Fair Division with money and prices

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Abstract

We must divide a finite number of indivisible goods and cash transfers between agents with quasi-linear but otherwise arbitrary utilities over the subsets of goods. We compare two division rules with cognitively feasible and privacy preserving individual messages.

In *Sell&Buy* agents bid for the role of Seller or Buyer: with two agents the smallest bid defines the Seller who then charges any a price constrained only by her winning bid.

In *Divide&Choose* agents bid for the role of Divider, then everyone bids on the shares of the Divider's partition.

S&B dominates D&C on two counts: its guaranteed utility in the worst case rewards (resp. penalises) more subadditive (resp. superadditive) utilities; playing safe is never ambiguous and is also better placed to collect a larger share of the efficient surplus.

Key words: ex ante fairness, guarantees, safe play, Sell & Buy, Divide & Choose

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1 Introduction

The fair allocation of indivisible objects is greatly facilitated if the agents who get few good objects or many bad ones accept compensations in cash or any other transferable and divisible commodity (workload, stocks, caviar, bitcoin). Examples of this common practice include the classic rent division problem ([14], Spliddit.org), the dissolution of a partnership ([9], the Texas Shoot Out clause

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to terminate a joint venture[1]), and the NIMBY problem (the allocation of a noxious facility between several communities [17]).

The familiar assumption that utilities are quasi-linear – each agent can attach to each bundle of objects a personal "price" and switching from one bundle to another is exactly compensated by the difference in their prices – yields a versatile fair division model that the economic literature discussed with any depth only in the special case of the *assignment* problem where each of the n agents must receive at most one object (see the literature review in section 2).

We initiate the discussion of fair division rules with a finite number of indivisible *goods* (freely disposable objects) and money. Utilities are weakly increasing over subsets of goods but externalities across goods are arbitrarily complex, exactly like in the combinatorial auction problem ([10]): if we distribute the m goods in a set A, a full description of an agent's utility measured in money is a vector of dimension $2^{|A|} - 1$.

The key design constraint of such rules is the simplicity of individual messages: even with a handful of objects it is not cognitively feasible to report a vector of such large dimension. A dual argument against against eliciting a full report, even when A is small, is privacy protection: a message of dimension (much) smaller than $2^{|A|}$ is an advantage in future bargaining interactions. That a single cut of the cake reveals so little about the Divider's preferences, and even less of the Chooser's, is what makes Divide&Choose so appealing.

In the new *Sell and Buy* rule (S&B) we propose, the messages are prices. The fair division rule implemented on the Spliddit platform routinely elicits price reports which we take as evidence of the practical applicability of the price-based S&B rule.

If only two agents divide m goods they bid first to determine the roles of Seller and Buyer in the next stage. A bid is interpreted as the price the Seller can charge for all the goods; the agent with the *smallest* bid x takes that role. The Seller then chooses a price for every good so that their sum is x and the Buyer buys at those prices any subset of goods,including all goods or none. The remaining goods go the the Seller, along with the cash from the Buyer's purchase.

It is also possible to adapt the *Divide&Choose* rule (D&C) to our model with cash transfers. The first round of bids picks the Divider, this time because her bid y is the highest. After paying $\frac{1}{n}y$ to each of the other $n-1$ participants, the Divider selects an allocation in which each of the n shares is a set of goods (possibly empty) plus a cash transfer, such that all goods are distributed and the sum of the cash transfers is zero.[2]

Both rules elicit cognitively realistic, if very different, messages. Our results focus on their performances with respect to ex ante fairness and the efficiency of the "safe play" by the participants. A message is safe if it maximises the worst

[1] Both parties submit sealed bids and the party who makes the higher bid buys the company at that price.

[2] We also discuss a version of D&C with the same guarantee but eliciting more information from the agents: first each agent proposes a partition of the objects in n shares after which they all bid to select the "best" partition: see section 6.

case utility the agent clueless about the preferences and behaviour of the other – possibly adversarial – participants. More sophisticated strategies exploiting some information about other participants' behaviour or preferences are unsafe inasmuch as this information may be wrong.

That worst case utility is called the *guarantee* offered by the rule and the first step toward its ex ante fairness is that the guarantee should be the same for two participants with identical utility functions. The next step is to make the guarantee as "efficient" as possible in the sense that the sum of individual guarantees, and/or the outcome of independent safe play captures a large share of the efficient surplus.

To appreciate the difficulty of taking into account arbitrary complex externalities, consider on the contrary the case of utilities *additive* over perfectly substitutable objects and in money too: they are described by an m-dimensional vector $(u_a)_{a \in A}$ and utility for the subset S of A is $u(S) = \sum_{a \in S} u_a$. The compelling division rule is the familiar *Multi Auction* (MA): each agent i places a bid $\beta_{ia}$for each good a, the highest bidder i^* on a gets this object and pays $\frac{1}{n}\beta_{i^*a}$ to each of the $n-1$ other agents. Clearly the truthful bid u_{ia} on each a is the unique *safe* play. MA is fair ex ante: it guarantees to each agent i her *Fair Share* $\frac{1}{n}u_i(A)$ which is the *best* guarantee we can offer in the additive context. It is fair ex post: by playing safe I will not envy anyone else's allocation. And if all agents play safe the efficient allocation is selected.

MA can be used as well with our much more complex utilities. Now the marginal utility of adding good a to a subset of goods varies so there is no "truthful" bid on a. The safe bid solves a program similar to the one we shall use for the S&B rule, but delivers in all problems a very low guarantee, much below those of the S&B and D&C rules. It also fares badly in terms of efficiency and ex post fairness. See subsection 11.1 in the Appendix.

In our model it is still feasible to guarantee the utility $\frac{1}{n}u_i(A)$ to each agent i but it is a untenably coarse interpretation of ex ante fairness that fails to reward agents with subadditive utilities and to penalise those with superadditive utilities. This is the normative motivation of our analysis, captured by contrasting two extreme utilities that play a major role throughout.

Example 0. We divide $m \geq 2$ identical goods between two agents Frugal and Greedy with the following utilities

$$u_F(S) = 1 \text{ for all } S, \varnothing \neq S \subseteq A \text{ ; } u_F(\varnothing) = 0$$

$$u_G(S) = 0 \text{ for all } S, \varnothing \subseteq S \subsetneq A \text{ ; } u_G(A) = 1$$

Frugal is content with any single good – her utility is maximally sub-additive – while Greedy needs all goods to derive any utility – his utility is maximally super-additive.

We submit that it is not fair to offer ex ante the same guarantee $\frac{1}{2}$ to both agents – achieved by auctioning the bundled goods as in MA. Under the veil of ignorance where we (as impartial observer) don't know person X who will share the goods with Frugal, we should take into account that together Frugal and X can produce at least as much utility surplus – and often much more – than if X is paired with Greedy. For this reason we dismiss $\frac{1}{n}u_i(A)$ as a "repugnant" interpretation of ex ante fairness. This benchmark utility still plays a significant role in our analysis: we will call it agent i's *Familiar Share* (FS) and regard it as a compelling guarantee only for additive utilities.

