional duration, the autoregressive conditional intensity, and Poisson count models with time-varying mean. They illustrate the GAS framework by introducing a new model specification for a dynamic copula.[1] Based on simulation results and empirical evidence, they point out that the driving mechanism in Patton (2006) only captures some of the changes in the dependence coefficients. Specifically, it has shortcomings in tracking the upper and lower tail dependence dynamics simultaneously, since the constant mechanism applies to both types of dependence. Conversely, the GAS specification has better performance in capturing different types of dynamics. Therefore, the GAS model is becoming popular in both economics and finance applications (see Oh and Patton, 2013; Lucas et al., 2014; Creal et al., 2014, among many others). Thus, our study adopts it as the driving mechanism to update copula parameters.

Recently, Christoffersen and Langlois (2013) study how the extreme dependence structure is related to Fama-French factors and address its role in broad areas of finance, such as asset pricing, portfolio analysis, and risk management. They also emphasize the importance of the

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[1]Harvey (2013) proposes a similar approach for modeling time-varying parameters, which he calls a "dynamic conditional score (DCS)" model

copula modeling of the extreme dependence structure. Chung et al. (2006) argue that the Fama-French factors are closely related to higher-order comoments such as coskewness and cokurtosis.[2] They suggest the use of Fama-French factors as good proxies for higher-order comoments on the grounds that the latter are, in practice, difficult to accurately estimate. These interesting studies have initiated research investigating the dependence structure between portfolios sorted by higher-order comoments. Several studies show that tail dependence between portfolios has a close relationship with, not only beta, but also coskewness. For example, Garcia and Tsafack (2011), in their international bond and equity market portfolio analysis, show that a strong dependence in lower returns creates a large negative coskewness. Chabi-Yo et al. (2014) also show that a strong lower tail dependence creates a large negative coskewness. In addition they show that beta is monotonically increasing with respect to the lower tail dependence.

In line with the asset pricing literature cited above, we recognize that the Fama-French factors are closely related to higher-order comoments. Moving beyond the asset pricing literature, we argue that it is important to investigate how higher-order comoments are related to the dependence structure of equity portfolios. The latter would be a key input when investors manage the market risk from portfolios constructed using the Fama-French factors or from higher-order comoments. Hence, we use equity portfolios sorted on beta and higher-order comoments. We empirically investigate if the dynamic asymmetric copula, combining generalized hyperbolic skewed *t* copula and GAS, significantly improves the modeling and forecasting of the market risk of the equity portfolios. We consider two popular market risk measures, Value-at-Risk (hereafter VaR) and Expected Shortfall (hereafter ES), both of which are very sensitive to the dynamics and extreme dependence structure of asset returns.

Our study makes three contributions. First, we provide a comprehensive study of the dynamic evolution of dependence in equity markets. We find striking evidence that the dependence structures between characteristic-sorted portfolios, such as the high beta portfolio and the low beta portfolio, significantly changed after the start of the global financial crisis of 2007-2009. Second, we provide new empirical evidence of asymmetric dependence in the US and

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[2]The importance of higher-order comoments have been demonstrated in many asset pricing literature (see Harvey and Siddique, 2000; Dittmar, 2002; Bakshi et al., 2003; Ang et al., 2006; Guidolin and Timmermann, 2008; Conrad et al., 2013).

UK equity markets. In general, we show that the coefficients of lower tail dependence (LTD) are greater than the coefficients of upper tail dependence (UTD) and that this asymmetry is statistically significant. Third, while estimation of portfolio VaR and ES has been widely studied in the literature, there have been relatively few studies examining portfolio VaR and ES forecasting, especially forecasting through asymmetric copula. Using the characteristic-sorted portfolios, we evaluate the statistical significance of incorporating asymmetric and dynamic dependence into VaR and ES forecasts, and show ignorance of dependence asymmetry and dynamics is costly in risk management. The backtesting results provide solid evidence that the dynamic asymmetric copula model can consistently provide better VaR and ES forecasts than alternative benchmark models, especially at the 99% level. And we also find that semiparametric dynamic asymmetric copula models perform better than full parametric dynamic copula models.

The remainder of this paper is organized as follows. In Section 2, we detail the methods we employ for portfolio sorting, and we provide an overview of copula theory and computation methods for tail dependence coefficients. Then, we present the dynamic copula model and its estimation methodology. The data used in the paper, summary statistics and univariate model estimations are in Section 3. In Section 4, we focus on testing whether the dependence structures between characteristic-sorted portfolios are statistically dynamic and asymmetric, especially during the global financial crisis of 2007-2009 and the Euro Sovereign Debt crisis of 2010-2011, and then discuss the possible reasons for different kinds of dependence. In Section 5, we predict portfolio VaR and ES using dynamic copulas and benchmark models and report the comparison results of backtesting. Finally, conclusions are given in Section 6. All the tables and figures used in this paper are presented in the Appendix.

2. Model Specification

In this section, we detail the models and portfolio construction that we use in this paper.

2.1. Portfolio Construction

The return on an asset is defined as the first difference of the log price, $r_t = \log P_t - \log P_{t-1}$. We construct portfolios sorted on beta, coskewness and cokurtosis separately. Following the

definition in Bakshi et al. (2003) and Conrad et al. (2013), the market beta, coskewness and cokurtosis are defined as:

$$BETA_{i,t} = \frac{\mathbb{E}\left[(r_{i,t} - \mathbb{E}[r_{i,t}])(r_{m,t} - \mathbb{E}[r_{m,t}])\right]}{Var(r_{m,t})}, \tag{1}$$

$$COSK_{i,t} = \frac{\mathbb{E}\left[(r_{i,t} - \mathbb{E}[r_{i,t}])(r_{m,t} - \mathbb{E}[r_{m,t}])^2\right]}{\sqrt{Var(r_{i,t})Var(r_{m,t})}}, \tag{2}$$

$$COKT_{i,t} = \frac{\mathbb{E}\left[(r_{i,t} - \mathbb{E}[r_{i,t}])(r_{mt} - \mathbb{E}[r_{m,t}])^3\right]}{Var(r_{i,t})Var(r_{m,t})}. \tag{3}$$

All stocks are sorted on the three characteristics above and divided into five groups based on the 20th, 40th, 60th and 80th percentiles. We estimate beta, coskewness and cokurtosis each year using all the daily data within this year. Then, we form annually rebalanced portfolios, value weighted based on the capitalization of each stock.[3] We denote by BETA1 (COSK1, COKT1) the portfolio formed by stocks with the highest beta (respectively, coskewness, cokurtosis), and BETA5 (COSK5, COKT5) denotes the portfolio formed by stocks with the lowest beta (coskewness, cokurtosis).

We then take a long position in the stocks falling in the highest beta (coskewness, cokurtosis) quintile and a short position in the stocks falling in the lowest beta (coskewness, cokurtosis) quintile to construct a high-minus-low (HML) portfolio. It is not our intention to gain high excess returns from this trading strategy. We simply generate a portfolio by combining two extreme characteristics (highest and lowest) using a popular HML strategy. We expect that these two extreme characteristics could create a portfolio with a strong extreme dependence structure. We define the HML portfolio return to be

$$r_{hml,t} = r_{h,t} - r_{l,t} \tag{4}$$

where $r_{h,t}$ and $r_{l,t}$ denote returns from the highest beta (coskewness, cokurtosis) and the lowest beta (coskewness, cokurtosis), respectively.

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[3]We compute the market capitalization of each company (stock price multiplied by the number of shares outstanding) and then use it to assign weights.

2.2. Modeling the Marginal Density

We allow each series ($r_{h,t}$ and $r_{l,t}$) to have time-varying conditional mean ($\mu_{i,t}$) and variance ($\sigma_{i,t}^2$), and we also assume that the standardized returns $z_{i,t} = (r_{i,t} - \mu_{i,t})/\sigma_{i,t}$ are identically distributed. We fit an AR model to the conditional mean

$$r_{i,t} = c_i + \sum_{k=1}^{p} \phi_{i,k} r_{i,t-k} + \varepsilon_{i,t}, \quad i = h, l, \quad , \text{ where } \varepsilon_{i,t} = \sigma_{i,t} z_{i,t} \tag{5}$$

and an asymmetric GARCH model, namely GJR-GARCH(1,1,1) (see Glosten et al., 1993), to the conditional variance

$$\sigma_{i,t}^2 = \omega_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2 + \gamma_i \varepsilon_{i,t-1}^2 I_{i,t-1} \tag{6}$$

where $I_{i,t-1} = 1$ if $\varepsilon_{i,t-1} < 0$, and $I_{i,t-1} = 0$ if $\varepsilon_{i,t-1} \geq 0$.

Let $z_{i,t}$ be a random variable with a continuous distribution F. For the parametric model, we assume that $z_{i,t}$ follows the skewed Student's t distribution of Hansen (1994):

$$z_{i,t} \sim F_{skew-t,i}(\eta_i, \lambda_i), \; u_{i,t} = F_{skew-t,i}(z_{i,t}; \eta_i, \lambda_i) \tag{7}$$

where $F_{skew-t,i}$ denotes the cumulative distribution function, η_i denotes the degrees of freedom, λ_i the skewness parameter, and $u_{i,t}$ the probability integral transformation. Hence, we can easily compute the probability given the estimates of parameters; $\hat{\mu}_{i,t}$, $\hat{\sigma}_{i,t}$, $\hat{\eta}_i$ and $\hat{\lambda}_i$. For the nonparametric model, we use the empirical distribution function to obtain the estimate of F_i:

$$\hat{F}_i(z) \equiv \frac{1}{T+1} \sum_{t=1}^{T} 1\{\hat{z}_{i,t} \leq z\}, \; \hat{u}_{i,t} = \hat{F}_i(\hat{z}_{i,t}). \tag{8}$$

We estimate all parameters in (6) – (7) using the maximum likelihood estimation. Then we generate each marginal density parametrically or nonparametrically for the purpose of copula construction.

2.3. Copulas

In this section, we provide a brief introduction to copulas. The Sklar (1959) theorem allows us to decompose a conditional joint distribution into marginal distributions and a copula. It allows considerable flexibility in modeling the dependence structure of multivariate data. Let $\mathbf{z} = (z_1, \ldots, z_d)'$, $d \geq 2$ be a d-dimensional random vector with joint distribution function

$\mathbf{F}(z_1, \ldots, z_d)$ and marginal distribution functions $F_i(z_i)$, $i = 1, \ldots, d$. According to Sklar's theorem, there exist a d-dimensional copula $C[0,1]^d \to [0,1]$ such that

$$\mathbf{F}(z_1, \ldots, z_d) = C(F_1(z_1), F_2(z_2), \ldots, F_d(z_d)), \tag{9}$$

and the copula $C(u_1, \ldots, u_d)$, $u_i \in (0,1)$ is unique if the marginal distributions are continuous. Let F_i^{-1} denote the generalized inverse distribution function of F_i, then $F_i^{-1}(u_i) = z_i$. The copula $C(u_1, \ldots, u_d)$ of a multivariate distribution $\mathbf{F}(z_1, \ldots, z_d)$ with marginals $F_i(z_i)$ is given by

$$C(u_1, \ldots, u_d) = \mathbf{F}\left(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_d^{-1}(u_d)\right) \tag{10}$$

If F_i has density f_i, the copula density c is given by

$$c(u_1, \ldots, u_d) = \frac{\mathbf{f}\left(F_1^{-1}(u_1), F_2^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right)}{\prod_{i=1}^d f_i\left(F_i^{-1}(u_i)\right)} = \frac{\partial^n C(u_1, \ldots, u_d)}{\partial u_1 \cdots \partial u_d} \tag{11}$$

Sklar's theorem implies that for multivariate distribution functions, the univariate marginals and the dependence structure can be separated. In our study, we only consider the case of a bivariate copula.

2.4. Computation of Asymmetric Dependence

A primary goal of our paper is to investigate how the characteristic-sorted portfolio returns covary and whether their dependence structures are asymmetric. Consequently, we consider three different dependence structures: The threshold correlation; the quantile dependence; and the tail dependence.

Following Longin and Solnik (2001) and Ang and Chen (2002), the threshold correlation for probability level p is given by

$$\rho^- = Corr\left(r_{h,t}, r_{l,t} | r_{h,t} \le r_h(p) \text{ and } r_{l,t} \le r_l(p)\right) \text{ if } p \le 0.5 \tag{12}$$

$$\rho^+ = Corr\left(r_{h,t}, r_{l,t} | r_{h,t} > r_h(p) \text{ and } r_{l,t} > r_l(p)\right) \text{ if } p > 0.5 \tag{13}$$

where $r(p)$ denotes the corresponding empirical percentile for asset returns $r_{h,t}$ and $r_{l,t}$. In words, we compute the correlation between two assets conditional on both of them being less (respectively, greater) than their pth percentile value when $p \le 0.5$ (respectively, $p > 0.5$). To examine whether this asymmetry is statistically significant, we consider a model-free test

proposed by Hong et al. (2007). If the null hypothesis of $\rho^+ = \rho^-$ is rejected, then there exists a linear asymmetric correlation between $r_{h,t}$ and $r_{l,t}$.

The quantile dependence provides a more precise measure of dependence structure than the threshold correlation, as it contains more detailed information. In addition, from the risk management perspective, tails are more important than the center. Following Patton (2012), the quantile dependence can be defined as

$$
\lambda^q =
\begin{cases}
\mathbb{P}\left\{u_{h,t} \leq q | u_{l,t} \leq q\right\} = \frac{C(q,q)}{q} & \text{if } 0 < q \leq 0.5 \\
\mathbb{P}\left\{u_{h,t} > q | u_{l,t} > q\right\} = \frac{1-2q+C(q,q)}{1-q} & \text{if } 0.5 < q \leq 1
\end{cases}
\tag{14}
$$

and nonparametrically estimated by

$$
\hat{\lambda}^q =
\begin{cases}
\frac{1}{Tq} \sum_{t=1}^{T} 1\left\{\hat{u}_{h,t} \leq q, \hat{u}_{l,t} \leq q\right\} & \text{if } 0 < q \leq 0.5 \\
\frac{1}{T(1-q)} \sum_{t=1}^{T} 1\left\{\hat{u}_{h,t} > q, \hat{u}_{l,t} > q\right\} & \text{if } 0.5 < q < 1.
\end{cases}
\tag{15}
$$

where C denotes the corresponding copula function.

