Sample mathematical exercises for engineering

Professor John H. Davies

2012 July 5

Here are a few exercises to help you to revise your mathematics before you come to university. All of the techniques should have been covered in Higher Mathematics but the questions are dressed up in the language of engineering, which may make them a lot more challenging! However, they will provide a good introduction to studying at university. The examples are from electronics because you may have encountered some of the material in Higher Physics at school. You will see plenty of applications to your discipline when you arrive here.

Please don't get the idea that the curriculum is dominated by mathematics: it is definitely engineering. However, professional engineers use mathematics as a tool to help them solve problems, which means that you must be able to do basic calculations quickly and reliably – almost automatically. You won't be able to concentrate on the engineering if it takes you half an hour to solve a quadratic equation, for instance. We will help you to develop this skill during your university course. It will be a challenge but very profitable for your future career.

I have included numerical answers to some of the questions. Full solutions are available on the School of Engineering's web site at www.gla.ac.uk/engineering/infopacks. Please don't look at these until you have tried the exercises yourself.

1. Figure 1(a) shows a widely used circuit called a *potential divider* formed by two resistors. The input and output voltages are given in terms of the resistances by

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} V_{\text{in}}.$$

   Use this to find the unknown quantities in figures 1(b)–(e). [0.5 V, 500 Ω, 12 V, 16 kΩ.]

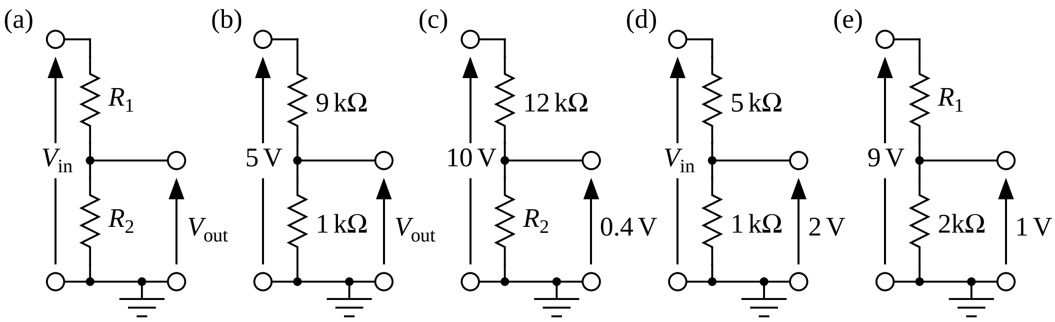

Figure 1: A selection of potential dividers.

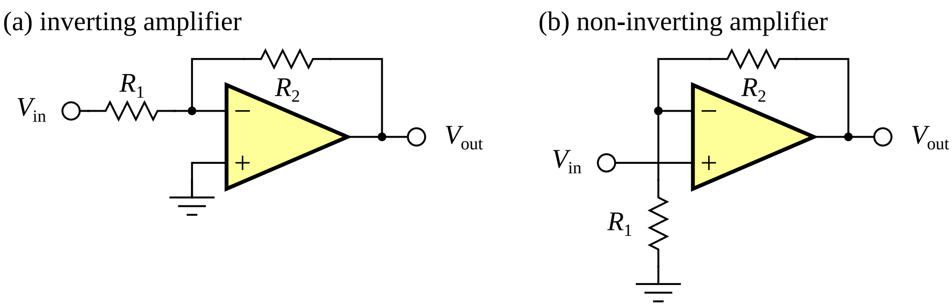

Figure 2: The classic inverting and non-inverting amplifier circuits.

2. A remote control draws 10 mA while it is being used and 10 μA when it is idle. (Make sure that you know the powers of 10 for the prefixes in mA and μA. How about kA and nA?) What is the average current drawn, assuming that it is used for 5 minutes per day? Which is more significant, the current drawn when it is operating or idle? [45 μA]

   The control's batteries are rated at 100 mAh. This means that the product of the current in mA and lifetime in hours is 100. For example, they will provide 100 mA for 1 hour or 0.1 mA for 1000 hours. How long will they last in the remote control? [About 3 months]

3. Figure 2 shows two classic circuits based on an operational amplifier – the component shown by the triangular symbol. They act as (a) inverting and (b) non-inverting amplifiers. You may have come across these circuits before but don't worry if you haven't because you only have to do the mathematics here, not the electronics! Standard circuit analysis leads to these equations for each circuit.

$$\text{(a)} \quad \frac{V_{\text{in}}}{R_1} + \frac{V_{\text{out}}}{R_2} = 0, \qquad \text{(b)} \quad \frac{-V_{\text{in}}}{R_1} + \frac{V_{\text{out}} - V_{\text{in}}}{R_2} = 0.$$

   Rearrange each of these equations to find expressions for $V_{\text{out}}/V_{\text{in}}$, called the voltage gain of the circuit. Watch the signs!