The *Responsiveness* property says that we should guarantee *strictly* more than her FS to Frugal, which implies that Greedy is guaranteed strictly less than his FS. The *Positivity* property, by contrast, protects Greedy: it requires to give him *some* positive guarantee because his equal rights to the goods should amount to something regardless of his uncompromising utility.

Divide&Choose guarantees to Greedy a utility of $\frac{1}{4}$, half of his FS. As the Divider he secures $\frac{1}{2}$ by offering to pay $\frac{1}{2}$ for the bundled objects, or be paid $\frac{1}{2}$ and get no object; as the Chooser being offered two non empty piles of objects with zero transfers, he will get no surplus at all; so his safe bid of $\frac{1}{4}$ to be the Divider guarantees a benefit of $\frac{1}{4}$ whatever his role in the second stage.

By contrast Frugal can secure a utility of $\frac{3}{4}$: as the Divider she gets the full benefit 1 by proposing two non empty piles and zero transfers; as a Chooser the sum of her utility for the two shares is at least 1 so one of them at least gives her utility $\frac{1}{2}$; then a bid of $\frac{1}{4}$ guarantees $1 - \frac{1}{4}$ if she Divides, and $\frac{1}{2} + \frac{1}{4}$ if she Chooses.

The guarantees offered by Sell&Buy to Frugal and Greedy are more nuanced: they take into account that the contrast between their preferences increases with the number m of goods to divide. Specifically Greedy's guaranteed utility is $\frac{1}{m+1}$ and Frugal's is $\frac{m}{m+1}$. S&B sensibly penalises Greedy and rewards Frugal more and more than D&C as m grows.

Frugal's safe bid $\frac{m}{m+1}$ in stage 1 goes as follows. As Seller she will charge $\frac{1}{m+1}$ per good and end up with utility $\frac{m}{m+1}$ if Buyer buys everything, and at least 1 if Buyer buys at least one good. As Buyer she will faces a total price of *at most* $\frac{m}{m+1}$ so at least one good will cost $\frac{1}{m+1}$ or less and buying just that good guarantees the benefit $\frac{m}{m+1}$.

Greedy's safe bid is $\frac{m}{m+1}$ as well. As Seller charging $\frac{1}{m+1}$ per good he will get $\frac{1}{m+1}$by selling one or more goods, and 1 by selling nothing; as Buyer the whole bundle will cost him at most $\frac{m}{m+1}$.

Responsiveness and Positivity of guarantees together reveal a surprising tension between ex ante and ex post fairness, when the latter is taken to mean – as in essentially all the fair division literature – that the final allocation is Envy Free. This simple fact is already apparent when we divide the m goods between Frugal and Greedy: it is easy to check that the envy free allocations are of two types: someone eats A and pays $\frac{1}{2}$ to the other; or they each get at least one good and no transfer takes place. The former violates Responsiveness because it gives their FS utility to both agents, and the latter violates Positivity for Greedy.

our punchlines Both the Sell&Buy and Divide&Choose division rules offer new interpretations of ex ante fairness rewarding sub-additive utilities and penalising super-additive ones. The price message in S&B allows its guarantee more flexibility than the partition in D&C does in response to the shape of utilities.

Playing safe in either rule cannot extract the efficient surplus because the messages reveal much less than full utilities. But D&C is particularly bad when utilities are nearly additive or when one agent values each good much more than any of the others, two problems that the S&B avoids.

Preliminary numerical experiments suggest S&B captures more of the efficient surplus than D&C, on average if not necessarily in the worst case. This is plausible because the S&B rule processes more information than D&C.

overview of the paper After the literature review in section 2 and the basics of the model in section 3, we define guarantees and the Positivity and Responsiveness properties in section 4. There we also describe simple auctions implementing the *fixed partition* guarantees, a key ingredient of the D&C rules. Section 5 introduces two critical utility levels: the *Maxmin utility* that an agent can secure as the Divider is an upper bound on *all* guarantees; the *minMax utility* that she can secure as a Chooser against an adversarial Divider is a lower bound on all reasonable guarantees: Proposition 1.

Section 6 defines the D&C rule and computes its safe play and guarantees: Proposition 2. Section 7 does the same for the S&B rule – Proposition 3 – and gives full computations in the simple case of two agents dividing identical goods – Proposition 4.

The systematic comparison of the S&B and D&C guarantees (and the FS) in section 8 starts by some common properties: they all increase in the number of goods and decrease in the number of agents: Proposition 5. Then Propositions 6 and 7 formalises their differences revealed in a handful of examples. Compared to the benchmark FS, the range of the S&B guarantee is much larger than that of the D&C one: Proposition 7. Unlike D&C, the S&B guarantee coincides with FS on a large subset of utilities with non empty interior: Proposition 6.

Section 9 offers two results toward the general statement, to be checked by systematic numerical experiments, that if all participants play safely the outcome in S&B collects on average a bigger share of the efficient surplus than in D&C.

With two agents and for both rules the profile of guarantees reduces the bargaining gap, i. e., it distributes at least the total utility at the worst partition of the goods: Lemma 10. But with three or more agents this is no longer true for D&C, while we conjecture that it is still true for S&B.

The final Proposition 8 considers the situation where agent 1's marginal utility for each good dominates that of every other agent: then the S&B safe play achieves full efficiency, i.e., gives all the goods to agent 1, while under D&C all but a $\frac{1}{n}$-th share of the efficient surplus can be lost.

All difficult proofs are gathered in the Appendix, section 11.

2 Relevant literature

Allowing cash compensations to smooth out indivisibilities of objects has been essentially ignored by the first four decades of the theoretical literature on fair division, if we except the cogent discussion by Steinhaus of what we call above the Multi Auction rule for additive utilities ([24] p. 317).

This changed with the microeconomic discussion of the assignment problem. Each agent wants at most one object and utilities are only assumed increasing in money but not necessarily quasi-linear; monetary compensations can restore fairness interpreted as Envy Freeness and even a version of the competitive equilibrium with equal incomes: [25] [1]]. The quasi-linear case of the model is discussed in [2] selecting a canonical envy free allocation, in [9] for the dissolution of partnership, in [17] for adressing the NIMBY problem, and currently implemented on the user-friendly Spliddit platform [15].

In the assignment problem ex ante fairness is captured by the *unanimous utility*: the best equal utility in the hypothetical problem where everyone else shares my preferences ([20], [26]). This is unambiguously the best possible guarantee and it is compatible with Envy Freeness.

In our model the set of allocations and utilities are vastly more complex than in an assignment problem and the unanimity utility – that we call the Maxmin utility – is an upper bound on guarantees but not itself a feasible guarantee. Our newfound critique of Envy Freeness (stated as Lemma 3 in section 4.2) complements the normative objections developed in [20].