The tail dependence coefficient (TDC) is a measure of the degree of dependence in the tail of a bivariate distribution (see McNeil et al., 2005; Frahm et al., 2005; Joe et al., 2010, among others). Let z_h and z_l be random variables with continuous distribution functions F_h and F_l. Then the coefficients of upper and lower tail dependence of z_h and z_l are

$$
\lambda^L = \lim_{q \to 0^+} \frac{\mathbb{P}\left\{z_h \leq F_h^{-1}(q), z_l \leq F_l^{-1}(q)\right\}}{\mathbb{P}\left\{z_l \leq F_l^{-1}(q)\right\}} = \lim_{q \to 0^+} \frac{C(q,q)}{q}
\tag{16}
$$

$$
\lambda^U = \lim_{q \to 1^-} \frac{\mathbb{P}\left\{z_h > F_h^{-1}(q), z_l > F_l^{-1}(q)\right\}}{\mathbb{P}\left\{z_l > F_l^{-1}(q)\right\}} = \lim_{q \to 1^-} \frac{1-2q+C(q,q)}{1-q}
\tag{17}
$$

The coefficients can be easily calculated when the copula C has a closed form. The copula C has upper tail dependence if $\lambda^U \in (0,1]$ and no upper tail dependence if $\lambda^U = 0$. A similar conclusion holds for the lower tail dependence. If the copulas are symmetric, then $\lambda^L = \lambda^U$, otherwise, $\lambda^L \neq \lambda^U$ (see Joe, 1997). McNeil et al. (2005) state that the copula of the bivariate *t* distribution is asymptotically dependent in both the upper and lower tail. The rotated Gumbel copula is an asymmetric Archimedean copula, exhibiting greater dependence in the negative tail than in the positive. Both of them allow heavier negative tail dependence than the Gaussian copula and are widely used in the finance literature. We use both the Student's *t* copula and the

rotated Gumbel copula to estimate the tail dependence coefficient between portfolios.

2.5. *Generalized Hyperbolic Skewed t Copulas*

In this section, we provide a brief introduction to the generalized hyperbolic (GH) skewed t distribution which we employ to capture asymmetric extreme dependence structure between equity portfolios in our study. It belongs to the class of multivariate normal variance mixtures and has the stochastic representation

$$\mathbf{X} = \mu + \gamma W + \sqrt{W}\mathbf{Z} \tag{18}$$

for a d-dimensional parameter vector γ. Further, W is a scalar valued random variable following an inverse gamma distribution $W \sim IG(\nu/2, \nu/2)$ and $\mathbf{Z}$ is a d- dimensional random vector following a normal distribution $\mathbf{Z} \sim N(\mathbf{0}, \Sigma)$ and is independent of W (see Demarta and McNeil, 2005).

The density function of multivariate GH skewed t distribution is given by

$$\mathbf{f}_{skt}(\mathbf{z}; \gamma, \nu, \Sigma) = \frac{2^{\frac{2-(\nu+d)}{2}} K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu + \mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*)\gamma\prime\Sigma^{-1}\gamma}\right) e^{\mathbf{z}^{*\prime}\Sigma^{-1}\gamma}}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}(\nu + \mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*)^{\frac{-\nu+d}{2}}\left(1 + \frac{1}{\nu}\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*\right)^{\frac{-\nu+d}{2}}} \tag{19}$$

where K_λ, ν and γ denote the modified Bessel function of the third kind, the degree of freedom and skewed parameter vector, respectively. The density of multivariate GH skewed t converges to the conventional symmetric t density when γ tends to $\mathbf{0}$. For the parametric case, we define the shocks $z_{i,t}^* = F_{skt,i}^{-1}(u_{i,t}) = F_{skt,i}^{-1}\left(F_{skew-t,i}(z_{i,t})\right)$ where $F_{skt,i}^{-1}(u_{i,t})$ denotes the inverse cumulative distribution function of the univariate GH skewed t distribution and it is not known in closed form but can be well approximated via simulation. $F_{skew-t,i}$ denotes the cumulative distribution function of skewed t distribution in Hansen (1994). Note that we use $z_{i,t}^*$ not the standardized return $z_{i,t}$. For the nonparametric case, we use the EDF to obtain the estimate of $u_{i,t}$. A more detailed discussion can be found in Christoffersen et al. (2012).

The probability density function of the GH skewed t copula defined from above multivariate

GH skewed t density of Eq. (19) is given by

$$c_{skt}(\mathbf{z};\gamma,\nu,\Sigma) = \frac{2^{\frac{(\nu-2)(d-1)}{2}} K_{\frac{\nu+d}{2}}\left(\sqrt{\left(\nu+\mathbf{z}^{*\prime}\Sigma_t^{-1}\mathbf{z}^*\right)\gamma^\prime\Sigma_t^{-1}\gamma}\right) e^{\mathbf{z}^{*\prime}\Sigma_t^{-1}\gamma}}{\Gamma\left(\frac{\nu}{2}\right)|\Sigma|^{\frac{1}{2}}\left(\nu+\mathbf{z}^{*\prime}\Sigma_t^{-1}\mathbf{z}^*\right)^{\frac{-\nu+d}{2}}\left(1+\frac{1}{\nu}\mathbf{z}^{*\prime}\Sigma_t^{-1}\mathbf{z}^*\right)^{\frac{-\nu+d}{2}}}$$
$$\times\prod_{i=1}^{d}\frac{\left(\sqrt{\left(\nu+(z_i^*)^2\right)\gamma_i^2}\right)^{-\frac{\nu+1}{2}}\left(1+\frac{1}{\nu}(z_i^*)^2\right)^{\frac{\nu+1}{2}}}{K_{\frac{\nu+1}{2}}\left(\sqrt{\left(\nu+(z_i^*)^2\right)\gamma_i^2}\right)e^{z_i^*\gamma_i}} \quad (20)$$

where Σ_t is the time-varying covariance matrix. Specifically, $\Sigma_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$, where $\mathbf{D}_t$ is an identity matrix in copula modeling and $\mathbf{R}_t$ is the time-varying correlation matrix. Note that Christoffersen et al. (2012) applied the GH skewed t copula by constraining all the margins to have the same asymmetry parameter. Different from their model, our model consider a more generalized case by allowing the copula to have the different asymmetry parameters across margins. Although our model can be used for high-dimensional copula modeling, in this paper, only the bivariate case is considered as modeling the dependence and market risk of long-short portfolio is our main task. Figure 6 shows the probability contours for bivariate GH skewed t copula with different asymmetric parameters.

2.6. Generalized Autoregressive Score (GAS) Model

We estimate the dynamic copula model based on the Generalized Autoregressive Score (GAS) model of Creal et al. (2013). We assume that the correlation parameter δ_t [4] is dynamic and is updated as function of its own lagged value. To make sure that it always lies in a pre-determined range (*e.g.* $\delta_t \in (-1,,1)$), the GAS model utilizes a strictly increasing transformation. Following Patton (2012), the transformed correlation parameter is denoted by g_t:

$$g_t = h(\delta_t) \Leftrightarrow \delta_t = h^{-1}(g_t), \quad (21)$$

where $\delta_t = (1-e^{-g_t})/(1+e^{-g_t})$. Further, the updated transformed parameter g_{t+1} is a function of a constant $\bar{\omega}$, the lagged transformed parameter g_t, and the standardized score of the

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[4]In the baivariate case, the copula correlation is a scalar and it can be obtained from the correlation matrix $\mathbf{R}_t = \begin{bmatrix} 1 & \delta_t \\ \delta_t & 1 \end{bmatrix}$ estimated in Section 2.5.

copula log-likelihood $Q_t^{-1/2}\mathbf{s}_t$:

$$g_{t+1} = \bar{\omega} + \eta Q_t^{-1/2}\mathbf{s}_t + \varphi g_t, \tag{22}$$

where

$$\mathbf{s}_t \equiv \frac{\partial \log c\left(u_{h,t}, u_{l,t}; \delta_t\right)}{\partial \delta_t} \text{ and } Q_t \equiv \mathbb{E}_{t-1}\left[\mathbf{s}_t \mathbf{s}_t'\right].$$

Since the GAS model is an observation driven model, we estimate the parameters using the maximum likelihood estimation

$$\widehat{\delta_t} = \operatorname*{argmax}_{\delta_t} \sum_{t=1}^{n} \log c\left(u_{h,t}, u_{l,t}; \delta_t\right). \tag{23}$$

The dynamic copulas are parametrically estimated using maximum likelihood estimation. When the marginal distributions are estimated using the skewed t distribution, the resulting joint distribution is fully parametric. When the marginal distribution is estimated by the empirical distribution function, then the resulting joint distribution is semiparametric. More details can be found in the appendix.

2.7. Value-at-Risk and Expected Shortfall Forecasts

We now turn to VaR and ES forecasts of the HML portfolio defined in (4). The ex ante VaR of the HML portfolio at time t and given nominal probability $\alpha \in (0,1)$, is defined as:

$$VaR_{hml,t}\left(\alpha\right) = \inf\left\{x \mid \mathbb{P}\left(r_{hml,t} \leq x | F_{t-1}\right) \leq 1 - \alpha\right\}, \tag{24}$$

$$ES_{hml,t}\left(\alpha\right) = \mathbb{E}\left[r_{hml,t} \mid r_{hml,t} < VaR_{hml,t}\left(\alpha\right)\right], \tag{25}$$

where F_{t-1} represents the information set available at $t-1$. In our study, α is assumed to be either 0.95 or 0.99, and we report results focusing on 0.99 which is the most widely used value for market risk management. Once the dynamic copula parameters have been estimated, Monte Carlo simulation is used to generate 5000 values of $r_{h,t}^{(s)}$ and $r_{l,t}^{(s)}$ and, hence, of $r_{hml,t}^{(s)}$. From the empirical distribution of $r_{hml,t}^{(s)}$, the desired quantile VaR and ES are estimated.

3. Data and Marginal Distribution Modeling

3.1. Description of Data

Stock prices are obtained from Datastream. Daily returns of the 500 stocks listed in the S&P 500 and the 100 stocks listed in FTSE 100 are used to construct portfolios. Our data,

spanning the period of the global financial crisis of 2007-2009 and European sovereign debt crisis of 2010-2011, go from January 4, 2000 to December 31, 2012, resulting in 3,268 daily observations for each stock in US and 3,283 daily observations for each stock in UK.

Given the one-year estimation period, we estimate beta, coskewness and cokurtosis using daily data (250 days) for each stock.[5] We rank securities by the estimates of beta (coskewness, cokurtosis) and form into five portfolios, highest (1st) – lowest (5th). Then we calculate daily returns for each portfolio within the estimation period.[6] In this way, we construct thirty different portfolios - fifteen for US equities and fifteen for the UK. The fifteen consist of one for each of the three characteristics (beta, coskewness and cokurtosis), divided into five portfolios. We annually rebalance all the portfolios and calculate 12-month daily returns.[7] The definitions of HML portfolios are presented in Table 1.

[INSERT TABLE 1 ABOUT HERE]

Summary statistics for the high and low portfolio returns are presented in Panel A of Table 2. We find that the portfolio constructed from high beta stocks (i.e. BETA5) tends to offer relatively lower average returns than the portfolio constructed from low beta stocks (i.e. BETA1). This anomaly is likely driven by the fact that the US and UK equity markets have fallen 2.02% and 16.13% from 2000 to 2012. Also, it can be explained by the "betting against beta" factor and the funding constraint model in Frazzini and Pedersen (2014). The skewness of the portfolio returns are non-zero while the kurtosis of the portfolio returns are significantly higher than 3 indicating that the empirical distributions of returns display heavier tails than a Gaussian distribution. Using the Ljung-Box Q-test, the null hypothesis of no autocorrelation is rejected at lag 5 and lag 10 for all the portfolios. The ARCH test of Engle (1982) indicates the significance of ARCH effects in all the series. We also find similar results for the HML portfolios in Panel B of Table 2. Overall, the summary statistics show the nonnormality, asymmetry, autocorrelation

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[5]Since we estimate a factor beta on the daily return, we use a short sample period. We also consider alternative longer estimation periods (3 years and 5 years) and find consistent results with a one year estimation period.

[6]We also calculate daily reruns for the next 12-months, which are forward looking portfolio returns, and find similar forecasting results. Since we are interested in how beta and higher-order comoments are related to the extreme dependence structure, we prefer portfolio returns calculated within the estimation period to forward looking returns.

[7]We also consider monthly rebalancing of portfolios and find results consistent with annual rebalancing.

and heteroscedasticity of portfolio returns.

[INSERT TABLE 2 ABOUT HERE]

Figure 1 displays the scatter plots of the high and low portfolio pairs; (BETA1, BETA5), (COSK1, COSK5) and (COKT1, COKT5). Further it provides threshold correlation coefficients at the center and at both the upper and lower tails of the empirical distribution. Beta portfolios have larger correlations at both tails than the correlations at the center in both stock markets. Coskewness portfolios of the US stock market have smaller correlation at the center than the correlation at the lower tail while those of the UK stock market have larger correlations at both tails than the correlation at the center. Cokurtosis portfolios show similar patterns to coskewness portfolios. The common feature is that the lower tail correlation is larger than the upper tail correlation. This stylized fact is consistent with previous research (see Patton, 2006; Hong et al., 2007; Christoffersen and Langlois, 2013, etc). Overall, the scatter plots and the threshold correlation coefficients clearly show that the correlations between the respective high and low portfolios are nonlinear and asymmetric.

[INSERT FIGURE 1 ABOUT HERE]

Before modeling the joint distribution of portfolio returns, it is necessary to select a suitable model for the marginal return distribution, because misspecification of the univariate model can lead to biased copula parameter estimates. To allow for autocorrelation, heteroskedasticity and asymmetry, we use the models introduced in Section 2 in Eq. (5) to (8).

First, we use the Bayesian Information Criterion (BIC) to select the optimal order of the AR model for the conditional mean up to order 5. Second, to allow for the heteroskedasticity of each series, we consider a group of GARCH models as candidates and find that the asymmetric GARCH model of Eq. (6) is preferred to the others based on their likelihood values. Thus, we consider the GJR-GARCH class of up to order (2,2,2) and select the optimal order by using BIC again. The model parameters are estimated by using maximum-likelihood estimation (MLE) and the results of AR and GARCH estimations are presented in Panel A of Table 3. For each series, the variance persistence implied by the model is close to 1. For all the series, the

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leverage effect parameters γ are significantly positive implying that a negative return on the series increases volatility more than a positive return with the same magnitude.