   What would the gain be in each case if $R_1 = 1\,\text{k}\Omega$ and $R_2 = 10\,\text{k}\Omega$? [−10, 11]

4. When designing a circuit, a resistance R is required to obey the equation

$$R^2 - 2R_1R + R_1^2 - R_2^2 = 0$$

   where the values of R_1 and R_2 are known. What sort of equation for R is this? Find an algebraic expression for the possible values of R.

   If $R_1 = 10\,\Omega$ and $R_2 = 30\,\Omega$, find the value of R. Why is there only one? [40 Ω.]

5. A signal with voltage $V(t)$ is processed through two circuits. The first adds a constant 5 V and the second squares its input. Find an expression for the final output $V_{\text{out}}(t)$. What is the output when the input is $V = 1\,\text{V}$? [36 V]

Sample mathematical exercises for engineering
Solutions

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Here are a few exercises to help you to revise your mathematics before you come to university. All of the techniques should have been covered in Higher Mathematics but the questions are dressed up in the language of engineering, which may make them a lot more challenging! However, they will provide a good introduction to studying at university. The examples are from electronics because you may have encountered some of the material in Higher Physics at school. You will see plenty of applications to your discipline when you arrive here.

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1. Figure 1(a) shows a widely used circuit called a *potential divider* formed by two resistors. The input and output voltages are given in terms of the resistances by

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} V_{\text{in}}.$$

Use this to find the unknown quantities in figures 1(b)–(e). [0.5 V, 500 Ω, 12 V, 16 kΩ.]

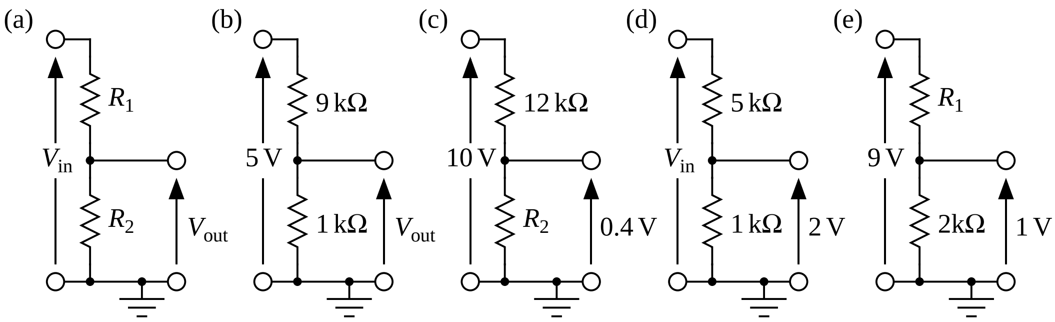

Figure 1: A selection of potential dividers.

Solution. We just have to rearrange the given equation in the four possible ways to get the different unknown quantities. This isn't necessary for (b) because the equation gives V_{out} directly. To get the others it is a good idea to multiply throughout by $(R_1 + R_2)$, which gets rid of the fraction:

$$R_1 V_{\text{out}} + R_2 V_{\text{out}} = R_2 V_{\text{in}}.$$

I'll start from here to solve (c)–(e).