The search for a practical and appealing guarantee started the mathematical cake cutting literature ([23], [18]) and is a prominent theme in the vibrant 21st century algorithmic literature on fair division surveyed in [21], [4] and [27]. The standard model has utilities additive over objects and no cash transfers or lotteries, so the definition of a convincing guarantee is complicated by the presence of "un-smoothable" indivisibilities.Our Maxmin and minMax utilities (section 5) are the counterpart of, respectively, the influential MaxMinShare due to Buddish [8] and its dual MinMaxShare [7]. The MaxMinShare is *almost* a feasible guarantee (it is not feasible in extremely rare configurations [22]); its dual minMax version is strongly unfeasible: the exact opposite holds in our model (Lemma 4 section 5), as well as when we divide a non atomic cake and utilities are continuous but otherwise arbitrary: see [6], [3].

Other definitions of guarantees are also discussed in the algorithmic literature, e. g. [5], as are guarantees adjusted to the granularity of the utilities in [13].

In the first of our two n-person versions of the Divide&Choose rule (section 6) the participants bid first for the role of Divider, which is similar to and inspired by a similar auction in [11] and [12] for implementing the egalitarian-equivalent division rule when we distribute Arrow-Debreu commodities.

3 Basic definitions and notation

objects and money The finite set A with cardinality m and generic elements $a, b, \cdots$, contains the indivisible objects that must **all** be distributed between the n agents in the set N with generic elements $i, j, \cdots$.

A n-partition π of A is a list $\pi = \{S_k\}_{k=1}^n$ of **possibly empty** and pairwise disjoint subsets of A such that $A = \cup_{k=1}^n S_k$: up to $n-1$ shares can be empty. The set of n-partitions is $\mathcal{P}(n; A)$ if the shares S_k are not assigned to specific agents, and $\mathcal{P}(N; A)$ if they are.

Money is available in unbounded quantities to perform transfers $t = (t_i)_{i \in N}$ that are balanced, $\sum_N t_i = 0$; the set of such transfers is $\mathcal{T}(N)$. An *allocation* is a pair $(\pi, t) \in \mathcal{P}(N; A) \times \mathcal{T}(N)$.

utilities Each agent i is endowed with a quasi-linear utility $u_i \in \mathbb{R}^{2^A}$ over shares, with the important normalisation $u_i(\varnothing) = 0$: her utility from the allocation (π, t) is $u_i(S_i) + t_i$. The marginal utility of object a at $S \subseteq A$ for utility u is $\partial_a u(S) = u(S \cup a) - u(S \diagdown a)$. We assume throughout the paper that all objects are *goods*: $\partial_a u_i(S) \geq 0$ for all $S \subseteq A$; utility functions can be any (weakly) inclusion increasing non negative function on 2^A, and $\mathcal{M}^+$ is our notation for this domain.

The utility u is additive if for all $a \in A$ the marginal $\partial_a u(S) = u_a$ is independent of S; in this case we write $u_S = \sum_S u_a$ instead of $u(S)$.

We often use the following *cover* operation to generate examples in the domain $\mathcal{M}^+$. Fix a subset $\{S_k; 1 \leq k \leq K\}$ of $2^A \diagdown \varnothing$ and K positive utilities v_k; the *cover* of the subset $\{(S_k, v_k)\}$ of $2^A \diagdown \varnothing \times \mathbb{R}_+$ is the smallest utility u in $\mathcal{M}^+$ such that $u(S_k) = v_k$ for all k:

$$u(S) = \max_{k: S_k \subseteq S} v_k \text{ ; } u(S) = 0 \text{ if } S_k \nsubseteq S \text{ for all } k$$

For instance the Greedy utility u_G in section 1 is the cover of $\{(A, 1)\}$ while the Frugal utility u_F is the cover of $\{(a, 1); a \in A\}$.

efficiency A N-profile of utilities is $\overrightarrow{u} = (u_i)_{i \in N} \in \mathcal{M}^{+N}$ and if $\pi \in \mathcal{P}(N; A)$ we write $\overrightarrow{u}(\pi) = \sum_{i \in N} u_i(S_i)$. The *unanimity* profile where all agents have the same utility u is written $[u]$.

The *efficient surplus* at profile $\overrightarrow{u}$ is $\mathcal{W}(\overrightarrow{u}) = \max_{\pi \in \mathcal{P}(N;A)} \overrightarrow{u}(\pi)$. Recall an easy but critical consequence of the quasi-linearity assumption*: the allocation $(\pi^*, t) \in \mathcal{P}(N; A) \times \mathcal{T}(N)$ is efficient (Pareto optimal: PO) if and only if π^* maximises $\overrightarrow{u}(\pi)$ over $\mathcal{P}(N; A)$*: Pareto optimality is independent of the balanced cash transfers.

implementation **I**n an arbitrary n-agent mechanism agent i's strategy is *safe* if it delivers to i the largest "worst case" utility against all other agents who play adversarially against i after seeing i's strategy. That utility is the *guarantee* implemented by this mechanism: it only depends upon the mechanism including n and agent i's utility function.

In our model as always many rules can implement the same guarantee: an example is the two versions of Divide&Choose in section 6. We systematically omit many tie-breaking details from the description of rules, and the reader will find it easy to check that they (the details) never affect the guarantee they implement.

We will also discuss the efficiency performance of a rule at the outcome resulting from the simultaneous safe play by all agents; if multiple choices of safe play or tie-breaking details affect the outcome, we focus on the least efficient possible outcome. This performance is meaningful in the fully decentralised context where no agent's choice of strategy uses any information about other agents. Naturally this restriction does not apply to a guarantee: its ex ante protection applies against all possible moves of the other agents including highly strategic moves based on knowledge of my own utility.

4 Guarantees, Positive and Responsive

Definition 1: *An n-person guarantee is a mapping* $\mathcal{M}^+ \ni u \to \Gamma_n(u) \in \mathbb{R}_+$ *such that*

$$\sum_{i \in N} \Gamma_n(u_i) \leq \mathcal{W}(\overrightarrow{u}) \text{ for each } \overrightarrow{u} \in \mathcal{M}^{+N} \tag{1}$$

The set of n-guarantees on A is written $\mathcal{G}(A;n)$.

Guarantees are *anonymous* by construction: they do not discriminate between agents on the basis of their name. The three guarantees getting most of our attention, Familiar Share, Sell&Buy, Divide&Choose are also *neutral*, i. e., oblivious to the name of the objects in A: the definition of a neutral guarantee only depends upon the numbers of objects and agents.

The simplest guarantee is the Familiar Share $\Gamma_n^{FS}(u) = \frac{1}{n}u(A)$: indeed $u_i(A) = \overrightarrow{u}(\pi)$ for the partition π giving all goods to i.

More generally to any $\pi \in \mathcal{P}(n;A)$ we associate the π*-guarantee*

$$\Gamma_n^{\pi}(u) = \frac{1}{n}[u](\pi) = \frac{1}{n}\sum_{k=1}^{n} u(S_k) \text{ for all } u \in \mathcal{M}^+$$

and we call Γ_n^{π} a *fixed partition* guarantee.