[INSERT TABLE 3 ABOUT HERE]

The obvious skewness and high kurtosis of returns leads us to consider the skewed Student's t distribution of Hansen (1994) for residual modeling. We report the estimation results in Table 3. To evaluate the goodness-of-fit for the skewed Student's t distribution, the Kolmogorov-Smirnov (KS) and Cramer-von Mises (CvM) tests are implemented and the p-values from these two tests are reported in Table 3.[8] Our results suggest that the skewed Student's t distribution is suitable for residual modeling. Thus, in general, the diagnostics provides evidences that our marginal distribution models are well-specified and therefore, we can reliably use the combination of AR, GARCH and skewed Student's t distribution, allied to copulas to model the dependence structure.

4. Dependence: Dynamics and Asymmetry

This section seeks to accomplish two tasks. First, we describe the dynamic evolution of dependence between the high beta (respectively, coskewness, cokurtosis) portfolios and the low beta (coskewness, cokurtosis) portfolios, and examine whether it is statistically time-varying. If the variation of dependence between portfolio returns were not to be statistically significant, then there would be no reason to implement a dynamic model (due to its increased computational complexity). In addition, we want to test whether the dependence structure has dramatically changed after the start of the global financial crisis of 2007-2009 and after the start of the European sovereign debt crisis of 2010-2011. Second, we measure asymmetric dependence using threshold correlation, copula-based quantile dependence and tail dependence and we test whether this asymmetry is significant.

4.1. Time-varying Dependence

There is considerable evidence that the conditional mean and conditional volatility of financial time series are time-varying. This, possibly, suggests the reasonable inference that the

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[8]The p-values are obtained based on the algorithm suggested in Patton (2012)

conditional dependence structure may also change through time. To visualize this variation, Figure 2 depicts two time series plots of average rolling 250-day rank correlation between the high and low portfolios in both US market and UK market for each year. The average rolling rank correlations for all the equity portfolios increase significantly during 2000-2002, which is probably caused by the early 2000s economic recession that affected the European Union during 2000 and 2001 and the United States in 2002 and 2003, and the bursting of the dot com bubble. In general, all the rolling window rank correlations between the high and low portfolios increase from 2000 to 2012.

[INSERT FIGURE 2 ABOUT HERE]

We now consider three tests for time-varying dependence. The first one is a naïve test for a break in rank correlation at specified points in the sample, see Patton (2006). A noticeable limitation of this test is that the break point of dependence structure (e.g. a specified date) must be known *a priori*. The second test for time-varying dependence allows for a break in the rank correlation coefficient at some prior unspecified date, see Andrews (1993). The third test is the ARCH LM test for time-varying volatility, see Engle (1982). The critical values for the test statistic can be obtained by using a *iid* bootstrap algorithm, see Patton (2012). The results of the above tests for time-varying dependence are summarized in Table 4. Suppose there is no *a priori* date for the timing of a break, we first consider naïve tests for a break at three chosen points in our sample, at t*/ T $\in \{0.15, 0.50, 0.85\}$, which corresponds to the dates 10-Dec-2001, 03-Jul-2006, and 17-Jan-2011. Then we consider another test in Andrews (1993) for a dependence break of unknown timing. As can be seen from Table 4, for almost all the equity portfolios, the *p*-value is significant at the 5% significance level showing clear evidence against a constant rank correlation with a one-time break. To detect whether the dependence structures between the high and low portfolios significantly changed during the global financial crisis of 2007-2009 and the European sovereign debt crisis of 2010-2011, we use 15-Sep-2008 (the collapse of Lehman Brothers) and 01-Jan-2010 (EU sovereign debt crisis) as two break points. We find that the dependence between BETA1 and BETA5 significantly changed around those dates, as all the *p*-values are fairly small. For other portfolio pairs, time homogeneity of the dependence structure is rejected by at least one test.

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[INSERT TABLE 4 ABOUT HERE]

Overall, we find evidence against time homogeneity of the dependence structure between the standardized residuals of portfolios. This result shows that the standard portfolio diversification and risk management techniques based on constant correlations (or dependence) are inadequate, especially during financial crisis. Thus, the heterogeneity of dependence provides us a strong motivation to introduce a dynamic copula model for financial forecasting.

4.2. Asymmetric Dependence

Standard models fail to take into account a noteworthy feature during financial crisis that asset returns often become more highly correlated (in magnitude). To test for the presence of this feature, we use threshold correlations, Eq. (12) – (13). Figure 3 shows the lower and upper threshold correlations for the high portfolio versus low portfolio. The lower threshold correlations are always greater than the upper threshold correlation indicating that portfolios are more correlated when both of them perform poorly. From a portfolio management perspective, this feature is extremely important. For instance, for UK market, the correlation between BETA1 and BETA5 is relatively low suggesting that diversification is high, but when both BETA1 and BETA5 have poor performances, their correlation can go up to more than 0.61. Therefore, the bivariate normal distribution cannot well describe the "true" dependence for the following reasons: First, the normal distribution is symmetric. Second, in the bivariate normal distribution, the threshold correlation approaches 0 when the threshold is asymptotically close to 0 or 1. To find out whether this asymmetry is statistically significant, we perform the symmetry tests of Hong et al. (2007). Table 5 reports the test results and shows that, as measured by threshold correlation, half of the portfolios are significantly asymmetric: $HML(Beta, US/UK)$ and $HML(Cokt, UK)$.

[INSERT FIGURE 3 AND TABLE 5 ABOUT HERE]

Although threshold correlation offers some insights, it is still based on (linear) correlation and, therefore, does not take into account nonlinear information. To capture nonlinear dependence, we consider copula-based quantile dependence and tail dependence. Compared with

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(linear) correlation, the key advantage of copulas is that they are a "pure measure" of dependence, which cannot be affected by the marginal distributions (see Nelsen, 2006).

Quantile dependence measures the probability of two variables both lying above or below a given quantile (e.g. upper or lower tail) of their univariate distributions. Examining different quantiles allows us to focus on different aspects of the relationship. In Figure 4, we present the quantile dependence between the high beta (coskewness, cokurtosis) portfolios and the low beta (coskewness, cokurtosis) portfolios as well as the difference in upper and lower quantile dependence. For every portfolio pair, Figure 4 shows the estimated quantile dependence plot, for $q \in [0.025, 0.975]$, along with 90% (pointwise) *i.i.d.* bootstrap confidence intervals. As expected, the confidence intervals are narrower in the middle of the distribution (values of q close to 1/2) and wider near the tails (values of q near 0 or 1). Figure 4 also shows that observations in the lower tail are slightly more dependent than observations in the upper tail, with the difference between corresponding quantile dependence probabilities being as high as 0.3. The confidence intervals show that these differences are borderline significant at the 10% significance level, with the upper bound of the confidence interval on the difference lying around zero for most values of q. From the perspective of risk management, the dynamics implied by our empirical results may be of particular importance in the lower tails, because of its relevance for the portfolio VaR and ES.

[INSERT FIGURE 4, 5 AND TABLE 6 ABOUT HERE]

Next, we consider the tail dependence, which is a copula-based measure of dependence between extreme events. We employ the rotated Gumbel copula and the Student's *t* copula to estimate the tail dependence coefficients. All the coefficients are estimated by both parametric and semiparametric copula methods (detailed in the appendix). To avoid possible model misspecification, we use the nonparametric estimation method proposed by Frahm et al. (2005) as a robustness check and the results are consistent with results generated by the parametric and semiparametric methods.

Table 6 reports the coefficients of lower tail dependence (LTD) and upper tail dependence (UTD) and the difference between them. The coefficients are estimated using McNeil et al.

(2005). For example, the lower tail coefficient estimated by rotated Gumbel copula (respectively, Student's t copula) for BETA1 and BETA5 in the US equity market is 0.256 (respectively, 0.171) and the upper tail coefficient estimated by rotated Gumbel copula (Student's t copula) is 0.099 (respectively, 0.018). Then we find the significant difference between the upper and lower tail dependence coefficients. In the UK equity market, we also find evidence of asymmetric dependence in that all the portfolio pairs exhibit greater correlation during market downturns than market upturns. This finding about asymmetric dependence between the high beta (coskewness, cokurtosis) portfolio and the low beta (coskewness, cokurtosis) portfolios is new. It is possibly associated with the fact that investors have more uncertainty about the economy, and therefore pessimism and panic spread from one place to another more quickly during market downturns. Another possible explanation is the impact of liquidity risk. Some "uncorrelated" liquid assets suddenly become illiquid during market downturns, and, therefore, even a small trading volume can lead to huge co-movements.

The semiparametric tail dependence approach (that is nonparametric approach for the marginal distributions and and parametric for the copula estimation) and the nonparametric tail dependence approach of Frahm et al. (2005) are used as robustness checks and both of them provide similar results to the parametric approach.

We present the dynamic evolution of tail dependence coefficient (TDC) between the standardized residuals of the high and low portfolios in Figure 5. The dependence between portfolios, such as BETA1 and BETA5, is quite low in 2003 and has significantly increased since then. In the US equity market, the lower tail dependence (LTD) is relatively close to or even lower than the upper tail dependence before the global financial crisis of 2007-2009. However, the LTD has become greater than UTD following 2007. In the UK market, the LTD is always greater than the UTD. This phenomenon can be interpreted from behavioural finance theory that myopic loss aversion investor has a greater sensitivity to losses than to gains. In general, for US portfolio pairs, the asymmetries (differences between LTD and UTD) become more significant during the global financial crisis of 2007-2009 but the differences are relatively stable for UK portfolio pairs during the full sample period. Thus, we can reject the null hypothesis of symmetric dependence and conclude that for, most portfolio pairs, dependence is significantly

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asymmetric.

5. Forecasting Portfolio Risk with Dynamic Asymmetric Copula

In the previous section, the significance of dynamic dependence and asymmetries has been verified. In this section, we evaluate the statistical significance of the dynamic asymmetric copula model by forecasting our portfolio VaR and ES.[9]

We consider 10 copulas including Normal, Student's t, generalized hyperbolic (GH) skewed Student's t, Clayton, Rotated Clayton, Gumbel, Rotated Gumbel, Plackett, Frank and Symmetrized Joe-Clayton,[10] as candidates to model the dependence between BETA1 (COSK1, COKT1, OF1) and BETA5 (COSK5, COKT5, OF5). All the copula parameters are estimated by maximizing the log-likelihood function of Eq. (35) for the parametric case, and Eq. (38) for the semiparametric case. Standard errors are estimated using Chen and Fan (2006). Computing the log-likelihood of each copula in constant case, we find that the Student's t copula and the generalized hyperbolic skewed Student's t copula provide the highest likelihoods over the in-sample period in most cases.[11] In addition, we also find that all the log-likelihoods of dynamic copula models are significantly higher than constant cases.

Figure 6 shows the probability contour plots for bivariate normal, Student's t and GH skewed t copula with different asymmetric parameters. It is clear that the skewed t copula exhibits great flexibility of modeling asymmetric dependence for both upper and lower tails. Thus, we employ t copula and skewed t copula to model the dependence and combine with GAS model to forecast our portfolio VaR and ES. A detailed algorithm for dynamic copula-based forecasting can be found in Appendix C.

In order to evaluate VaR and ES forecasts, we use a rolling window instead of the full

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[9]We use the HML portfolio returns to forecast VaR and ES. This is important for the following reasons: First, the HML portfolio returns have different characteristics compared to simple long or short return series; Second, the HML portfolios have recently become increasingly popular in studies of asset pricing; Third, modeling the VaR and ES of the HML portfolios is of interest to practitioners as HML strategies are widely used in the financial industry.

[10]The reason that we consider so many copula candidates is because different copula could capture different dependence structure across assets. The analytical forms of Normal, Student's t, Clayton, Gumbel, Plackett, Frank and Symmetrized Joe-Clayton can found in (Patton, 2004). More details about GH skewed t copula can be found in Demarta and McNeil (2005) and Christoffersen et al. (2012). See Lucas et al. (2014) for a detailed discussion of GAS dynamics for the correlation matrix of GH skewed t copula.

[11]For the sake of simplicity, we call the Student's t copula and the generalized hyperbolic skewed t copula as t copula and skewed t copula, respectively.

sample period and the rolling window size is set at 250 (one trading year) for all the data sets.[12] All the models are recursively re-estimated throughout the out-of-sample period and the correlation coefficients of copulas are forecasted by the GAS model. For the purpose of comparison, we also consider the most successful univariate model, filtered historical simulation (FHS; Barone-Adesi, et al., 2002),[13] and three simulation-based multivariate GARCH models, namely, BEKK-GARCH, CCC-GARCH, DCC-GARCH.

The backtesting evaluates the coverage ability and the statistical accuracy of the VaR models. The coverage ability is evaluated by the empirical coverage probability (hereafter ECP) and Basel Penalty Zone (hereafter BPZ). The statistical accuracy is evaluated by the conditional coverage test (hereafter CC test; Christoffersen, 1998) and the dynamic quantile test (hereafter DQ test; Engle and Manganelli, 2004).

We first define the failure of the VaR model as the event that a realized return r_s is not covered by the predicted VaR. We identify it by the indicator function taking the value unity in the case of failure:

$$I_s = 1 \left\{ r_s < \widehat{VaR}_s \left(\alpha | F_{s-1} \right) \right\}, \ s = 1, \ldots, N, \tag{26}$$

where $\widehat{VaR}_s \left(\alpha | F_{s-1} \right)$ is the VaR forecast based on the information set at $s-1$, denoted by F_{s-1}, with a nominal coverage probability α. Henceforth, we abbreviate the notation $\widehat{VaR}_s \left(\alpha | F_{s-1} \right)$ to $\widehat{Var}_s \left(\alpha \right)$.

ECP is calculated by the sample average of I_s, $\hat{\alpha} = N^{-1} \sum_{s=1}^{N} I_s$ which is a consistent estimator of the coverage probability. The VaR model for which ECP is closest to its nominal coverage probability is preferred. BPZ is suggested by Basel Committee on Banking and Supervision (1996). It describes the strength of the VaR model through the test of failure rate. It records the number of failures of the 99 percent VaR in the previous 250 business days. One

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[12]The reason we use a moving window of 250 days instead of other window length or expending window is because a moving window of 250 days is the standard estimation period by the Basel accord. In practice the selection of an optimal sample size is a nontrivial issue. As the window size increases, estimation and forecasting precision generally improves. On the other hand it also raises uncertainty about the latent market regimes caused by a sequence of rare or extreme shocks hitting the market in which case it would be more desirable to select the shorter and homogeneous sample rather than longer and heterogeneous ones.

