A Note on the Baxter-King Filter*

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Abstract

Recently, Baxter and King (1995) developed a bandpass filter which overcomes to some extent the well known drawbacks of the Hodrick-Prescott filter. In this paper, the circumstances under which the Baxter-King filter is preferable are identified, and a modification is presented which takes into account spurious side lobes generated by this method.

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1 Introduction

Following Baxter and King (1995), p. 3, a useful detrending method should fulfil 6 requirements: (i) The filter should extract a cyclical component within a specified range of periodicities, and leave the characteristics of this component undistorted; (ii) there should be no phase shift, i.e. the filter should not change the timing of the turning points in the series under analysis; (iii) The filter should be an optimal approximation to the "ideal" filter; (iv) the filter should have trend-reducing properties; (v) the filter should yield business cycle components unrelated to the length of the observation period; and (vi) the method must be operational. They proposed a new digital filter, the derivation of which is explicitly based on these requirements.[1]

Recently it was demonstrated by Cogley and Nason (1995),[2] that the widely used Hodrick-Prescott filter (Hodrick and Prescott 1980) is likely to generate spurious cyclical structure at business cycle frequencies if applied to difference stationary series. Similar points can be made with respect to the Baxter-King Filter (Guay and St-Amant 1997), and to moving-average filters in general (Osborn 1995). Moreover, there is the danger of spurious correlation between Hodrick-Prescott filtered series (Harvey and Jaeger 1993).

The crucial problem in filtering of economic time series is that in most cases, we cannot reliably distinguish between types of stationarity, and hence are not able to choose the "correct" filtering method. A possible pragmatic solution is to asses the robustness of the results by comparing the outcome of a number of filtering methods. It will be shown that the Baxter-King filter

[1] An application can be found in Baxter (1994), and, more recently, in Stock and Watson (1998).

[2] See also King and Rebelo (1993) and Harvey and Jaeger (1993).

has characteristics which make it an attractive alternative to the Hodrick-Prescott filter. However, despite its advantages, the filter can produce spurious cycles. A modification will be presented which largely overcomes this undesirable effect.

2 The Baxter-King Filter

2.1 Derivation of the Filter

Based on the 6 requirements listed in the introduction, Baxter and King (1995) construct a bandpass filter of finite order K which is optimal in the sense that it is an approximate bandpass filter with trend-reducing properties and symmetric weights, which ensure that there is no phase shift in the filter output. In time domain, the impact of the filter on an input series y_t is given by the finite moving average[3] $\tilde{y}_t = \sum_{j=-K}^{K} a_j L^j y_t$. In frequency domain, the filter is characterised by its Fourier transform $\alpha(\omega)$.[4] To find the weights a_j, one solves the minimisation problem

$$\min_{a_j} Q = \int_{-\pi}^{\pi} |\beta(\omega) - \alpha(\omega)|^2 \, d\omega, \text{ s.t. } \alpha(0) = 0; \qquad (1)$$

where $|\beta(\omega)|$ is the "ideal" filter gain with cut-off frequencies ω_1 and ω_2.[5] The constraint ensures that the resulting filter has trend reducing properties (requirement iv).[6]

[3] L denotes the backshift operator ($L^n y_t = y_{t-n}$).

[4] See e.g. Koopmans (1974), p. 165 ff.

[5] The gain of a filter measures the change in the amplitude of the input components if transformed by the filter. The ideal bandpass filter gain $|\beta(\omega)|$ takes the value 1 in the frequency interval $[\omega_1, \omega_2]$ and 0 outside this interval.

[6] In order to remove the component with the frequency $\omega = 0$ from the series, the filter weights must sum to zero (Baxter and King 1995, p. 26).