To check inequality (1) at profile $\overrightarrow{u} = (u_i)_{i \in N}$ assign the shares of π to agents in N arbitrarily, pick a circular permutation σ of N and sum up the inequalities $\sum_{i \in N} u_i(S_{\sigma^k(i)}) \leq \mathcal{W}(\overrightarrow{u})$ for all $k = 0, \cdots, n-1$.

The Familiar Share terminology comes from the special case of additive utilities where we have long known that it is the compelling guarantee (as explained in section 5 after Proposition 1).

4.1 three simple auction rules

The simple **Bundled Auction (BA)** implements Γ_n^{FS}: each agent i submits a non negative bid β_i that the rule interprets as this agent's utility for the entire set A; (one of) the highest bidder(s) i^*gets A and pays $\frac{1}{n}\beta_i$ to each of the $n-1$ other agents.

The only safe bid in BA is the truthful one $\beta_i = u_i(A)$: it guarantees to agent i her Familiar Share $\frac{1}{n}u_i(A)$ while any other bid risks delivering a smaller benefit: this is clear for a winning overbid, and for an underbid losing to a bid between $\beta_i$and $u_i(A)$.[3]

The π-auction and averaging-auction we introduce now are the key ingredients of the Divide&Choose rules in section 6.

The **π-auction** implements the π-guarantee**.** Given $\pi = \{S_k\}_{k=1}^n$ and the set N, each agent i reports a vector $t^i = (t^i_k)_{k\in[n]} \in \mathcal{T}(n)$ of balanced transfers over those shares (with the notation $[n] = \{1, \cdots, n\}$).

Write $\mathcal{C}$ for the set of bijections σ of N into $[n]$. An *optimal assignment* of π at $t = (t^i)_{i\in N}$ is (one of) the bijection(s) $\sigma^* \in \mathcal{C}$ minimising $\sum_{i\in N} t^i_{\sigma(i)}$ over $\mathcal{C}$ (the interpretation of optimality comes below). Because each vector t^i is balanced, the *slack* $\delta = \sum_{j\in N} t^j_{\sigma^*(j)}$is negative or zero. After each agent j is "paid" $t^j_{\sigma^*(j)}$ (a cash handout if $t^j_{\sigma^*(j)} > 0$, a tax if $t^j_{\sigma^*(j)} < 0$) the remaining cash surplus δ is divided equally between all agents. Agent i's final allocation is $(S_{\sigma^*(i)}, t^i_{\sigma^*(i)} - \frac{1}{n}\delta)$.

What is the agent i's worst utility by reporting $t^i \in \mathcal{T}(n)$? Check first that any σ in $\mathcal{C}$ can be selected as uniquely optimal for some reports of the other agents. This happens if we pick some $\varepsilon > 0$ and the reports $t^j, j \neq i$ are

$$t^j_{\sigma(j)} = t^i_{\sigma(j)} - \varepsilon \text{ ; } t^j_k = t^i_k + \frac{1}{n-1}\varepsilon \text{ for all } k \neq \sigma(j)$$

In this case the slack is $\delta = -(n-1)\varepsilon$ and our agent's final utility is $u_i(S_{\sigma(i)}) + t^i_{\sigma(i)} + \frac{n-1}{n}\varepsilon$. As σ and ε were arbitrary we conclude that i's utility could be as low as $\min_{k\in[n]} u_i(S_k) + t^i_k$. The unique choice of t^i maximising the latter equalises i's utility across these shares

$$u_i(S_k) + t^i_k = u_i(S_\ell) + t^i_\ell \text{ for all } k, \ell \in [n] \tag{2}$$

and secures the utility $\frac{1}{n}\sum_{k\in[n]} u_i(S_k) = \frac{1}{n}[u_i](\pi)$, while any other report is unsafe. We just proved

Lemma 1 *The π-auction implements the π-guarantee, and the unique safe play is to report the transfers equalising one's utility across the shares of π (as in (2)).*

If all agents report safely, an assignment of the shares in π is efficient if and only if the corresponding transfers in (2) have the smallest sum. Thus the π-auction interprets the report t^i as revealing the utilities $u_i(S_k), k \in [n]$, up to an additive constant (except of course if agent i reports − unsafely − different transfers for two empty shares).

The set $\mathcal{G}(A; n)$ of n-guarantees is clearly convex (it is defined by inequalities (1)) and the canonical way to implement the average $\frac{1}{\rho}\sum_{r=1}^{\rho} \Gamma^r_n$ of an arbitray

[3] The tie break rule is irrelevant and the safe strategy does not change if the winner only pays $\frac{1}{n}$-th of the second highest price to each loser.

set of guarantees is the **averaging-auction**. Each agent $i \in N$ must report a vector $t^i = (t^i_r)_{r \in [\rho]} \in \mathcal{T}(\rho)$ of balanced transfers over those guarantees. The rule interpretes t^i as equalising the utilities $\Gamma^r_n(\widetilde{u}_i)$

$$\text{for all } r, r' \colon \; \Gamma^r_n(\widetilde{u}_i) + t^i_r = \Gamma^{r'}_n(\widetilde{u}_i) + t^i_{r'} = \frac{1}{\rho} \sum_{r=1}^{\rho} \Gamma^r_n(\widetilde{u}_i) \tag{3}$$

and it selects a guarantee $\Gamma^{r^*}_n$ such that
$r^* \in \arg\min_{r \in \rho} \sum_{i \in N} t^i_r = \arg\max_{r \in \rho} \sum_{i \in N} \Gamma^r_n(\widetilde{u}_i)$. The slack $\theta = \sum_{j \in N} t^j_{r^*}$ is negative or zero, each agent i receives the transfer $t^i_{r^*} - \frac{1}{n}\theta$ and the guarantee $\Gamma^{r^*}_n$ is implemented.

Lemma 2 *The averaging-auction implements the average guarantee $\frac{1}{\rho}\sum_{r=1}^{\rho} \Gamma^r_n$, and the unique safe play is to report the transfers equalising one's utility across guarantees (as in (3)).*

The omitted proof is similar to that of Lemma 1.

Remark: It is just as easy to implement any convex combination of guarantees $\sum_{r=1}^{\rho} \lambda_r \Gamma^r_n$ where each λ_r is positive and $\sum_{r=1}^{\rho} \lambda_r = 1$. Each agent i reports a vector of λ-balanced transfers t^i, $\sum_{r=1}^{\rho} \lambda_r t^i_r = 0$, and the rule proceeds as before: it implements $\Gamma^{r^*}_n$ where r^* minimises $\sum_{i \in N} t^i_r$ so that the slack $\theta = \sum_{i \in N} t^i_{r^*}$ is still non positive and i receives $t^i_{r^*} - \frac{1}{n}\theta$. The safe strategy is to choose λ-balanced transfers t^i equalising utilities as in (3).