[13]We also evaluate other univariate models such as Historical Simulation, RiskMetrics, GARCH, GJR-GARCH, Extreme Value Theory model, and CAViaR. We find that FHS strongly outperforms others for all backtestings. Hence, we include only FHS as a univariate model. The results of other univariate models are available in the Internet Appendix.

may expect, on average, 2.5 failures out of the previous 250 VaR forecasts given the correct forecasting model. The Basel Committee rules that up to four failures are acceptable for banks and defines the range as a "Green" zone. If the failures are five or more, the banks fall into a "Yellow" (5–9) or "Red" (10+) zone. The VaR model for which BPZ is in the "Green" zone is preferred.

Accurate VaR forecasts should satisfy the condition that the conditional expectation of the failure is the nominal coverage probability:

$$\mathbb{E}\left[I_s | \mathcal{F}_{s-1}\right] = \alpha. \tag{27}$$

Christoffersen (1998) shows that it is equivalent to testing if $I_s | \mathcal{F}_{s-1}$ follows an i.i.d. Bernoulli distribution with parameter α:

$$H_0 : I_s | \mathcal{F}_{s-1} \sim i.i.d.\, Bernoulli(\alpha). \tag{28}$$

The CC test uses the LR statistic which follows the chi-squared distribution with two degrees-of-freedom under the null hypothesis, Eq. (28). The DQ test is a general extension of the CC test allowing for more time-dependent information of $\{I_s\}_{s=1}^N$. The out-of-sample DQ test is given by

$$DQ = \frac{(\tilde{\mathbf{I}}'\mathbf{Z})\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\left(\mathbf{Z}'\tilde{\mathbf{I}}\right)}{\alpha\left(1-\alpha\right)} \overset{d}{\sim} \chi^2_{p+2}, \tag{29}$$

where $\tilde{\mathbf{I}} = \left(\tilde{I}_{p+1}, \tilde{I}_{p+2}, \ldots, \tilde{I}_N\right)'$, $\tilde{I}_s = I_s - \alpha$, $\mathbf{Z} = \left(\mathbf{z}_{p+1}, \ldots, \mathbf{z}_N\right)'$ and $\mathbf{z}_s = \left(1, \tilde{I}_{s-1}, \ldots, \tilde{I}_{s-p}, \widehat{VaR}_s(\alpha)\right)'$. We use the first four lags for our evaluation, i.e., $\mathbf{z}_s = \left(1, \tilde{I}_{s-1}, \ldots, \tilde{I}_{s-4}, \widehat{VaR}_s(\alpha)\right)'$.

Backtesting of ES is not a straightforward task because it fails to satisfy elicitability (see Gneiting, 2011). We consider a backtesting for the ES forecast given the sample of N ES forecasts,

$$\left\{\widehat{ES}_1(\alpha), \ldots, \widehat{ES}_N(\alpha)\right\},$$

where $\widehat{ES}_s(\alpha)$ is the ES forecast based on the information set at $s-1$. We simply evaluate the ES forecast based on a loss function which enables researchers to rank the models and specify a utility function reflecting the concern of the risk manager. We define two loss functions:

$$\text{Absolute error} := \left|r_s - \widehat{ES}_s(\alpha)\right| I_s, \quad \text{Squared error} := \left(r_s - \widehat{ES}_s(\alpha)\right)^2 I_s,$$

where $I_s = 1\left\{r_s < \widehat{VaR}_s(\alpha)\right\}$. In order to rank the models, we compute the mean absolute error (MAE) and the mean squared error (MSE):

$$MAE = \frac{1}{N}\sum_{s=1}^{N}\left|r_s - \widehat{ES}_s(\alpha)\right|I_s, \tag{30}$$

$$MSE = \frac{1}{N}\sum_{s=1}^{N}\left(r_s - \widehat{ES}_s(\alpha)\right)^2 I_s. \tag{31}$$

This evaluation is in line with the framework proposed by Lopez (1999) for the VaR evaluation. The smaller value indicates more accurate forecast.

For the UK portfolios, we estimate the VaR and ES models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent VaR and ES for 18 Dec. 2000. We conduct rolling forecast by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts.

5.1. Backtesting of Value-at-Risk

We evaluate the coverage ability by ECP and BPZ as follows: We calculate ECP for each portfolio and then report bias and Root Mean Square Error (RMSE). Bias is the average deviation of ECP from the nominal coverage probability (1% in our case). The smaller the bias is, the more accurate the VaR forecast is. RMSE is the average of the squared deviation. It shows the dispersion of ECP from the nominal coverage probability. It makes up for the defect of bias due to the offset of positive and negative deviations. Financial regulators would prefer a VaR model with, simultaneously, a small bias and small RMSE. BPZ describes the coverage ability of the VaR model through the test of failure rate. It counts the number of failure over the previous 250 business days.

5.1.1. Empirical Coverage Probability

Table 7 presents the ECPs of the VaR models. First, the bias of the parametric dynamic copula models is -0.03% and the bias of the semiparametric dynamic copula models is -0.02% which are much smaller than those of the other models. It shows that the ECPs of the dynamic

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copula models are very close to the nominal one. In addition, the RMSEs of dynamic skewed *t* copula are significantly smaller than the others. The bias of the static skewed *t* copula is 0.05% which is greater than those (magnitude) of the dynamic skewed *t* copula models, and the RMSE is more than two times the RMSE of the dynamic *t* copula. This is clear evidence of the superiority of the dynamic copula model. Although, the biases of dynamic *t* copulas and their skewed version are similar, the RMSEs of dynamic skewed *t* copula are clearly smaller than those of dynamic *t* copula, implying the importance of incorporating asymmetric dependence in risk forecasting. Second, FHS shows a large positive bias (0.31%) which implies the under-forecasts of VaR. Its RMSE is also large - twice greater than the dynamic *t* and skewed *t* copula models. Finally, the bias of the multivariate GARCH models range from 0.26% to 0.36%. These are greater than for the dynamic copula models. Their RMSEs are also much greater than those of the dynamic copula models.

[INSERT TABLE 7 ABOUT HERE]

5.1.2. Basel Penalty Zone

Table 8 presents the BPZ of the VaR models. We find that most of the models achieve 12 Green zone using the framework of Basel committee. The static *t* copula, static skewed *t* copula and BEKK GARCH achieve 11 Green zone and 1 Yellow zone. This result is not surprising as the "traffic lights" backtest is not as rigorous as other statistical tests such as CC test and DQ test.

[INSERT TABLE 8 ABOUT HERE]

Next, we evaluate the statistical accuracy by the CC test and the DQ tests as follows: We calculate both statistics for each portfolio and test them at the 5% significance level. Then we report the number of rejected portfolios.

5.1.3. Conditional Coverage Test

Table 9 reports the CC test results. First, the dynamic *t* copula models are rejected for 2 (parametric) and 1 (semiparametric) portfolios whilst the static *t* copula is rejected for 5 portfolios. Also, the dynamic skewed *t* copula (parmaetric and semiparametric) models are rejected for 2 portfolios whilst the static skewed *t* copula is rejected for 5 portfolios. Second,

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FHS is rejected for 3 portfolios which is slightly more than the dynamic copula models. Third, multivariate GARCH models are rejected for 3 (BEKK; DCC) and 4 (CCC) portfolios which are slightly more than the dynamic copula models.

[INSERT TABLE 9 ABOUT HERE]

5.1.4. Dynamic Quantile Test

Table 10 reports the results of the DQ test. Although the number of rejections increases, the results are qualitatively consistent with those of the CC test. First, the dynamic *t* copula models are rejected for 3 portfolios whilst the static *t* copula is rejected for 6 portfolios. Also, the dynamic skewed *t* copula models are rejected for 2 (parametric) and 3 (semiparametric) portfolios whilst the static skewed *t* copula is rejected for 6 portfolios. Second, FHS is rejected for 4 portfolios which is slightly more than the dynamic copula models. Finally, the multivariate GARCH models are more frequently rejected for the DQ test than the dynamic copula models. BEKK, CCC and DCC are rejected for 4, 8 and 4 portfolios, respectively.

[INSERT TABLE 10 ABOUT HERE]

5.2. Backtesting of Expected Shortfall

Table 11 reports the MAE results for ES. Dynamic copula models provide the most accurate forecasts (lowest MAEs) in 10 out of 12 portfolios. Also, the skewed *t* copula models generate lower average MAE in general comparing with *t* copula models, as it takes into account the asymmetric dependence between portfolios. As a robustness check, the MSE results reported in Table 12 also confirm this conclusion. The dynamic copula models have better performance than both univariate model and multivariate models in almost all cases. In general, the semiparametric skewed *t* copula model tends to outperform the parametric copula models, as it allows for more flexible assumptions regarding the true distribution.

[INSERT TABLE 11 AND 12 ABOUT HERE]

In sum, we have the following implications of the copula model from the backtesting results. First, the multivariate modeling of the tail dependence is more effective than the multivariate modeling of the central dependence (e.g. covariance) for the accurate extreme event forecast.

Further, it outperforms the most successful univariate model, FHS.[14] Second, the modeling of the tail dependence is more important than the modeling of the central dependence for improving the extreme event forecast. Third, a copula must take into account the dynamic nature of the tail dependence. The dynamic copula models strongly outperform the static copula models. Forth, the modeling of asymmetric tail dependence can help it to improve the extreme event forecast. The dynamic skewed *t* copula model tends to outperform the dynamic *t* copula models in the extreme event forecast. Finally, to check the robustness of our results, we also examine the predication performance of all the candidate models at 95% and 97.5% significance level. The consistent results confirm our conclusion and suggest that data mining are unlikely explanations.[15]

6. Conclusion

This paper empirically addresses three related questions to improve our understanding of the dependence structure between financial assets with different characteristics under various market conditions and shows the statistical significance of dynamic asymmetric copula-based models from a risk management perspective. Our findings are novel as we go beyond the earlier copula literature that investigates the dependence across single assets and explore dependence in a cross-sectional setting by forming characteristics-based portfolios of stocks in US and UK markets. We sort stocks listed on the S&P 500 and the FTSE 100 into portfolios based on their comoments including beta, coskewness and cokurtosis.

First, we provide empirical evidence that the dependence between characteristic-sorted portfolios is significantly time-varying. Using empirical data, spanning recent financial crises, we conclude that the returns of portfolios exhibit time-varying dependence and that the dependence has increased in recent years. Therefore, it provides strong support and motivation to apply dynamic copulas in dependence modeling.

Second, we use several tests to verify the presence of asymmetric dependence between high beta (coskewness, cokurtosis) portfolios and low beta (coskewness, cokurtosis) portfolios.

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[14]The multivariate GARCH models could not outperform FHS. Rather, FHS outperforms the multivariate GARCH models for some cases. It implies that the multivariate modeling of the central dependence cannot help it to improve the extreme event forecast.

[15]All the robustness checks are available on request from the authors.

Our empirical results confirm this asymmetry and show that most portfolio pairs have stronger dependence during market downturns than during market upturns. Our conclusion strongly confirms the results in the extant literature, see Patton (2006), Okimoto (2008), Chollete et al. (2011) and many others. It has wide implications for empirical asset pricing and asset allocation as well as for risk management.

Third, we apply a dynamic asymmetric copula framework based on Demarta and McNeil (2005) and Creal et al. (2013) to predict portfolio VaR and ES. This dynamic copula model has several attractive properties for VaR forecasting. The most attractive one is that it not only takes into account common features of univariate distributions, such as heteroscedasticity, skewness, fat tails, but also captures asymmetries and time-varying dependency between time series. All the models are estimated either parametrically, with the marginal distributions and the copula specified as belonging to parametric families, or semiparametrically, where the marginal distributions are estimated nonparametrically. Several widely used univariate and multivariate VaR and ES models are also considered for comparison. Backtestings are included in the evaluation process as well. To evaluate the predictions of ES, we consider a test in line with the one proposed for backtesting VaR in Lopez (1999). Overall, our study provides new evidence that the dynamic asymmetric copula model can offer more accurate VaR and ES forecasts.

Taken together, these empirical findings indicate the statistical significance of incorporating asymmetric and dynamic dependence in risk management. They can help investors better understand the co-movement between portfolios with different characteristics, and control portfolio risk more effectively under different market conditions. Moreover, we empirically prove that the dynamic asymmetric copula-based model can provide both the Basel committee and financial institutions with a more powerful and precise tool to forecast market risk and adjust minimum capital requirements.