Solving the minimisation problem leads to the following results:[7]

$$
\begin{aligned}
a_j &= b_j + \theta;\ \ j = 0, \pm 1, \ldots, \pm K; \\
b_j &= \begin{cases} \dfrac{\omega_2 - \omega_1}{\pi} & \text{if } j = 0 \\ \frac{1}{\pi j}\left(\sin \omega_2 j - \sin \omega_1 j\right) & \text{if } j = \pm 1, \pm 2, \ldots \end{cases}; \\
\theta &= \frac{-\sum_{j=-K}^{K} b_j}{2K+1};
\end{aligned}
\tag{2}
$$

Based on the experience with US business cycle stylised facts, Baxter and King (1995) propose for quarterly data the following set of parameters: $K = 12$, $\omega_1 = 2\pi\frac{1}{32}$, and $\omega_2 = 2\pi\frac{1}{6}$ or $2\pi\frac{1}{2}$; for annual data, they suggest $K = 3$, $\omega_1 = 2\pi\frac{1}{8}$, and $\omega_2 = \pi$. But of course these suggestions depend on the length of the observation period and the frequency band one is interested in. An example for the power transfer function (*ptf*) of the Baxter-King filter is displayed in Figure 1 (quarterly filter).[8]

[Figure 1 about here.]

2.2 Smoothing the Baxter-King Filter

As seen in Figure 1, the *ptf* of the Baxter-King filter oscillates around the *ptf* of the ideal filter. This undesirable property, which applies to all digital filters of finite length, is known as *Gibb's phenomenon.* It is due to the fact that the ideal filter, which is a discontinuous function of ω, is approximated by a finite Fourier series. This approximation leads to the side lobes observed in Figure 1 (Priestley 1981, p. 561-3, Koopmans 1974, p. 187-9). These side

[7] The filter is symmetric (i.e. $a_j = a_{-j}$), and therefore does not impose a phase shift on the output (requirement (ii)).

[8] The *ptf* of a filter is its squared gain, and allows to judge the filter impact on the spectrum of the input series.

lobes clearly violate requirement (i) in the list of Baxter and King (1995). While the relative contribution of some components for the overall variance of the series is exaggerated (i.e. they are multiplied by a gain greater than 1), other components are suppressed (i.e. multiplied by a gain less than 1).

An obvious solution to this problem is to increase the filter length. But since we are restricted by the limited availability of economic data, there is not much to be gained from changing the length of the filter. A more appropriate solution is to apply spectral windows.[9] As an example, consider the so called *Lanczos's σ factors* (Bloomfield 1976, p. 129-137). We replace the truncated weights of the optimal filter b_j in equation (2) by the modified weights $b_j^\star$, which are obtained from

$$b_j^\star = b_j \frac{\sin\left((2\pi j)/(2K+1)\right)}{(2\pi j)/(2K+1)}; |j| = 1, \ldots, K. \qquad (3)$$

After this step, the modified filter weights of the Baxter-King filter $a_j^\star$ can be calculated as demonstrated above. This procedure ensures that the modified filter still fulfils requirements (ii) and (iv) (no phase shift; stationary output series). The corresponding *ptf* for the modified Baxter-King filter is displayed in Figure 2.

[Figure 2 about here.]

The modified filter in Figure 2 has a *ptf* without pronounced sidelobes. On the other hand, the modified filter is no longer optimal in a least squares sense: requirements (i) and (iii) seem to be mutually exclusive. The following section demonstrates under which circumstances the modified filter is preferable.

[9]For an overview of widely used window functions in the design of digital filters see Mitra and Kaiser (1993), p. 180-189, and Antoniou (1993), p. 282-293.

2.3 A Comparison of the Hodrick-Prescott and Baxter-King Filter

Assume that the time series under analysis is a random walk, i.e. $(1-L)y_t = u_t;\ u_t \sim N(0,\sigma^2)$, and consider the resulting *ptf*. It has been shown that in this case, the Hodrick-Prescott filter creates spurious cyclical structure at the business cycle frequency range,[10] and that the same is true for the Baxter-King filter. The spectrum of a Baxter-King filtered random walk is given by[11]

$$f(\omega) = \left|\alpha^{I(1)}(\omega)\right|^2 \frac{\sigma^2}{2\pi},\ \omega \in [-\pi,\pi],$$
$$\text{with } \left|\alpha^{I(1)}(\omega)\right|^2 = \left(2\sin\left(\frac{\omega}{2}\right)\sum_{j=1}^{K} a_j \sum_{h=-(j-1)}^{j-1} (j-|h|)\cos(\omega h)\right)^2. \tag{4}$$

However, the possible damage in the case of a Baxter-King filtered series is obviously less serious than in the case of the Hodrick-Prescott filter, because peak indicating a spurious cyclical component is smaller. A further reduction can be achieved by using the modification proposed above (Figure 3).

[Figure 3 about here.]