4.2 Positivity and Responsiveness

Definition 2 *The n-guarantee $\Gamma_n \in \mathcal{G}(n, A)$ is*
Positive if for all $u \in \mathcal{M}^+ : \Gamma_n(u) > 0 \Longrightarrow u(A) > 0$
Responsive if $\Gamma_n(u_F) > \frac{1}{n} u_F(A)$

Positivity applies to all utilities except the null one, so if Γ_n is weakly increasing in u (as are all the guarantees we discuss: Proposition 4 section 8), it simply requires $\Gamma_n(\lambda u_G) > 0$ for all $\lambda > 0$.

Responsiveness implies the strictly weaker inequality $\Gamma_n(u_G) < \frac{1}{n} u_G(A)$: simply apply (1) to a n-profile with one u_F and the rest u_G utilities, where theefficient surplus is 1.

Among the fixed partition guarantees, only Γ^{FS}_n is Positive; all fixed partition guarantees are Responsive, with the single exception of Γ^{FS}_n.[4]

The next result formalises the important trade-off of our two normative requirements with Envy Freeness, the standard interpretation of ex post fairness. Recall that the allocation (π, t) is Envy Free if $u_i(S_i) + t_i \geq u_i(S_j) + t_j$ for all $i, j \in N$.

Lemma 3 *If $|A| \geq 2$ and the n-guarantee Γ_n* in $\mathcal{M}^+$ *is Positive and Responsive, it is incompatible with Envy Freeness: at some profile $\overrightarrow{u} \in \mathcal{M}^{+N}$ at least one agent is envious in every envy free allocation.*

[4] So a convex mixture of Γ^{FS}_n with some other π-guarantees meets both properties, for instance $\frac{1}{2n} u(A) + \frac{1}{2n} \sum_A u(a)$ if $n = m$.

Proof: in the two agent problem (Greedy, Frugal) we saw that envy free allocations either give the utility $\frac{1}{2}$ to both agents, or give zero to Greedy. For a general n the argument is the same in the problem with $n-1$ Greedy and one Frugal agents.

Note that the safe play in the BA ensures Envy Freeness for that agent, whether other agents are playing safe or not.

5 Maxmin and minMax utilities

The recent literature on fair division pays close attention to these two canonical utility levels inspired by Divide&Choose for cake-cutting, but playing a role in many other models. We adapt them as follows. Recall the notation $[u]$ for the unanimity profile where all agents have utility u.

Definition 3 *Fix* A, n *and* $u \in \mathcal{M}^+$.
i) the Maxmin utility at u *is* $Maxmin_n(u) = \frac{1}{n}\max_{\pi\in\mathcal{P}(n;A)}[u](\pi)$*; it is the largest utility agent* u *can secure by choosing an (anonymous) allocation* $(\pi, t) \in \mathcal{P}(n; A) \times \mathcal{T}(n)$ *and eating his worst share* (S_k, t_k) *of that allocation.*
ii) the minMax utility at u *is* $minMax_n(u) = \frac{1}{n}\min_{\pi\in\mathcal{P}(n;A)}[u](\pi)$*; it is the largest utility agent* u *can secure by picking her best share in the worst possible (anonymous) allocation* $(\pi, t) \in \mathcal{P}(n; A) \times \mathcal{T}(n)$.

Given an n-partition $\pi = \{S_k\}_{k\in[n]}$ of A, the π-auction guarantees the utility $\frac{1}{n}[u_i](\pi)$ to each agent i: therefore i reaches her $Maxmin$ utility if she can choose π, and at least her $minMax$ one if the choice of π is adversarial.

Lemma 4 *Fix* A, n and the domain $\mathcal{M}^+$
i) $\mathcal{M}^+ \ni u \to Maxmin_n(u)$ *is not a* n*-guarantee (inequality (1) fails)*
ii) but it is an upper bound for every guarantee $\Gamma_n \in \mathcal{G}(A; n)$:

$$\Gamma_n(u) \leq Maxmin_n(u) \text{ for all } u \in \mathcal{M}^+$$

iii) $\mathcal{M}^+ \ni u \to minMax_n(u)$ is a n-guarantee: $minMax_n(\cdot) \in \mathcal{G}(A; n)$

The proof is easy.[5] For *ii*) we fix Γ_n, u,and n. Inequality (1) at the unanimity profile $[u]$ for all i is $n\Gamma_n(u) \leq \max_{\pi\in\mathcal{P}(n;A)} u(\pi)$ as desired. For *i*) one checks easily

$$Maxmin_n(u_F) = \min\{1, \frac{m}{n}\} \text{ and } Maxmin_n(u_G) = \frac{1}{n}$$

so (1) fails at the n-profile with one u_F agent and all others are u_G because $W(\overrightarrow{u}) = 1$. Note that this is not a knife edge situation: the set of profiles where the corresponding profile of Maxmin utilities is not feasible is open in $\mathbb{R}_+^{2^A}$.

For *iii*) note that for any fixed partition guarantee $\Gamma_n^\pi(u) = \frac{1}{n}[u](\pi) \geq minMax_n(u)$ for all u, so that (1) holds for the latter as well. This also shows

[5] All three statements hold in the cake-cutting model (Bogomolnaia and Moulin (201X)), but there the proof of *iii*) is hard!

that the minMax guarantee can easily be improved for instance $minMax_2(u_G) = 0$; $minMax_2(u_F) = \frac{1}{2}$. ■

Our next result, technically very simple, shows an important benefit of choosing a guarantee in the "duality interval" $[minMax_n(u), Maxmin_n(u)]$. Write $\mathcal{S}up$ and $\mathcal{S}ub$ for the sub-domains of superadditive and subadditive utilities: in $\mathcal{S}up$ we have $u(S) + u(T) \leq u(S \cup T)$ for all disjoint S, T ; and the opposite inequality in $\mathcal{S}ub$.

Proposition 1 *Suppose the guarantee Γ_n is such that*

$$\Gamma_n(u) \in [minMax_n(u), Maxmin_n(u)] \text{ for all } u \in \mathcal{M}^+ \tag{4}$$

Then $\Gamma_n(u) = \frac{1}{n}u(A)$ if u is additive; $\Gamma_n(u) \geq \frac{1}{n}u(A)$ if u is subadditive; and $\Gamma_n(u) \leq \frac{1}{n}u(A)$ if u is superadditive.

Clearly if $u \in \mathcal{S}ub$ (resp. $u \in \mathcal{S}up$) we have $u(A) = \min_{\pi \in \mathcal{P}(n;A)}[u](\pi)$ (resp. $\max_{\pi \in \mathcal{P}(n;A)}[u](\pi)$) therefore (4) implies the desired inequalities.

If u is additive $Maxmin_n(u) = \frac{1}{n}u(A)$, so by statement *ii*) in Lemma 4 the Familiar Share is the best possible guarantee and the compelling interpretation of ex ante fairness. The other two inequalities are a weak form of the normative principle conveyed by the Responsiveness property.