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Appendix

Appendix A. Estimation of Parametric Copula Model

The log-likelihood of a fully parametric copula model for conditional distribution of $\mathbf{z}_t$ takes the form:

$$
\begin{aligned}
L(\theta) &= \prod_{t=1}^{T} \mathbf{f}(\mathbf{z}_t | F_{t-1}; \theta) \\
&= \prod_{t=1}^{T} \left[c_t \left(u_{1,t}, ..., u_{d,t} | F_{t-1}; \theta_C \right) \prod_{i=1}^{N} f_{i,t} \left(z_{i,t} | F_{t-1}; \theta_i \right) \right]
\end{aligned}
\tag{32}
$$

with log-likelihood

$$
\begin{aligned}
\sum_{t=1}^{T} \log \mathbf{f}(\mathbf{z}_t | F_{t-1}; \theta) &= \sum_{t=1}^{T} \sum_{i=1}^{d} \log f_{i,t} \left(z_{i,t} | F_{t-1}; \theta_i \right) \\
&+ \sum_{t=1}^{T} \log c_t \left(F_{1,t} \left(z_{1,t} | F_{t-1}; \theta_1 \right), \ldots, F_{d,t} \left(z_{d,t} | F_{t-1}; \theta_d \right) | F_{t-1}; \theta_C \right)
\end{aligned}
\tag{33}
$$

where θ denotes the parameter vector for the full model parameters, θ_i denotes the parameters for the ith marginals, θ_C denotes the parameters of copula model and F_{t-1} denotes the information set at time $t-1$. Following the two-stage maximum likelihood estimation (also known as the Inference method for marginals) of Joe and Xu (1996), we first estimate the parameters of marginal models using maximum likelihood:

$$
\hat{\theta}_{\mathbf{i}} = \arg\max_{\theta_i} \sum_{t=1}^{T} \log f_{i,t} \left(z_{i,t} | F_{t-1}; \theta_i \right), \ i = 1, ..., N
\tag{34}
$$

and then using the estimations in the first stage, we calculate $F_{i,t}$ and estimate the copula parameters via maximum likelihood:

$$
\hat{\theta}_{\mathbf{C}} = \arg\max_{\theta_C} \sum_{t=1}^{T} \log c_t \left(F_{1,t} \left(z_{1,t} | F_{t-1}; \theta_1 \right), ..., F_{d,t} \left(z_{d,t} | F_{t-1}; \theta_d \right) | F_{t-1}; \theta_C \right)
\tag{35}
$$

Appendix B. Estimation of Semiparametric Copula Model

In the semiparametric estimation (also known as Canonical Maximum Likelihood Estimation), the univariate marginals are estimated nonparametrically using the empirical distribution

function and the copula model is again parametrically estimated via maximum likelihood.

$$\hat{F}_i(z) \equiv \frac{1}{T+1} \sum_{t=1}^{T} 1\{\hat{z}_{i,t} \leq z\} \tag{36}$$

$$\hat{u}_{i,t} \equiv \hat{F}_i(z) \sim Unif(0,1), \ i = 1, 2, ..., N \tag{37}$$

$$\hat{\theta}_{\mathbf{C}} = \arg\max_{\theta_{\mathbf{C}}} \sum_{t=1}^{T} \log c_t(\hat{u}_{1,t}, ..., \hat{u}_{i,t} | \mathcal{F}_{t-1}; \theta_{\mathbf{C}}) \tag{38}$$

where $z_{i,t}$ are the standardized residuals of the marginal model and $\hat{F}_i$ is different from the standard empirical CDF by the scalar $\frac{1}{n+1}$ (in order to ensure that the transformed data cannot be on the boundary of the unit interval $[0,1]$).

Appendix C. Algorithm for VaR and ES Forecasting Using Dynamic GH Skewed t Copula Model

Step 1 Determine the in sample and out-of-sample period for VaR and ES forecasting.

Step 2 We predict conditional mean and conditional volatility from the prespecified time series model on rolling window and do one step ahead forecasting for each margins;

Step 3 Estimate the density model to get the probabilities for each forecasted margin. We consider both parametric (univariate skewed *t*) and nonparametric (EDF) estimation on sliding window.

Step 4 Estimate the parameters for full parametric and semiparametric copulas using using maximum likelihood estimation (see Appendix A and B).

Step 5 Using the estimated parameters in Step 4 as initial values, we estimate time-varying dependence parameters for asymmetric (GH skewed *t*) copulas based on the GAS framework, see Equation (22).

Step 6 With the estimated time-varying copula parameters in hand, we can apply Monte Carlo simulation to generate N samples of shocks and then portfolio returns.

Step 7 Based on the empirical $\alpha-$quantile of forecasted portfolio return, it is straightforward to forecast corresponding VaR.

Step 8 Given the N simulated portfolio returns, we can also calculate $\alpha-$quantile expected shortfall using Equation (25).

Step 9 Use the realized portfolio returns to backtest VaR and ES predications.

Figure 1: The Scatter Plots for Portfolio 1 (High) and Portfolio 5 (Low)

Panel A: US Stock Market

Panel B: UK Stock Market

$\rho_L = 0.63$, $\rho_C = 0.42$, $\rho_U = 0.52$
Beta Portfolio

$\rho_L = 0.61$, $\rho_C = 0.29$, $\rho_U = 0.30$
Beta Portfolio

$\rho_L = 0.70$, $\rho_C = 0.63$, $\rho_U = 0.46$
Coskewness Portfolio

$\rho_L = 0.61$, $\rho_C = 0.46$, $\rho_U = 0.53$
Coskewness Portfolio

$\rho_L = 0.65$, $\rho_C = 0.61$, $\rho_U = 0.41$
Cokurtosis Portfolio

$\rho_L = 0.67$, $\rho_C = 0.47$, $\rho_U = 0.68$
Cokurtosis Portfolio
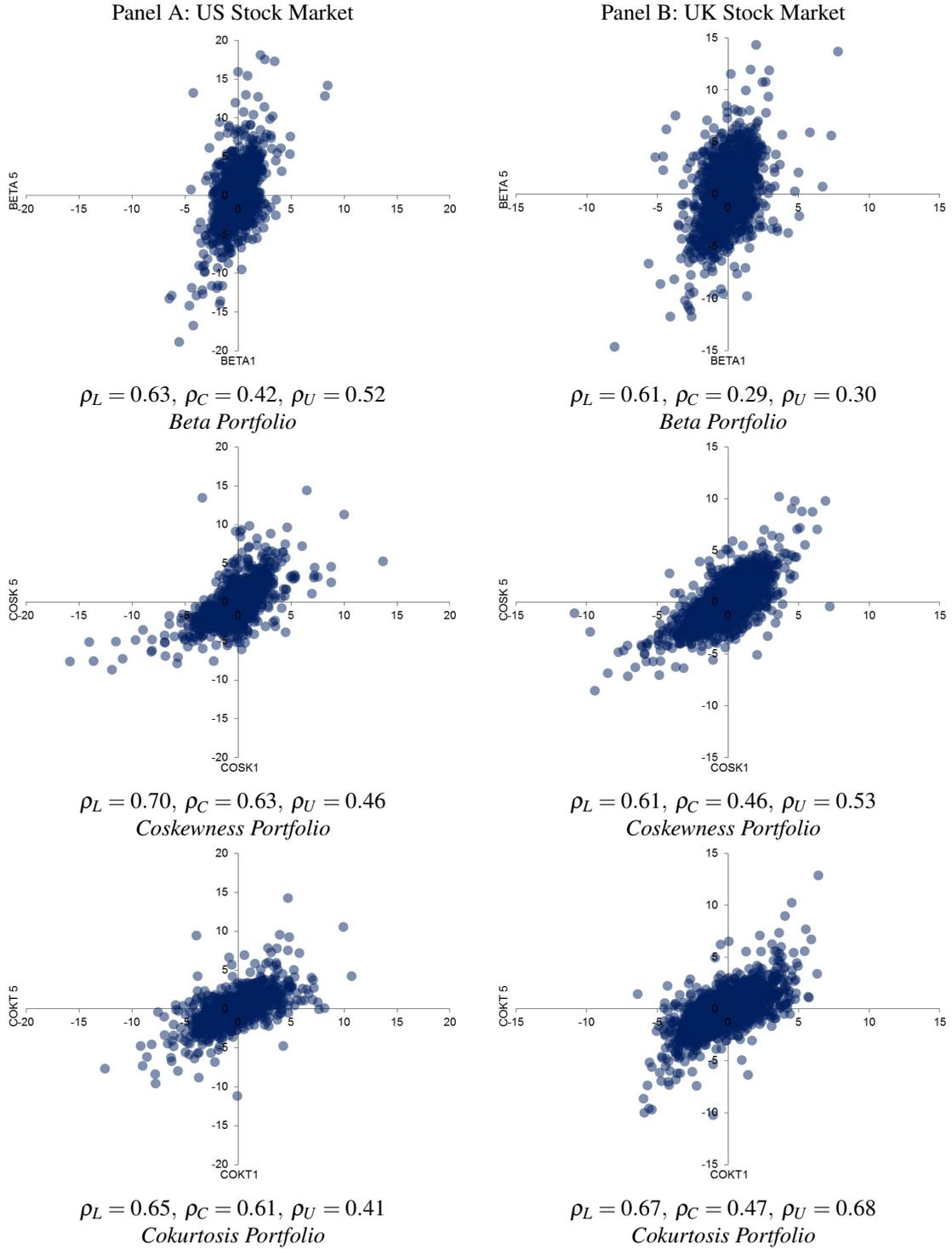
Note: This figure shows the scatter plots for different portfolio pairs, including $(BETA1, BETA5)$, $(COSK1, COSK5)$, and $(COKT1, COKT5)$. Three threshold correlation coefficients are used to demonstrate the asymmetric dependence between the portfolios:

$$\rho_L = Corr\left(r_1, r_5 | r_1 \leq F_1^{-1}(0.15), r_5 \leq F_5^{-1}(0.15)\right),$$
$$\rho_U = Corr\left(r_1, r_5 | F_1^{-1}(0.85) < r_1, F_5^{-1}(0.85) < r_5\right),$$
$$\rho_C = Corr\left(r_1, r_5 | F_1^{-1}(0.15) < r_1 \leq F_1^{-1}(0.85), F_5^{-1}(0.15) < r_5 \leq F_5^{-1}(0.85)\right),$$

where ρ_L, ρ_U and ρ_C denote the correlation coefficients at the lower tail, upper tail and center, respectively, and F^{-1} denotes the inverse cumulative probability density function.

Figure 2: Time-varying Rank Correlation for High versus Low Portfolios

Panel A: US Stock Market

Panel B: UK Stock Market
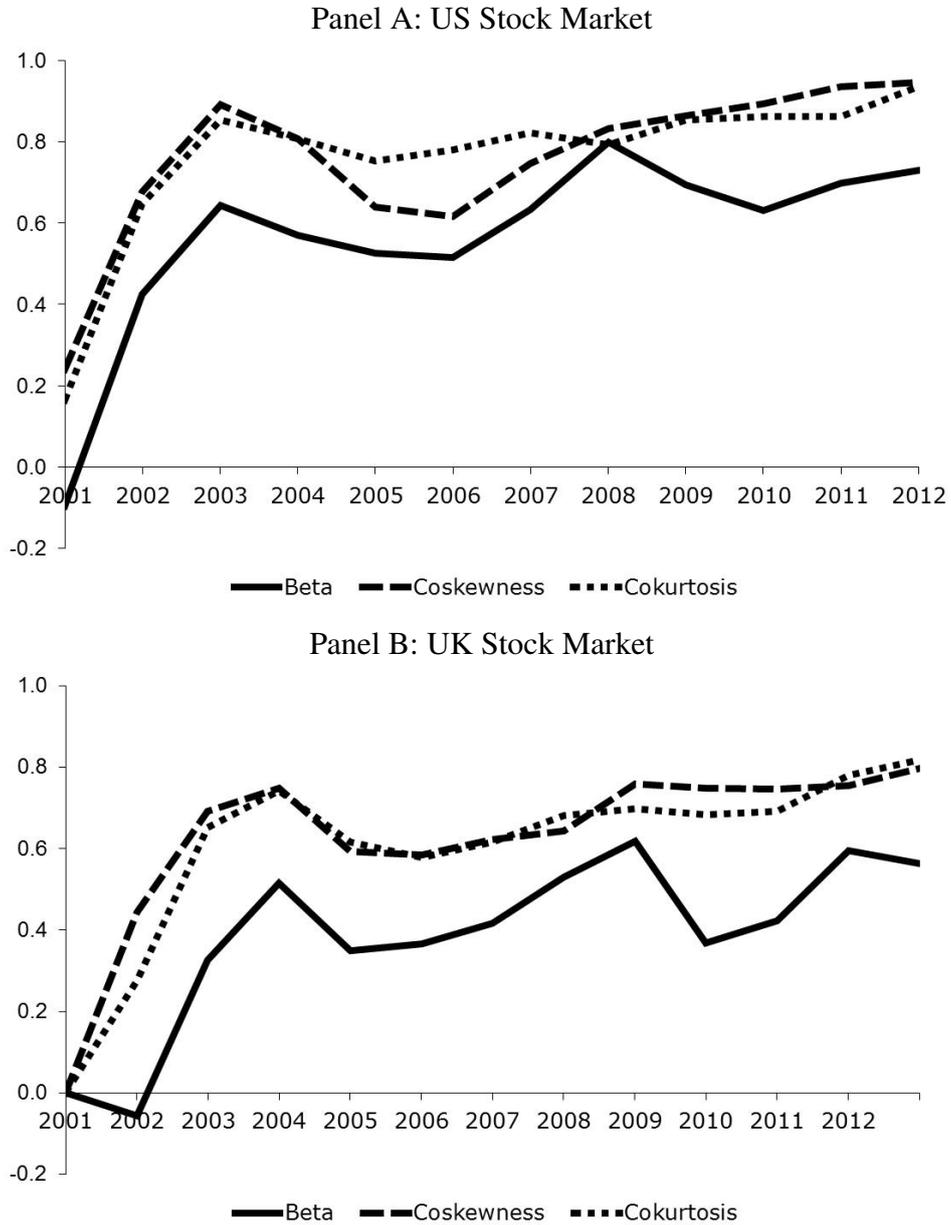
Note: This figure depicts two time series plots of average rolling 250-day rank correlation between the high and low portfolios; ($BETA1, BETA5$), ($COSK1, COSK5$), and ($COKT1, COKT5$), for each year.