For (c) we need to find R_2. Take the term $R_2 V_{\text{out}}$ from the left to the right-hand side, changing the sign.

$$R_1 V_{\text{out}} = R_2 V_{\text{in}} - R_2 V_{\text{out}}.$$

Now take out the common factor of R_2.

$$R_1 V_{\text{out}} = R_2 (V_{\text{in}} - V_{\text{out}}).$$

Finally, divide both sides by $(V_{\text{in}} - V_{\text{out}})$.

$$\frac{R_1 V_{\text{out}}}{V_{\text{in}} - V_{\text{out}}} = R_2.$$

For (d) we need to find V_{in}. Take out the common factor of V_{out} on the left.

$$V_{\text{out}}(R_1 + R_2) = R_2 V_{\text{in}}.$$

Divide by R_2 and we are there.

$$\frac{V_{\text{out}}(R_1 + R_2)}{R_2} = V_{\text{in}}.$$

Finally, (e) needs R_1. This is almost the same as finding R_2 down to the line

$$R_1 V_{\text{out}} = R_2 (V_{\text{in}} - V_{\text{out}}).$$

This time, divide both sides by V_{out}.

$$R_1 = \frac{R_2 (V_{\text{in}} - V_{\text{out}})}{V_{\text{out}}}.$$

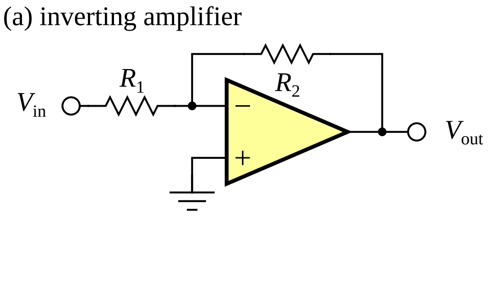

Figure 2: The classic inverting and non-inverting amplifier circuits.

2. A remote control draws 10 mA while it is being used and 10 µA when it is idle. (Make sure that you know the powers of 10 for the prefixes in mA and µA. How about kA and nA?) What is the average current drawn, assuming that it is used for 5 minutes per day? Which is more significant, the current drawn when it is operating or idle? [45 µA]

   The control's batteries are rated at 100 mAh. This means that the product of the current in mA and lifetime in hours is 100. For example, they will provide 100 mA for 1 hour or 0.1 mA for 1000 hours. How long will they last in the remote control? [About 3 months]

Solution. Prefixes: k = kilo = 10^3, m = milli = 10^{-3}, µ = micro = 10^{-6}, n = nano = 10^{-9}, p = pico = 10^{-12}.

The average current is given by

$$I_{\text{average}} = \frac{I_{\text{operating}} \times T_{\text{operating}} + I_{\text{idle}} \times T_{\text{idle}}}{T_{\text{operating}} + T_{\text{idle}}}.$$

This is much the same as finding the average velocity for a journey. Let's use µA for all currents and minutes for all times. There are $60 \times 24 = 1440$ minutes in a day and the remote control is used for 5 minutes so it is idle for 1435 minutes.

$$\begin{aligned} I_{\text{average}} &= \frac{10\,000 \times 5 + 10 \times 1435}{5 + 1435} \\ &= \frac{50\,000 + 14350}{1440} \\ &= 35 + 10 = 45\,\mu\text{A}. \end{aligned}$$

This shows that the average current is made up of 35 µA from the time when the remote control is operating and 10 µA when idle. Thus the operating current is more significant although the system spends most of its time idle.

3. Figure 2 shows two classic circuits based on an operational amplifier – the component shown by the triangular symbol. They act as (a) inverting and (b) non-inverting amplifiers. You may have come across these circuits before but don't worry if you haven't because you only have to do the mathematics here, not the electronics! Standard circuit analysis leads to these equations for each circuit.

$$\text{(a)} \quad \frac{V_{\text{in}}}{R_1} + \frac{V_{\text{out}}}{R_2} = 0, \qquad \text{(b)} \quad \frac{-V_{\text{in}}}{R_1} + \frac{V_{\text{out}} - V_{\text{in}}}{R_2} = 0.$$

Rearrange each of these equations to find expressions for $V_{\text{out}}/V_{\text{in}}$, called the voltage gain of the circuit. Watch the signs!

What would the gain be in each case if $R_1 = 1\,\text{k}\Omega$ and $R_2 = 10\,\text{k}\Omega$? [−10, 11]

Solution. Start with (a), the inverting amplifier. Take the term with V_{in} over to the right-hand side, changing the sign.

$$\frac{V_{\text{out}}}{R_2} = -\frac{V_{\text{in}}}{R_1}.$$

Now multiply both sides by R_2.

$$V_{\text{out}} = -\frac{R_2}{R_1}V_{\text{in}}\,.$$

Finally, divide both sides by V_{in} to get the voltage gain.

$$\text{voltage gain} = \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_2}{R_1}.$$

This is negative, which is why it is called an inverting amplifier. The gain is -10 with the values given. This means that an input of 1 V gives an output of $-10\,\text{V}$, for instance.