Note that property (4) is not very restrictive: it is clearly satisfied by the Familiar Share, the fixed partitions guarantees Γ_n^π, the D&C and S&B guarantees defined shortly, and their convex combinations.

In subsection 11.1 of the Appendix we describe the plausible Multi Auction rule and show that its guarantee falls outside the duality interval, often strictly smaller than the $minMax$ guarantee. We dismiss that rule for this very reason.

Remark: there is a precise connection between the duality interval in (4) and Envy Freeness, confirming the trade-off between ex ante and ex post fairness in Lemma 3 above. At an envy free allocation, it is clear that every agent i gets at least her $minMax_n(u_i)$ utility. Conversely if the single-valued rule $(\mathcal{M}^+)^N \ni (u_i)_{i\in N} \to (\pi, t) \in \mathcal{P}(N;A) \times \mathcal{T}(N)$ is efficient and envy-free, then it implements precisely the $minMax$ guarantee. For each utility function u_i we can complete a profile (u_i, u_{-i}) at which the rule gives to agent i precisely his $minMax_n(u)$ utility.[6]

6 Two Divide&Choose rules

The two rules have the same guarantee, their building blocks are the π-auction and averaging-auction in subsection 4.1.

Definition 4 *Divide&Choose*$_n^{\mathbf{1}}$

[6] Proof: Fix $u_1 \in \mathcal{M}^+$ and $\pi = (S_k)_{k=1}^n$ achieving $\min_{\pi \in \mathcal{P}(n;A)}[u_1](\pi)$, and a positive number δ. Construct a profile where the common utility v of the $n-1$ other agents is the cover of the sequence $\{(S_k, u_1(S_k) + \delta); k \in [n]\}$. If δ is very large any assignment of the shares S_k to the agents is efficient (and any other efficient partition distributes the same utilities pre-transfers). By the construction of utility v, at an envy free and efficient allocation the transfers make agent 1 indifferent between all the shares so her utility is $minMax_n(u)$.

stage 1: run a simple auction for the role of Divider; the winner i's bid $\beta_i \geq 0$ *is (one of) the highest bid(s)*
stage 2: agent i pays $\frac{1}{n}\beta_i$ *to every other agent and picks a partition* $\pi = \{S_k\}_{k=1}^n$ *in* $\mathcal{P}(n, A)$
stage 3: run the π*-auction.*

In the second rule the initial bidding to become the Divider is replaced by a stage in which each agent i proposes her own partition (as if she was the Divider).

Definition 5 *Divide&Choose*$_n^{\mathbf{2}}$
stage 1: each agent i picks a partition π^i *in* $\mathcal{P}(n, A)$*; those partitions are publicly revealed*
stage 2: run the averaging auction between the guarantees $\Gamma_n^{\pi_i}, i \in [n]$.

The D&C$_n^2$ rule takes longer to run than D&C$_n^1$: once the agents submit balanced transfers between the proposed partitions π^i and the rule selects the efficient π_{i^*} – maximising $\sum_i \Gamma_n^{\pi_{i^*}}(u_i)$ according to the reported transfers –, we still need to run the π_{i^*}-auction.

Proposition 2
In the D&C$_n^1$ *rule, agent i's play is safe **iff** he bids* $\beta_i = Maxmin_n(u_i) - minMax_n(u_i)$ *in stage 1, chooses a partition* π_i *maximising* $[u_i](\pi)$ *in stage 2, and reports truthfully equalising transfers across the shares of* π *in stage 3*
In the D&C$_n^2$ *rule, agent i's play is safe **iff** she proposes a partition* π_i *maximising* $[u_i](\pi)$ *in stage 1, then reports truthfully equalising transfers across the guarantees* $\Gamma_n^{\pi_j}(u_i), j \in N$*, and finally reports truthful transfers in the final* π_{i^*}*-auction;*
Both rules implement the guarantee

$$\Gamma_n^{DC}(u) = \frac{1}{n} Maxmin_n(u) + \frac{n-1}{n} minMax_n(u)$$
$$= \frac{1}{n^2} \max_{\pi \in \mathcal{P}(n,A)} u(\pi) + \frac{n-1}{n^2} \min_{\pi \in \mathcal{P}(n,A)} u(\pi) \qquad (5)$$

Proof
For D&C$_n^1$. By Lemma 1 the worst drop in guaranteed utility to agent i between the roles of Divider and Chooser (non-Divider) is $\delta_i = Maxmin_n(u_i) - minMax_n(u_i)$. Therefore the safe bid in stage 1 is precisely δ_i. If i becomes the Divider she pays $\frac{n-1}{n}\delta_i$ then secures $Maxmin_n(u_i)$ in the π-auction; otherwise she is paid at least $\frac{1}{n}\delta_i$ and secures $minMax_n(u_i)$ in the π-auction.
For D&C$_n^2$. By Lemma 2 agent i's guaranteed utility in stage 2 is $\frac{1}{n}\sum_j \frac{1}{n}[u_i](\pi_j)$ where the worst case will be if $\frac{1}{n}[u_i](\pi_j) = minMax_n(u_i)$ for each $j \neq i$; therefore by proposing in stage 1 a partition π_i maximising $[u_i](\pi_i)$ she secures the utility (5).

It is easy to check that no other play is safe. ■

Lemma 5
i) the guarantee Γ_n^{DC} *is Positive, Responsive, and in the duality interval (4)*

ii) if u is additive the safe bid in D&C[1] *is zero and any partition is a safe proposal in both rules.*

The easy proof is omitted.

Statement *i*) says that the D&C rules are "reasonable" in terms of Proposition 1. Statement *ii*) on the contrary is an unappealing feature of D&C[1,2]: if my utility is additive all partitions are equally safe choices they convey no information relevant to the selection of an efficient allocation.

In the reporting stages common to both rules I only reveal the relative utilities between the shares of certain partitions but the level of my absolute utility remains private: this increases privacy but is detrimental to efficiency. If a certain agent i's utility is everywhere higher than other utilities, efficiency means that all the goods should go to i: the reports in D&C[1,2] fail to recognise this feature. On the contrary in the Sell&Buy rule individual messages are related to the absolute utilities and avoid this type of inefficiencies: Proposition 8 in section 9.

7 the Sell&Buy rule

7.1 defining the rule and its guarantee

A non negative price $p \in \mathbb{R}_+^A$ is formally an additive utility; we use the same notation $p_S = \sum_{a \in S} p_a$. Next $\Delta(x)$ is the simplex of prices such that $p_A = x$, the whole bundle costs x. Because the recursive definitions of the S&B rule and its guarantee works over shrinking subsets of objects, we make explicit their dependence on the set A.