Figure 3: Threshold Correlation for High versus Low Portfolios

Panel A: US Stock Market

Panel B: UK Stock Market

Beta Portfolio

Beta Portfolio

Coskewness Portfolio

Coskewness Portfolio

Cokurtosis Portfolio

Cokurtosis Portfolio
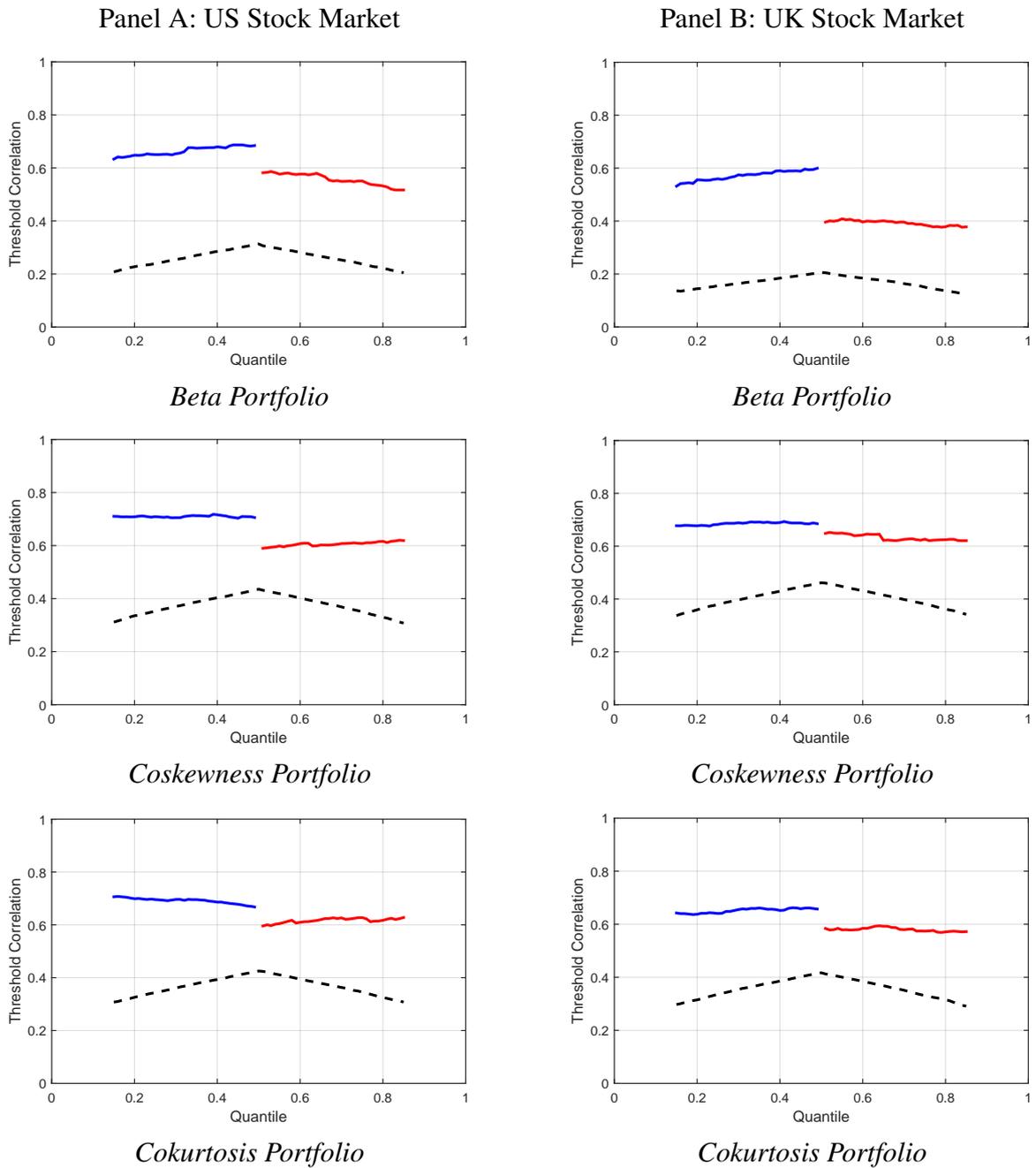
Note: This figure shows the threshold correlation (or exceedance correlation) between high beta (coskewness and cokurtosis) portfolio and low beta (coskewness and cokurtosis) portfolio. The threshold correlation measures the linear correlation between two assets when both assets increases or decreases of more than specified quantiles (see Longin and Solnik, 2001; Ang and Bekaert, 2002; Ang and Chen, 2002). A solid blue line denotes (12) and a solid red line denotes (13), respectively. The dash line represents the threshold correlations implied by the bivariate normal distribution with a linear correlation rho from the data.

Figure 4: Quantile Dependence between the Standardized Residuals of High and Low Portfolios

Panel A: US Stock Market
Quantile Dependence

Panel B: UK Stock Market
Quantile Dependence

Beta Portfolio

Beta Portfolio

Coskewness Portfolio

Coskewness Portfolio

Cokurtosis Portfolio

Cokurtosis Portfolio
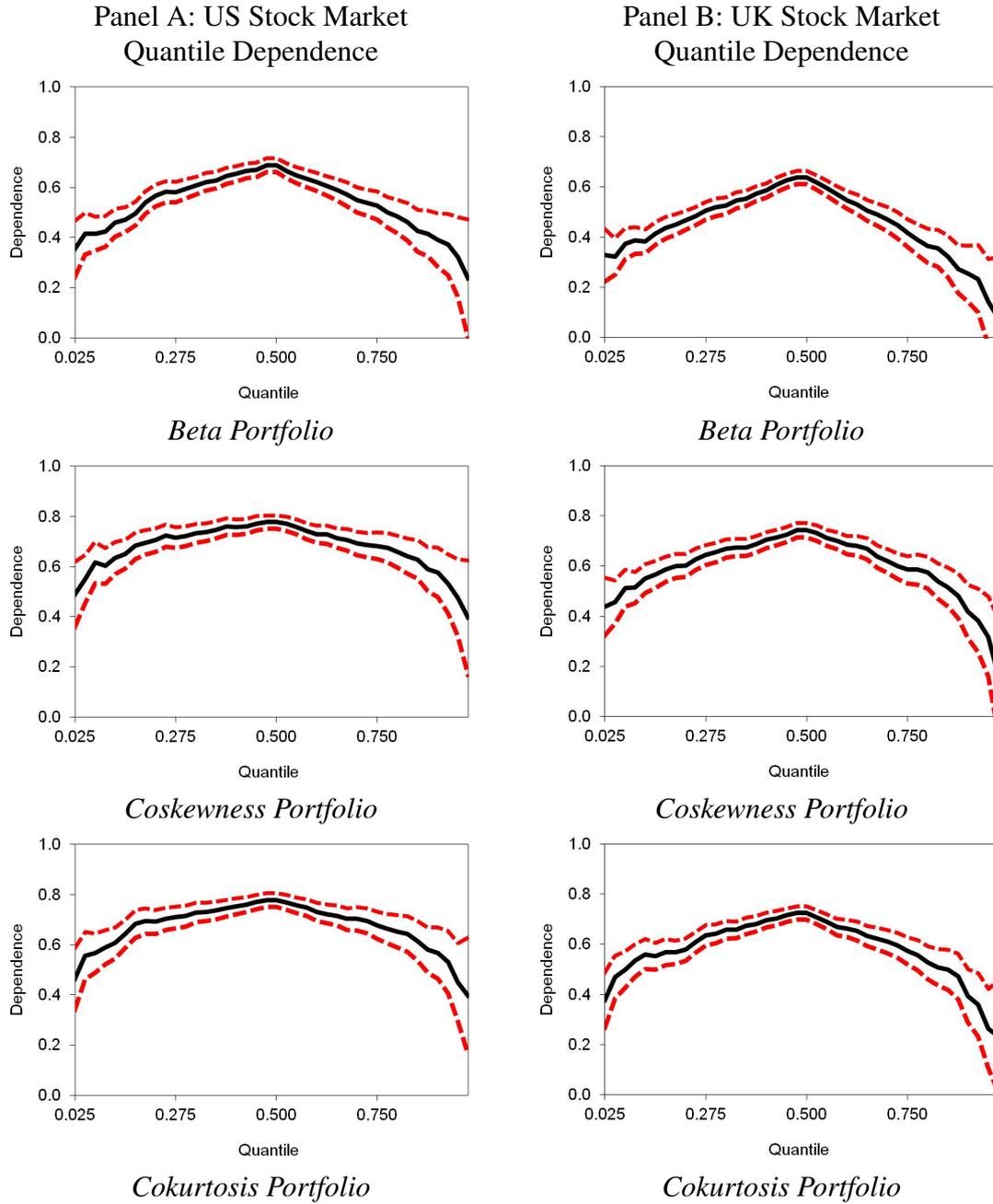
Note: This figure presents the estimated quantile dependence between the standardized residuals for high beta (coskewness and cokurtosis) portfolio and low beta (coskewness and cokurtosis), and the difference in upper and lower quantile dependence. The red dash lines are 90% bootstrap confidence interval for the dependence and lower-upper difference. A solid black line denotes a quantile dependence and dot red lines denote 90% bootstrap confidence interval.

Figure 5: Time-varying Asymmetric Tail Dependence

Panel A: US Stock Market

Beta Portfolio

Coskewness Portfolio

Cokurtosis Portfolio

Panel B: UK Stock Market

Beta Portfolio

Coskewness Portfolio

Cokurtosis Portfolio
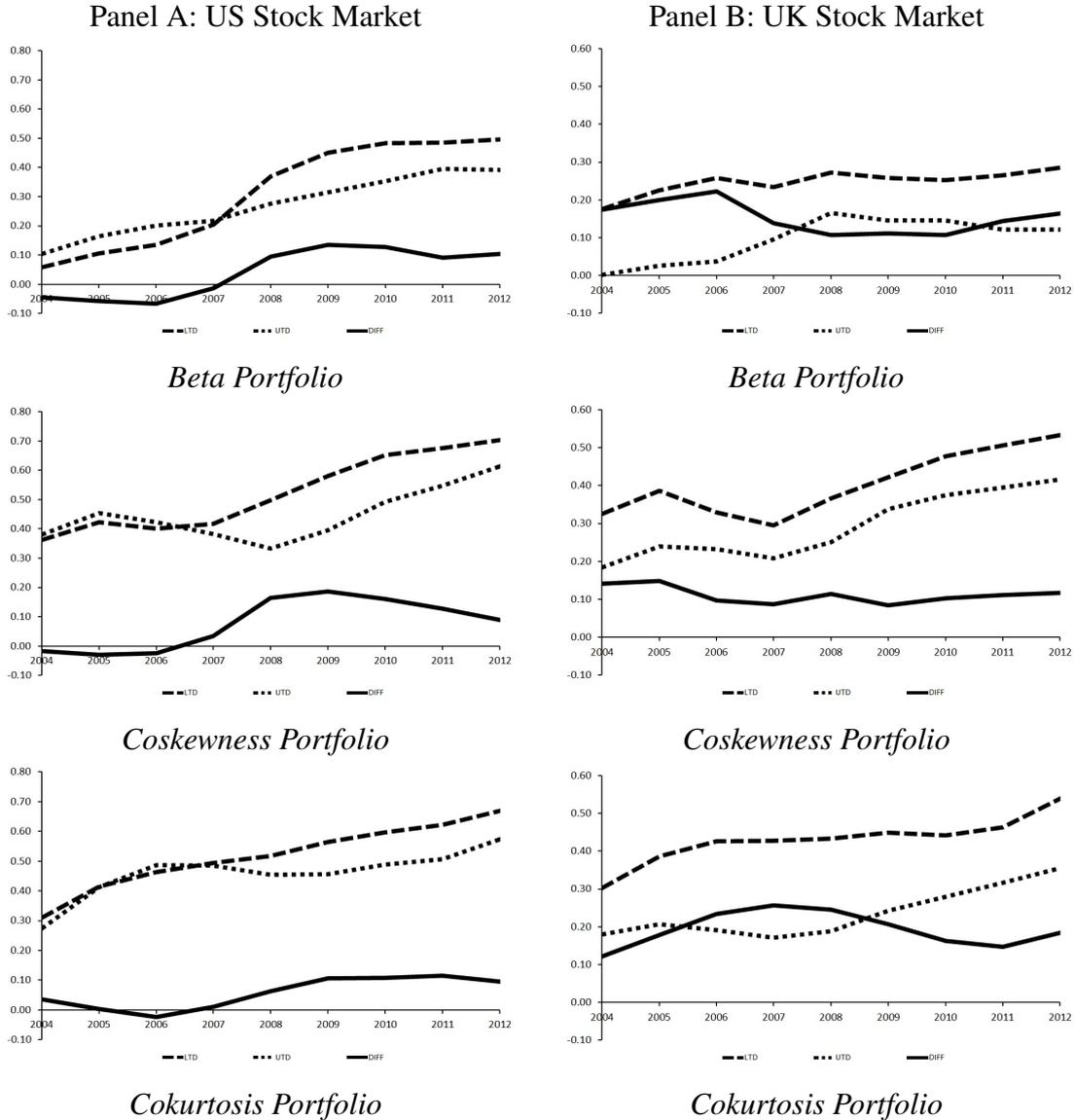
Note: This figure shows the dynamic evolution of average tail dependence coefficient (TDC). TDC is estimated by rotated Gumbel copula from rolling window with window length of 1,000 observations for all the portfolio pairs and we take the average of TDC for each year. The TDCs between portfolios generally increase over time, especially during recent financial crisis. In the US market, lower tail dependence (LTD) is relatively close to or even lower than the upper tail dependence before the financial crisis. However, the LTD has become greater than upper tail dependence (UTD) since the outbreak of the US subprime mortgage crisis in 2007. In the UK market, the LTD is always greater than UTD. Note that DIFF denotes the difference between LTD and UTD ($DIFF = LTD - UTD$).