Now the non-inverting case. Multiply throughout by R_1R_2 (the lowest common denominator) to eliminate the fractions.

$$-R_2V_{\text{in}} + R_1(V_{\text{out}} - V_{\text{in}}) = 0.$$

Expand the brackets.

$$-R_2V_{\text{in}} + R_1V_{\text{out}} - R_1V_{\text{in}} = 0.$$

Collect the two terms with V_{in}.

$$-R_2V_{\text{in}} - R_1V_{\text{in}} + R_1V_{\text{out}} = 0.$$

Take out the common factor of $-V_{\text{in}}$.

$$-V_{\text{in}}(R_2 + R_1) + R_1V_{\text{out}} = 0.$$

Watch the sign here! Both terms are negative, so the sign in the bracket is positive because of the minus sign in front. Take this negative term on to the other side. This makes it positive.

$$R_1V_{\text{out}} = V_{\text{in}}(R_2 + R_1).$$

Now divide by R_1 and by V_{in} to get the voltage gain.

$$\text{voltage gain} = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_2 + R_1}{R_1} = 1 + \frac{R_2}{R_1}.$$

This time the gain is positive. The numerical value is 11.

4. When designing a circuit, a resistance R is required to obey the equation

$$R^2 - 2R_1R + R_1^2 - R_2^2 = 0$$

where the values of R_1 and R_2 are known. What sort of equation for R is this? Find an algebraic expression for the possible values of R.

If $R_1 = 10\,\Omega$ and $R_2 = 30\,\Omega$, find the value of R. Why is there only one? [40 Ω.]

Solution. There are several ways of solving the quadratic equation for R. I think that the best way is to spot that $R^2 - 2R_1R + R_1^2 = (R - R_1)^2$. This is the standard result for the expansion of $(A-B)^2$. Then

$$(R - R_1)^2 - R_2^2 = 0.$$

Take R_2^2 over to the right-hand side.

$$(R - R_1)^2 = R_2^2\,.$$

Now take the square root of both sides. The only tricky point is the sign. Remember that the square roots of A^2 are $\pm A$. Here we should put a $\pm$ sign on both side but there are really only two possibilities: either the two sides have the same sign or the opposite signs.

$$R - R_1 = \pm R_2\,.$$

Finally, take R_1 across to get the solutions.

$$R = R_1 \pm R_2\,.$$

There are plenty of other ways of solving this. Completing the square gives exactly the same working. You can also use the standard formula for the roots of $ax^2 + bx + c = 0$. Here $x = R$, $a = 1$, $b = -2R_1$ and $c = R_1^2 - R_2^2$. The standard formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Substitute our expressions into this.

$$R = \frac{2R_1 \pm \sqrt{(-2R_1)^2 - 4 \times 1 \times (R_1^2 - R_2^2)}}{2 \times 1}.$$

Simplify this.

$$\begin{aligned}
R &= \frac{2R_1 \pm \sqrt{4R_1^2 - 4(R_1^2 - R_2^2)}}{2} \\
&= \frac{2R_1 \pm \sqrt{4R_1^2 - 4R_1^2 + 4R_2^2}}{2} \\
&= \frac{2R_1 \pm \sqrt{4R_2^2}}{2} \\
&= \frac{2R_1 \pm 2R_2}{2} = R_1 \pm R_2\,.
\end{aligned}$$

It is a lot less effort to spot the pattern $(A-B)^2$!

The numerical values give $10 \pm 30\,\Omega = 40\,\Omega$ or $-20\,\Omega$. The second result is negative and you can't buy negative resistors!

5. A signal with voltage $V(t)$ is processed through two circuits. The first adds a constant 5 V and the second squares its input. Find an expression for the final output $V_{\text{out}}(t)$. What is the output when the input is $V = 1\,\text{V}$? [36 V]

Solution. The output is $V_{\text{out}} = (V+5)^2$ because the addition is done first, then the squaring. (It would be $V_{\text{out}} = V^2 + 5$ if the operations were done in the reverse order.)