Definition 6 *Sell&Buy, S&B$_2(A)$ for two agents*

stage 1: the agents bid to become the Seller: this is (one of) the lowest bidder(s) with bid x

stage 2: the Seller chooses a price p in $\Delta(x)$

stage 3: the Buyer can buy any share S of objects (possibly $\varnothing$) at this price and the Seller eats the unsold objects

$$\text{final allocation: Buyer } (S, -p_S) \text{ ; Seller } (A \diagdown S, p_S)$$

To understand how to bid safely we compute first the worst utility $W_2(u; x|A)$ an agent with utility u can get by bidding x and becoming the Seller

$$W_2(u; x|A) = \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (u(T) + p_{A \diagdown T}) = \max_{p \in \Delta(x)} \{ \min_{\varnothing \subseteq T \subseteq A} (u(T) - p_T) + x \} \tag{6}$$

because he must expect the worst purchase from the Buyer.

We compare it with the worst utility $L_2(u; x|A)$ of this agent if her bid x loses *by a hair* (to a bid just below x) and she is offered the worst possible price but a total price no larger than x:

$$L_2(u; x|A) = \min_{p \in \Delta(x)} \max_{\varnothing \subseteq S \subseteq A} (u(S) - p_S) \tag{7}$$

Clearly $W_2(u;x|A)$ increases in x while $L_2(u;x|A)$ decreases hence the safe bid is where they intersect which we show below is an unambiguous bid x^*; the common value is the guarantee $\Gamma_2^{SB}(u|A)$.

With n agents there are at most $n-1$ rounds of bidding in S&B$_n(A)$: in each round one agent becomes the Buyer and all others become Sellers; the Buyer leaves after being offered to buy some goods from all the Sellers.

Definition 7 *Sell&Buy, S&B$_n(A)$ recursive definition*
stage 1: each agent i bids x_i to become Seller or Buyer; (one of) the highest bidders becomes the Buyer
stage 2: each of the $n-1$ Sellers j chooses a price p_j in $\Delta(x_j)$
stage 3: the Buyer buys a share S of goods (possibly $\varnothing$) by paying $p_j(S)$ to each Seller and leaves; the rule stops if $S = A$, otherwise we go to
stage 4: the remaining agents play S&B$_{n-1}(A \diagdown S)$

The worst utility $W_n(u;x|A)$ from becoming a Seller after bidding x is now

$$W_n(u;x|A) = \max_{p\in\Delta(x)} \min_{\varnothing\subseteq T\subseteq A}(\Gamma_{n-1}^{SB}(u|T)+p_{A\diagdown T}) = \max_{p\in\Delta(x)} \min_{\varnothing\subseteq T\subseteq A}(\Gamma_{n-1}^{SB}(u|T)-p_T)+x \quad (8)$$

and the worst utility as a Buyer after bidding x is

$$L_n(u;x|A) = \min_{p\in\Delta((n-1)x)} \max_{\varnothing\subseteq S\subseteq A}(u(S)-p_S) = \min_{p\in\Delta(x)} \max_{\varnothing\subseteq S\subseteq A}(u(S)-(n-1)p_S) \quad (9)$$

because the worst case is when the $n-1$ other bids are just below x.

Lemma 6 *For any non null utility $u \in \mathcal{M}^+$ the recursive programs (8),(9), together with the initial pair (6), (7), define unambiguously*
the function $W_n(u;x|A)$ concave and strictly increasing in x from 0 to $u(A)$
the function $L_n(u;x|A)$ convex and strictly decreasing in x from $u(A)$ to 0
and the guarantee $\Gamma_n^{SB}(u)$ at their intersection: $W_n(u;x^*|A) = L_n(u;x^*|A) = \Gamma_n^{SB}(u|A)$

These properties imply: $0 < \Gamma_n^{SB}(u|A) < u(A)$.

Proposition 3 *the guarantee Γ_n^{SB} is Positive, Responsive, and in the duality interval (4).*

The proof of the key Lemma 6 and its corollary Proposition 3 is an application of the minimax theorem, in subsection 11.2 of the Appendix.

7.2 computing the guarantee Γ_n^{SB}

The recursive computation of Γ_n^{SB} is not easy in the general case, but greatly facilitated when some goods are identical.

Lemma 7 *If the goods a, b enter symmetrically in u – $u(S-b+a) = u(S)$ if $a \notin S \ni b$ – the optimal price in $W_n(x;u|A)$ and the worst price in $L_n(x;u|A)$ can be taken equal.*

Proof in the Appendix subsection 11.3 where we state a stronger group version of this property.

Our first example illustrates the recursive computation of Γ_n^{SB} in a simple family of fully symmetric step functions connecting the Frugal and Greedy utilities.

Example 1 The m objects are identical and for each integer $\theta \in [m]$ the dichotomous utility u^θ requires no less and no more than θ goods:

$$u^\theta(S) = 1 \text{ if } |S| \geq \theta \text{ ; } u^\theta(S) = 0 \text{ if } |S| < \theta$$

So u^1 is Frugal and u^m is Greedy.

For $t \in [m]$ write $\Gamma_n(\theta|t) = \Gamma_n^{SB}(u^\theta|T)$ the n person S&B-guarantee when there are only t goods to divide and note that $\Gamma_n(\theta|t) = 0$ if $t < \theta$. We compute first $\Gamma_2(\theta|t)$ for $t \geq \theta$:

$$W_2(\theta; x|t) = \min\{1, \frac{t-\theta+1}{t}x\} \text{ ; } L_2(\theta; x|t) = \max\{0, 1 - \frac{t}{m}x\}$$

$$\Longrightarrow \Gamma_2(\theta|t) = \frac{t-\theta+1}{t+1} \text{ for } \theta \leq t \leq m$$

For $n = 3$ equation *(9)* is simply: $L_3(\theta; x|m) = \max\{0, 1 - \frac{2\theta}{m}x\}$. By the concavity of $t \to \Gamma_2(\theta|t)$ *(8)* becomes

$$W_3(\theta; x|m) = \min_{\theta-1 \leq t \leq m}\{\Gamma_2(\theta|t) + \frac{m-t}{m}x\} = \min\{\frac{m-\theta+1}{m}x, \frac{1}{\theta+1} + \frac{m-\theta}{m}x\}$$

after which one checks that the graph of L_3 intersects that of W_3 on the line $x \to \frac{m-\theta+1}{m}x$, and finally $\Gamma_3(\theta|m) = \frac{m+1-\theta}{m+1+\theta}$ with the optimal bid $x^* = \frac{m}{m+1+\theta}$. The general inductive step works in exactly the same way: the optimal bid is $x^* = \frac{m}{m+1+(n-2)\theta}$ and the guarantees are

$$\Gamma_n^{SB}(u_F) = \frac{m}{m+n-1} \geq \Gamma_n^{SB}(u^\theta) = \frac{m+1-\theta}{m+1+(n-2)\theta} \geq \Gamma_n^{SB}(u_G) = \frac{1}{m(n-1)+1}$$

For instance the ratio $\frac{\Gamma_n^{SB}(u_F)}{\Gamma_n^{SB}(u_G)}$ is already 10 with four agents and five goods.