Figure 6: Contour Probability Plots for Copulas

Normal Copula, $\rho = 0.5$

Student's Copula, $\rho = 0.5, \nu = 10$

SkT Copula, $\rho = 0.5, \nu = 6, \lambda_1 = -0.5, \lambda_2 = -0.5$

SkT Copula, $\rho = 0.5, \nu = 6, \lambda_1 = 0.5, \lambda_2 = 0.5$

SkT Copula, $\rho = 0.5, \nu = 6, \lambda_1 = -0.5, \lambda_2 = 0.5$

SkT Copula, $\rho = 0.5, \nu = 6, \lambda_1 = 0.5, \lambda_2 = -0.5$
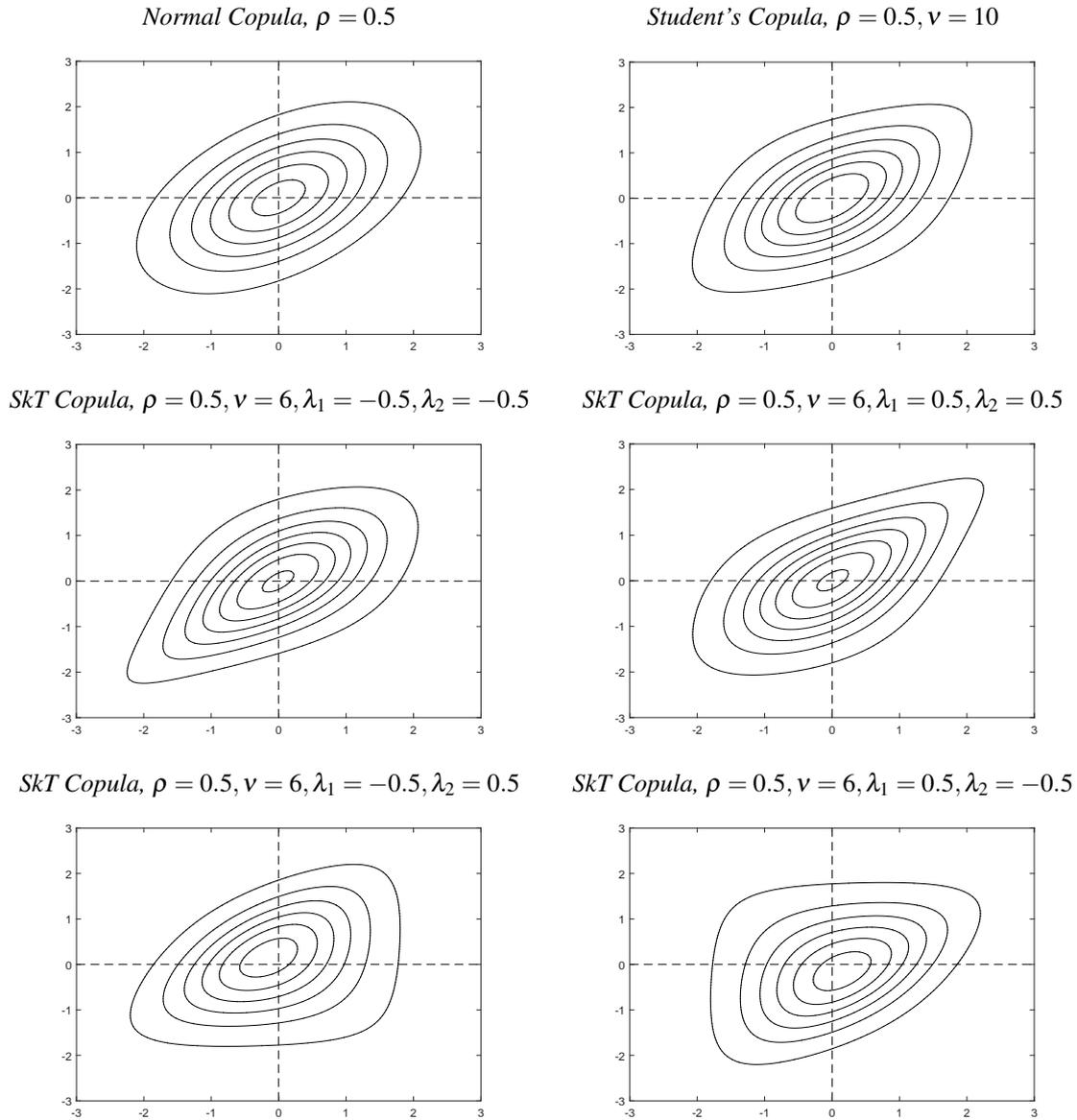
Note: This figure shows contour probability plots for the normal, Student's *t*, and asymmetric skewed *t* copulas. The probability levels for each contour are held fixed across six panels. The marginal distributions are assumed to be normally distributed. ρ denotes the correlation coefficient, ν denotes the degree of freedom, and λ denotes the asymmetric parameters of copulas.

Table 1: Definitions of Portfolios

This table describes the 12 HML portfolios that we constructed for the purpose of empirical analysis in our study. Portfolios are sorted by market beta, coskewness and cokurtosis. All the portfolios are annually rebalanced.

Portfolio	Market	Description
$HML(Beta, L/S; US)$	US Stock Market	Long (short) BETA5 and short (long) BETA1
$HML(Cosk, L/S; US)$		Long (short) COSK5 and short (long) COSK1
$HML(Cokt, L/S; US)$		Long (short) COKT5 and short (long) COKT1
$HML(Beta, L/S; UK)$	UK Stock Market	Long (short) BETA5 and short (long) BETA1
$HML(Cosk, L/S; UK)$		Long (short) COSK5 and short (long) COSK1
$HML(Cokt, L/S; UK)$		Long (short) COKT5 and short (long) COKT1

Table 2: Descriptive Statistics for Returns on the Characteristic-sorted Portfolios

Panel A reports descriptive statistics for daily returns on the characteristic-sorted portfolios from January 4, 2000 to December 31, 2012, which correspond to a sample of 3,268 observations for US market and a sample of 3,283 observations for UK market. We sort stocks into quintiles according to market beta (coskewness, cokurtosis) and form five capitalization-weighted, annually rebalanced portfolios. BETA1 (COSK1, COKT1) contains the equities with the lowest beta (coskewness and cokurtosis), and portfolio 5 contains the equities with the highest beta (coskewness and cokurtosis). Panel B reports descriptive statistics for returns on the high minus low (HML) portfolios. Note that the means, standard deviations, minima, and maxima are reported in %. LB test lag 5 and 10 denote the p-values of the Ljung-Box Q-test for autocorrelation at lags 5 and 10, respectively. In addition, we report the p-values of Engle's Lagrange Multiplier test for the ARCH effect on the residual series.

Panel A: Charateristic-sorted Portfolio

	US Stock Market						UK Stock Market					
Statistics	BETA1	BETA5	COSK1	COSK5	COKT1	COKT5	BETA1	BETA5	COSK1	COSK5	COKT1	COKT5
Mean	0.023	-0.021	0.004	0.003	-0.011	0.024	0.039	-0.023	0.018	-0.002	0.009	0.006
Std dev	0.889	2.567	1.525	1.707	1.612	1.380	0.952	2.317	1.394	1.553	1.457	1.425
Skewness	0.045	-0.067	-1.181	0.733	-0.137	0.280	0.030	-0.099	-0.807	0.390	-0.018	-0.278
Kurtosis	12.413	12.447	19.667	10.315	8.763	16.443	10.804	9.144	8.785	7.349	4.768	12.906
Maximum	8.477	18.044	13.735	14.394	10.761	14.222	7.817	17.378	7.220	10.188	6.431	12.848
Minimum	-6.444	-20.238	-15.811	-8.734	-12.524	-11.224	-7.995	-16.927	10.188	-8.569	-6.356	-10.264
LB test lag 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
LB test lag 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Engle's test	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Panel B: HML Portfolio

Statistics	HML(Beta,L;US)	HML(Cosk,L;US)	HML(Cokt,L;US)	HML(Beta,L;UK)	HML(Cosk,L;UK)	HML(Cokt,L;UK)
Mean	-0.044	-0.001	0.035	-0.062	-0.020	-0.003
Std dev	2.206	1.312	1.235	2.108	1.152	1.193
Skewness	0.369	1.177	0.346	0.089	0.355	-0.041
Kurtosis	12.925	20.139	17.163	9.151	8.881	8.371
Maximum	17.389	16.719	13.292	15.970	9.767	7.769
Minimum	-18.409	-8.470	-11.182	-19.017	-7.658	-9.247
LB test lag 5	0.00	0.00	0.00	0.01	0.00	0.00
LB test lag 10	0.00	0.01	0.00	0.00	0.00	0.00
Engle's test	0.00	0.00	0.00	0.00	0.00	0.00

Table 3: Parameter Estimates and Goodness of Fit Test for the Univariate Modeling

This table reports parameter estimates from AR and GJR-GARCH models for conditional mean and conditional variance of portfolio returns. We estimate all parameters using the sample from January 4, 2000 to December 31, 2012, which correspond to a sample of 3,268 observations for US market and a sample of 3,283 observations for UK market. We use * to indicate the significance of estimate at the 5% significance level. We also report the p-values of two goodness-of-fit tests for the skewed Student's t distribution. We employ Kolmogorov-Smirnov (KS) and Cramer-von Mises (CvM) tests.

Porfolio	Panel A: US Stock Market						Panel B: UK Stock Market					
	BETA1	BETA5	COSK1	COSK5	COKT1	COKT5	BETA1	BETA5	COSK1	COSK5	COKT1	COKT5
ϕ_0	0.025	-0.021	0.004	0.006	-0.011	0.029	0.039*	-0.023	0.018	-0.001	0.009	0.008
ϕ_1	-0.064*	—	—	-0.091*	—	-0.085*	—	—	—	-0.045*	—	-0.060*
ϕ_2	—	—	—	-0.085*	—	-0.061*	—	—	—	—	—	—
ω	0.012*	0.048*	0.027*	0.015*	0.021*	0.016*	0.014*	0.036*	0.021*	0.019*	0.014*	0.028*
α	0.018	0.001*	0.0235*	0.000	0.023*	0.008	0.012	0.004	0.031*	0.006	0.016	0.047*
γ	0.122*	0.133*	0.104*	0.163*	0.126*	0.179*	0.111*	0.130*	0.092*	0.135*	0.103*	0.142*
β	0.901*	0.923*	0.903*	0.916*	0.903*	0.896*	0.913*	0.924*	0.911*	0.919*	0.925*	0.867*
ν	11.507	12.805	9.404	12.331	24.329	5.637	9.593	27.910	7.384	33.361	94.667	7.475
λ	-0.111	-0.072	-0.246	0.006	-0.099	-0.099	-0.076	-0.054	-0.168	0.025	-0.062	-0.076
KS	0.61	0.17	0.43	0.11	0.14	0.97	0.77	0.96	0.53	0.53	0.35	0.36
CvM	0.33	0.10	0.42	0.10	0.16	1.00	0.87	0.92	0.30	0.40	0.45	0.27

Table 4: Tests for Time-varying Dependence between High and Low Portfolios

We report the *p*-value from tests for time-varying rank correlation between the high portfolio (e.g. BETA5) and the low portfolio (e.g. BETA1). Having no a priori dates to consider for the timing of a break, we consider naive tests for breaks at three chosen points in sample period, at t*/ T ∈{0.15, 0.50, 0.85}, which corresponds to the dates 10-Dec-2001, 03-Jul-2006, 17-Jan-2011. The 'Any' column reports the results of test for dependence break of unknown timing proposed by Andrews (1993). To detect whether the dependence structures between characteristic-sorted portfolios significantly changed after the US and EU crisis broke out, we use 15-Sep-2008 (the collapse of Lehman Brothers) and 01-Jan-2010 (EU sovereign debt crisis) as two break points and the 'Crisis' panel reports the results for this test. The 'AR' panel presents the results from the ARCH LM test for time-varying volatility proposed by Engle (1982). Under the null hypothesis of a constant conditional copula, we test autocorrelation in a measure of dependence (see Patton, 2012).

	Panel A: Break				Panel B: Crisis		Panel C: AR(p)		
Portfolio	0.15	0.5	0.85	Any	US	EU	AR(1)	AR(5)	AR(10)
US BETA1&5	0.00	0.00	0.04	0.00	0.00	0.07	0.00	0.00	0.00
US COSK1&5	0.00	0.03	0.82	0.04	0.22	0.38	0.00	0.00	0.00
US COKT1&5	0.02	0.30	0.67	0.25	0.92	0.42	0.18	0.75	0.09
UK BETA1&5	0.00	0.00	0.17	0.00	0.04	0.08	0.00	0.12	0.00
UK COSK1&5	0.59	0.03	0.62	0.03	0.07	0.25	0.01	0.00	0.02
UK COKT1&5	0.98	0.24	0.36	0.24	0.24	0.13	0.00	0.00	0.00

Table 5: Testing the Significance of the Differences of Exceedence Correlations

This table presents the statistics and *p*-values from a model-free symmetry test proposed by Hong et al. (2007) to examine whether the exceedance correlations between low portfolio (i.e. BETA1) and high portfolio (i.e. BETA5) are asymmetric. We report *p*-values in [·]. The *J* statistics for testing the null hypothesis of symmetric correlation, $\rho^+(c) = \rho^-(c)$, is defined as

$$J_\rho = T \left(\hat{\rho}^+ - \hat{\rho}^- \right)' \hat{\Omega}^{-1} \left(\hat{\rho}^+ - \hat{\rho}^- \right)$$

where $\hat{\Omega} = \sum_{l=1-T}^{T-1} k(l/p) \hat{\gamma}_l$ and *k* is a kernel function that assigns a suitable weight to each lag of order l, and p is the smoothing parameter or lag truncation order (see Hong et al. (2007) for more details).

Panel A: US market			Panel B: UK market		
BETA1&5	COSK1&5	COKT1&5	BETA1&5	COSK1&5	COKT1&5
48.471	40.246	44.363	56.249	38.655	46.367
[0.06]	[0.25]	[0.13]	[0.01]	[0.31]	[0.09]

Table 6: Estimating Tail Dependence Using Parametric Rotated Gumbel and Student's *t* Copula.

This table reports the coefficients of lower tail dependence (LTD) and upper tail dependence (UTD) and the difference between them for all the portfolios pairs. The estimations are calculated by the parametric approach in McNeil et al. (2005). λ_L^G and λ_U^G denote the lower and upper tail dependence coefficients estimated by rotated Gumbel copula and λ_L^T and λ_U^T denote the lower and upper tail dependence coefficients estimated by *t* copula. The *p*-values of testing $\lambda_L = \lambda_U$ are computed by a bootstrapping with 500 replications and reported in $[\cdot]$.

	LTD		UTD		Difference	Difference
Portfolio	λ_L^G	λ_L^T	λ_U^G	λ_U^T	$\lambda_L^G - \lambda_U^G$	$\lambda_L^T - \lambda_U^T$
US BETA1&5	0.256	0.171	0.099	0.018	0.157 [0.00]	0.153 [0.02]
US COSK1&5	0.315	0.200	0.192	0.153	0.123 [0.01]	0.047 [0.53]
US COKT1&5	0.306	0.153	0.216	0.103	0.090 [0.12]	0.050 [0.13]
UK BETA1&5	0.165	0.104	0.024	0.018	0.141 [0.00]	0.086 [0.00]
UK COSK1&5	0.297	0.203	0.095	0.062	0.202 [0.00]	0.141 [0.01]
UK COKT1&5	0.209	0.137	0.088	0.052	0.121 [0.00]	0.085 [0.21]

Table 7: Backtesting of VaR: Empirical Coverage Probability

This table reports ECP for each HML portfolio and VaR model. *Bias* summarises the average deviation of 12 portfolios from the nominal coverage probability, 1%, for each VaR model, and *RMSE* (Root Mean Square Error) summarises the fluctuation of the deviation across 12 portfolios for each VaR model,

$$Bias = \frac{1}{12}\sum_{p=1}^{12}\left(ECP_p - 1\%\right), \quad RMSE = \sqrt{\frac{1}{12}\sum_{p=1}^{12}\left(ECP_p - 1\%\right)^2}.$$

For the UK portfolios, we estimate the VaR models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent VaR for 18 Dec. 2000. We conduct rolling forecasting by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts. D, (P) and (S) denote "Dynamic", "Parametric" and "Semiparametric", respectively.