Besides the problem of spurious cycles in the univariate case, Hodrick-Prescott filtered random walks might exhibit high cross correlations. Harvey and Jaeger (1993) show this for the Hodrick-Prescott filter by computing the correlations between two Hodrick-Prescott filtered, independent random walks. If we apply the same procedure to two Baxter-King filtered random

[10] See Harvey and Jaeger (1993), King and Rebelo (1993), and Cogley and Nason (1995).

[11] The derivation of equation (4) is based on Baxter and King (1995), Appendix A.2. $\left|\alpha^{I(1)}(\omega)\right|^2$ is the *ptf* of the Baxter-King filter, if applied to a series which is integrated of order 1.

walks, we obtain the results displayed in Table 1.

[Table 1 about here.]

The table can be interpreted in the following way: since the cross correlations are asymptotically normal, a standard deviation of 0.44 in the case of the Baxter-King filter (annual, sample size $T = 25$) means that there is a probability of about 25 per cent that the cross correlations between the filtered random walks exceed 0.5 in absolute value. We see that overall, the standard deviations are relatively high, and that for annual data, the Baxter-King filters are preferable by this criterion as well. For quarterly data, Baxter-King filters exhibit larger standard errors. Even here, however, the modifed version performs better than the original.

3 Conclusion

It has been shown that the Baxter-King filter has desirable properties which make it at least a reasonable alternative to the widely used Hodrick-Prescott filter, especially if the criterion of choice is to minimise the possible "damage", i.e. the danger of identifying spurious cyclical structure. Smoothing the filter using spectral windows improves the performance of the filter with respect to this criterion.

However, in the case of an integrated input series, the Baxter-King filter will produce a distorted output series. Facing the problem that we cannot reliably distinguish between types of non-stationarity, one should avoid using a single filtering method. A more appropriate approach is to apply a set of filters based on different stationarity assumptions in order to judge the robustness of the results (Canova 1998). This approach is the subject of ongoing research on business cycle stylised facts (Reiter and Woitek 1998).

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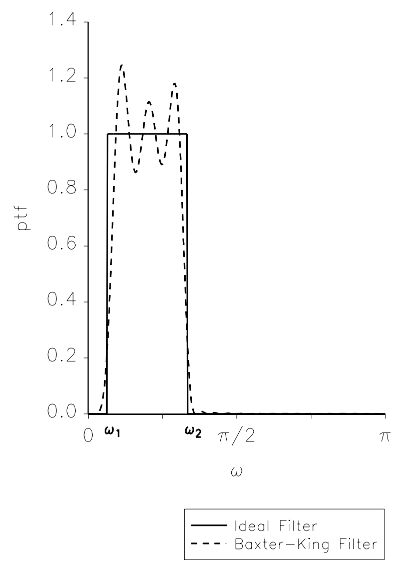
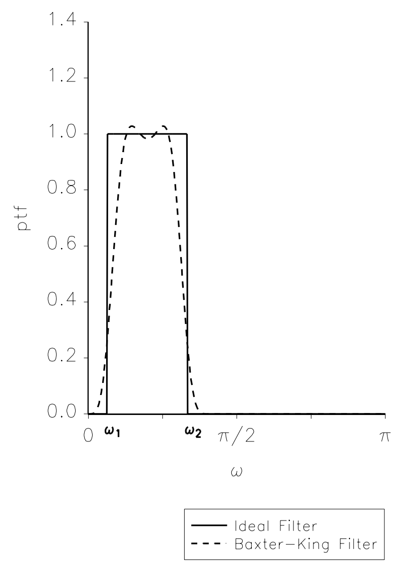
Figure 1: The Baxter-King Filter



Quarterly filter, with length $K = 20$

Figure 2: The Modified Baxter-King Filter



Quarterly filter, with length $K = 20$

Figure 3: Comparison of Spurious Cycles for the Hodrick-Prescott and the Baxter-King Filter