Portfolio	Student's t Copula			Skewed t Copula			Univariate Models	Multivariate GARCH		
	Static	D (P)	D(S)	Static	D (P)	D(S)	FHS	BEKK	CCC	DCC
HML (Beta,L;US)	0.89%	0.93%	0.86%	0.86%	0.93%	0.86%	1.09%	1.39%	1.23%	1.46%
HML (Cosk,L;US)	1.09%	0.86%	1.06%	1.03%	0.80%	1.03%	1.56%	1.66%	1.49%	1.36%
HML (Cokt,L;US)	1.03%	0.86%	0.89%	0.99%	0.86%	0.86%	1.36%	1.29%	1.36%	1.19%
HML (Beta,S;US)	0.56%	1.19%	1.19%	0.56%	1.16%	1.13%	1.26%	1.36%	1.19%	1.39%
HML (Cosk,S;US)	1.26%	0.96%	0.99%	1.19%	0.96%	0.93%	1.23%	1.52%	1.49%	1.42%
HML (Cokt,S;US)	0.96%	0.86%	0.86%	1.15%	0.96%	1.03%	1.29%	1.39%	1.36%	1.29%
HML (Beta,L;UK)	1.19%	0.92%	0.92%	0.93%	0.92%	0.92%	1.39%	1.06%	1.26%	1.13%
HML (Cosk,L;UK)	1.19%	1.06%	0.92%	1.09%	1.02%	0.96%	1.16%	1.26%	1.06%	1.19%
HML (Cokt,L;UK)	0.99%	0.86%	0.99%	0.96%	0.92%	0.99%	1.16%	1.16%	1.03%	1.19%
HML (Beta,S;UK)	1.02%	1.12%	0.92%	0.99%	1.12%	0.89%	1.52%	1.39%	1.52%	1.06%
HML (Cosk,S;UK)	1.81%	1.19%	1.25%	1.68%	1.12%	1.22%	1.33%	1.42%	0.99%	1.26%
HML (Cokt,S;UK)	1.22%	0.82%	0.96%	1.12%	0.82%	0.92%	1.33%	1.42%	1.13%	1.36%
Bias	0.10%	-0.03%	-0.02%	0.05%	-0.03%	-0.02%	0.31%	0.36%	0.26%	0.28%
RMSE	0.31%	0.14%	0.13%	0.26%	0.12%	0.11%	0.34%	0.39%	0.32%	0.30%

Table 8: Backtesting of VaR: Basel Penalty Zone

This table reports BPZ for each HML portfolio and counts the number of portfolios for each zone. BPZ counts the number of failures of the 99 percent VaR in the previous 250 VaR forecasts. Up to four failures, on average, the portfolio falls into the range of a "Green" zone. If the failures are five or more, the portfolio falls into a "Yellow" (5–9) or "Red" (10+) zone. The VaR model of which BPZ is "Green" zone is preferred. For the UK portfolios, we estimate the VaR models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent VaR for 18 Dec. 2000. We conduct rolling forecasting by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts. D, (P), and (S) denote "Dynamic", "Parametric", and "Semiparametric", respectively.

Portfolio	Student's t Copula			Skewed t Copula			Univariate Model	Multivariate GARCH		
	Static	D(P)	D(S)	Static	D(P)	D(S)	FHS	BEKK	CCC	DCC
HML(Beta,L;US)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Cosk,L;US)	Green	Green	Green	Green	Green	Green	Green	Yellow	Green	Green
HML(Cokt,L;US)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Beta,S;US)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Cosk,S;US)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Cokt,S;US)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Beta,L;UK)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Cosk,L;UK)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Cokt,L;UK)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Beta,S;UK)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
HML(Cosk,S;UK)	Yellow	Green	Green	Yellow	Green	Green	Green	Green	Green	Green
HML(Cokt,S;UK)	Green	Green	Green	Green	Green	Green	Green	Green	Green	Green
Green/Yellow/Red	11/1/0	12/0/0	12/0/0	11/1/0	12/0/0	12/0/0	12/0/0	11/1/0	12/0/0	12/0/0

Table 9: Bactesting of VaR: Conditional Coverage Test

This table presents the CC results. The CC test uses the LR statistic and it follows the Chi-squared distribution with two degrees-of-freedom under the null hypothesis. For the UK portfolios, we estimate the VaR models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent VaR for 18 Dec. 2000. We conduct rolling forecasting by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts. * indicates that the VaR model is rejected at the 5% significance level. D, (P), and (S) denote "Dynamic", "Parametric", and "Semiparametric", respectively.

Portfolio	Student's t Copula			Skewed t Copula			Univariate Model	Multivariate GARCH		
	Static	D (P)	D (S)	Static	D (P)	D (S)	FHS	BEKK	CCC	DCC
HML (Beta,L;US)	0.84	0.69	1.06	1.06	2.79	3.44	0.99	4.43	1.96	6.91*
HML (Cosk,L;US)	13.07*	6.07*	8.33*	13.83*	7.43*	8.60*	8.20*	12.23*	8.19*	6.83*
HML (Cokt,L;US)	8.60*	11.25*	5.52	8.96*	6.07*	6.07*	3.85	2.81	9.05*	1.67
HML (Beta,S;US)	7.09*	1.94	4.22	8.28*	1.56	4.01	2.34	3.82	1.94	4.43
HML (Cosk,S;US)	8.22*	1.20	1.05	7.97*	1.20	1.42	1.96	7.35*	6.54*	6.11*
HML (Cokt,S;US)	0.61	1.06	1.06	0.69	1.37	1.06	3.41	5.36	5.86	2.78
HML (Beta,L;UK)	1.88	0.87	1.10	1.51	0.71	0.71	4.31	0.78	2.78	1.15
HML (Cosk,L;UK)	4.18	0.78	0.71	3.99	0.66	0.62	4.05	4.64	0.97	4.18
HML (Cokt,L;UK)	1.06	1.10	0.60	1.22	0.71	0.74	1.34	1.51	0.66	1.88
HML (Beta,S;UK)	0.66	1.21	0.71	0.60	1.21	0.87	11.46*	6.26*	11.46*	0.78
HML (Cosk,S;UK)	16.35*	1.88	2.78	11.85*	1.51	2.30	3.18	4.96	4.43	2.78
HML (Cokt,S;UK)	4.38	1.42	0.62	3.99	1.42	0.71	3.18	5.98	1.21	4.55
Number of Rejection	5	2	1	5	2	2	3	3	4	3

Table 10: Backtesting of VaR: Dynamic Quantile Test

This table presents the DQ test results. The DQ test uses the Wald statistic and it follows the Chi-squared distribution with 6 degrees-of-freedom under the null hypothesis (see Eq, (29)). For the UK portfolios, we estimate the VaR models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent VaR for 18 Dec. 2000. We conduct rolling forecasting by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts. * indicates that the VaR model is rejected at the 5% significance level. D, (P) and (S) denote "Dynamic", "Parametric" and "Semiparametric", respectively.

Portfolio	Student's t Copula			Skewed t Copula			Univariate Model	Multivariate GARCH		
	Static	D (P)	D (S)	Static	D (P)	D (S)	FHS	BEKK	CCC	DCC
HML (Beta,L;US)	4.93	13.32*	5.49	5.49	8.69	9.75	16.73*	44.91*	41.50*	34.03*
HML (Cosk,L;US)	65.11*	13.69*	33.83*	58.64*	15.82*	35.42*	12.25*	27.26*	36.14*	19.74*
HML (Cokt,L;US)	50.07*	36.16*	14.17*	52.44*	17.75*	15.19*	6.72	5.25	20.58*	3.64
HML (Beta,S;US)	6.90	3.21	10.36	7.84	2.90	10.84	4.55	5.61	3.23	6.17
HML (Cosk,S;US)	21.01*	4.60	2.71	22.31*	4.60	3.18	4.14	9.14	11.32	7.67
HML (Cokt,S;US)	1.20	3.85	1.63	1.26	4.37	1.63	4.69	10.77	15.33*	4.86
HML (Beta,L;UK)	4.00	3.21	1.42	3.79	2.92	1.18	24.63*	2.59	10.12	3.64
HML (Cosk,L;UK)	27.31*	2.54	11.84*	31.43*	2.50	11.19*	10.67	15.31*	11.93*	11.87*
HML (Cokt,L;UK)	12.56*	1.42	2.54	13.29*	1.18	2.54	3.18	2.72	3.61	3.24
HML (Beta,S;UK)	3.78	3.65	4.71	4.00	3.65	5.20	27.39*	10.54	21.48*	3.65
HML (Cosk,S;UK)	24.04*	3.24	4.53	15.90*	2.72	3.85	5.71*	7.88*	12.44*	9.86
HML (Cokt,S;UK)	10.26	1.64	1.17	9.69	1.64	1.18	5.57*	38.86*	17.98*	25.44*
Number of Rejection	6	3	3	6	2	3	4	4	8	4

Table 11: Backtesing of ES: Mean Absolute Error

This table reports the mean absolute error (MAE) for each HML portfolio and ES model (see Eq, 30). For the UK portfolios, we forecast the ES models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent ES for 18 Dec. 2000. We conduct rolling forecasting by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts. The lowest MAE for each portfolio is given in bold. The average MAE and corresponding ranking are reported at the bottom of this table. D, (P), and (S) denote "Dynamic", "Parametric", and "Semiparametric", respectively.

Portfolio	Student's t Copula		Skewed t Copula		Univariate Model	Multivariate Models		
	D (P)	D (S)	D (P)	D (S)	FHS	BEKK	CCC	DCC
HML (Beta,L;US)	0.0064	0.0073	**0.0061**	0.0068	0.0100	0.0138	0.0093	0.0108
HML (Cosk,L;US)	0.0036	0.0042	0.0037	0.0047	0.0055	0.0036	0.0032	**0.0030**
HML (Cokt,L;US)	**0.0052**	0.0058	0.0078	0.0084	0.0081	0.0087	0.0067	0.0056
HML (Beta,S;US)	0.0170	0.0172	0.0169	**0.0054**	0.0112	0.0090	0.0137	0.0149
HML (Cosk,S;US)	0.0118	**0.0029**	0.0103	0.0112	0.0093	0.0113	0.0096	0.0094
HML (Cokt,S;US)	0.0105	0.0102	0.0089	0.0099	0.0069	0.0066	**0.0049**	0.0060
HML (Beta,L;UK)	0.0087	0.0071	0.0084	**0.0070**	0.0127	0.0105	0.0099	0.0073
HML (Cosk,L;UK)	0.0043	**0.0043**	0.0045	0.0048	0.0049	0.0055	0.0044	0.0054
HML (Cokt,L;UK)	**0.0045**	0.0072	0.0049	0.0071	0.0108	0.0108	0.0067	0.0096
HML (Beta,S;UK)	0.0081	0.0062	0.0079	**0.0058**	0.0136	0.0137	0.0122	0.0099
HML (Cosk,S;UK)	0.0077	0.0101	**0.0074**	0.0100	0.0139	0.0174	0.0090	0.0125
HML (Cokt,S;UK)	0.0049	0.0050	0.0047	**0.0046**	0.0064	0.0074	0.0063	0.0067
Average MAE	0.0077	0.0073	0.0076	**0.0071**	0.0094	0.0099	0.0080	0.0084
Ranking	4	2	3	**1**	7	8	5	6

Table 12: Backtesting of ES: Mean Squared Error

This table reports the mean squared error (MSE) for each HML portfolio and ES model (see Eq, 31). For the UK portfolios, we forecast the ES models using 250 business days over the period 4 Jan. 2000 - 15 Dec. 2000, and compute the one-day-ahead forecast of the 99 percent ES for 18 Dec. 2000. We conduct rolling forecasting by moving forward a day at a time and end with the forecast for 31 Dec. 2012. This generates 3,033 out-of-sample daily forecasts. Next we repeat the same process for the US portfolios. It starts with the forecast for 18 Dec. 2000 and ends with the forecast for 31 Dec. 2012. This generates 3,018 out-of-sample daily forecasts. The lowest MSE for each portfolio is given in bold. The average MSE and corresponding ranking are reported at the bottom of this table. D, (P), and (S) denote "Dynamic", "Parametric", and "Semiparametric", respectively.

Portfolio	Student's t Copula		Skewed t Copula		Univariate Model	Multivariate Models		
	D (P)	D (S)	D (P)	D (S)	FHS	BEKK	CCC	DCC
HML (Beta,L;US)	0.0145	0.0178	**0.0140**	0.0168	0.0237	0.0422	0.0172	0.0287
HML (Cosk,L;US)	0.0054	0.0054	0.0051	0.0053	0.0083	0.0022	0.0020	**0.0016**
HML (Cokt,L;US)	**0.0116**	0.0130	0.0161	0.0176	0.0234	0.0220	0.0179	0.0169
HML (Beta,S;US)	0.0499	0.0514	0.0490	0.0253	0.0286	**0.0237**	0.0599	0.0497
HML (Cosk,S;US)	0.0289	**0.0127**	0.0242	0.0282	0.0184	0.0218	0.0190	0.0155
HML (Cokt,S;US)	0.0215	0.0237	0.0176	0.0219	0.0089	0.0074	**0.0053**	0.0069
HML (Beta,L;UK)	0.0192	0.0121	0.0166	**0.0105**	0.0257	0.0305	0.0213	0.0200
HML (Cosk,L;UK)	**0.0028**	0.0033	0.0031	0.0045	0.0042	0.0050	0.0034	0.0044
HML (Cokt,L;UK)	**0.0087**	0.0178	0.0102	0.0148	0.0550	0.0522	0.0122	0.0330
HML (Beta,S;UK)	0.0204	0.0122	0.0185	**0.0119**	0.0346	0.0321	0.0180	0.0155
HML (Cosk,S;UK)	0.0152	0.0215	**0.0149**	0.0214	0.0394	0.0580	0.0223	0.0351
HML (Cokt,S;UK)	0.0047	0.0050	0.0047	**0.0046**	0.0063	0.0077	0.0065	0.0068
Average MSE	0.0169	0.0163	0.0162	**0.0152**	0.0230	0.0254	0.0171	0.0195
Ranking	4	3	2	**1**	7	8	5	6