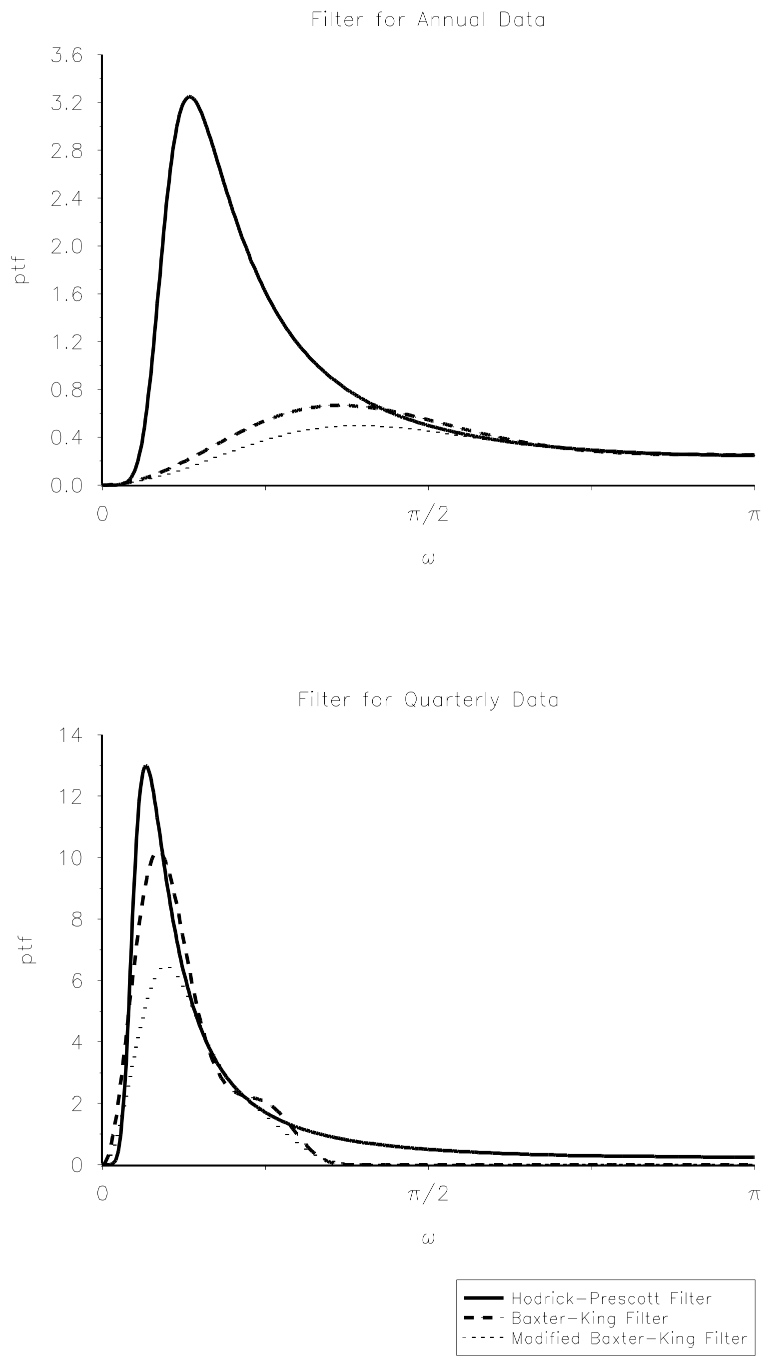


Table 1: Standard Deviations of Cross Correlations between Two Filtered Random Walks

sample size		25	50	100	200	500
Hodrick-Prescott Filter	annual	0.73	0.51	0.38	0.26	0.16
	quarterly	0.77	0.55	0.40	0.28	0.18
Baxter-King Filter	annual	0.44	0.32	0.22	0.16	0.10
	quarterly	0.87	0.61	0.44	0.32	0.20
Baxter-King Filter	annual	0.44	0.30	0.22	0.16	0.10
(mod.)	quarterly	0.83	0.59	0.42	0.30	0.18

Notes:
Smoothing parameter for the Hodrick-Prescott filter: $\mu = 100$ (annual), $\mu = 1600$ (quarterly).
The variance of the correlation between two independent random walks j and k is given by

$$\sigma^2_{\rho_{jk}} = N^{-1} \sum_{h=-\infty}^{\infty} \rho_{jj}(h)\rho_{kk}(h),$$

where N denotes the sample size. The autocorrelations $\rho_{jj}(h)$ and $\rho_{kk}(h)$ can be obtained by calculating the inverse Fourier transform of the spectrum of the filtered random walks. Following Harvey and Jaeger (1993), the infinite sum is approximated by the first 100 autocorrelations.