We conclude with two important special cases where Γ_n^{SB} is easy to describe.

Lemma 8 *if u is additive then $x^*(u) = \frac{1}{n}u_A = \Gamma_n^{SB}(u)$ and, if chosen as Seller the price $p = \frac{1}{n}u$ is safe.*

The omitted proof checks by induction that if u is additive $W_n(u; x) = \min\{x, u_A\}$ and $L_n(u; x) = \max\{u_A - (n-1)x, 0\}$.

Contrast this result with the fact that any play is safe in the D&C rules when my utility is additive (statement *ii*) in Lemma 5).

The final result in this section covers all problems with *fully symmetric goods and two agents.* The utility function takes the form $u(S) = u_s$ if $|S| = s$, $1 \leq s \leq m$ and $u_0 = 0$. We write the non negative marginals $\partial_k u_\ell = u_{\ell+k} - u_\ell$.

Proposition 4 *$n = 2$, identical goods*
Agent u's optimal bid in the S&B rule is

$$x^* = \max\{\frac{m}{m+k}\partial_k u_\ell | 0 \leq k, \ell \leq m \text{ and } 0 \leq \ell + k \leq m\} \qquad (10)$$

If $x^* = \frac{m}{m+k}\partial_k u_\ell$ *then*

$$\Gamma_2^{SB}(u) = \frac{\ell+k}{m+k}u_\ell + \frac{m-\ell}{m+k}u_{\ell+k} \tag{11}$$

Proof in subsection 11.4. The recursive formula for larger n is more complicated but still solvable numerically.

Corollary:
If in addition u *is convex then* $x^* = \max_{0\le k\le m}\{\frac{m}{m+k}(u_m - u_{m-k})\}$ *and* $\Gamma_2^{SB}(u) = \frac{m}{m+k}u_{m-k} + \frac{k}{m+k}u_m$
If in addition u *is concave then* $x^* = \max_{0\le k\le m}\frac{m}{m+k}u_k$ *and* $\Gamma_2^{SB}(u) = \frac{m}{m+k}u_k$

8 Comparing the S&B, D&C, and FS guarantees

Proposition 5
i) The three guarantees $\Gamma_n^{SB}(u), \Gamma_n^{DC}(u)$ *and* $\Gamma_n^{FS} = \frac{1}{n}u(A)$ *are continuous and weakly increasing in* u.
ii) They are also scale invariant ($\Gamma_n(\lambda u) = \lambda\Gamma_n(u)$ *for* $\lambda > 0$*) increasing in* A *and decreasing in* n*: for all* $A, a \notin A, n, u$ *we have*

$$\Gamma_n(u|A) \le \Gamma_n(u|A\cup a) \text{ and } \Gamma_{n+1}(u|A) \le \Gamma_n(u|A)$$

Proof in subsection 11.5.

We start our numerical examples with the simplest choices of n and m.

Example 2: two agents, two goods: $n = 2$, $A = \{a, b\}$
Here the S&B and D&C guarantees only depend upon $u(A)$ and $u(a) + u(b)$; by scale invariance we can normalise $u(A) = 1$ then set $z = u(a) + u(b) \in [0, 2]$; note that $z = 1$ means the utility is additive.

Clearly $\Gamma_2^{DC}(z) = \frac{1+z}{4}$ with a bid of $\frac{|z-1|}{2}$ in D&C[1]. The omitted computations are almost as easy for Γ_2^{SB} and the optimal bid:

z	0	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{3}{2}$	$\cdots$	2
Γ_2^{SB}	$\frac{1}{3}$	$\frac{z+1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}z$	$\frac{2}{3}$
x^*	$\frac{2}{3}$	$\frac{2-z}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}z$	$\frac{2}{3}$

So Γ_2^{SB} is more generous than Γ_2^{DC} for superadditive utilities, and less generous for subadditive ones.

Example 3: $n = 3, A = \{a, a\}$ *three agents, two identical goods*
We set $u(A) = u_2 = 1$ and $z = 2u_1 \in [0, 2]$ as above. Compute

$$\Gamma_3^{DC}(z) = \frac{1}{9}\min\{2z+1, z+2\} \text{ with the bid } \frac{|z-1|}{3} \text{ in D\&C}^1$$

z	0	$\cdots$	$\frac{2}{3}$	$\cdots$	$\frac{4}{3}$	$\cdots$	2
Γ_3^{SB}	$\frac{1}{5}$	$\frac{z+1}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{z}{4}$	$\frac{1}{2}$
x^*	$\frac{2}{5}$	$\frac{4-z}{10}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{z}{4}$	$\frac{1}{2}$

This time $\Gamma_3^{DC} > \Gamma_3^{SB}$ holds only in the interval $]1, \frac{8}{5}[$.

The striking feature in these two examples is that Γ^{SB} is exactly the Familiar Share on a large interval of the parameter z, while for Γ^{DC} this only happens for a single value of z.

Example 11.6.1 in the Appendix computes our guarantees with two agents and three identical goods and finds that the equality $\Gamma^{SB} = \Gamma^{FS}$ holds over 39% of the two parameter space. Here is the general fact explaining these observations.

Proposition 6
i) The guarantee Γ_n^{SB} (resp. Γ_n^{DC}) is identical to the Familiar Share on a subset $\mathcal{E}$ of $\mathcal{M}^+$ with non empty interior (resp. with empty interior)
ii) If the duality interval is not trivial ($minMax_n(u) < Maxmin_n(u)$) then Γ_n^{DC} is an interior point, but Γ_n^{SB} needs not be: it is one of the end points on a subset of $\mathcal{M}^+$ with non empty interior

Proof in subsection 11.7.

The first example in subsection 11.6 shows that the set of additive utilities is *not* entirely contained in the interior of $\mathcal{E}$: in words, arbitrarily small perturbations away from additivity can destroy this equality. The proof of Proposition 6 only shows that a neighborhood of the *uniform* additive utility ($\partial_a u = \partial_b u$ for all a, b) is contained in $\mathcal{E}$.

The same proof delivers a sufficient condition for the equality $\Gamma_n^{SB}(u) = \frac{1}{n}u(A)$ in two agent problems, and a complete characterisation if in addition the goods are identical.

Lemma 9 *for two agents $n = 2$*
i) Fix a utility $u \in \mathcal{M}^+$. If there exists a price vector p s.t.

$$p_A = \frac{1}{2}u(A) \text{ and } p_S \leq u(S) \leq p_S + \frac{1}{2}u(A) \text{ for all } S \subseteq A \tag{12}$$

then $\Gamma_2^{SB}(u) = \frac{1}{2}u(A)$ and p is a safe price for u if she is the Seller.
ii) If the goods are identical, the equality $\Gamma_2^{SB}(u) = \frac{1}{2}u_m$ holds true if u is bounded as follows:

$$\frac{1}{2}\frac{k}{m}u_m \leq u_k \leq \frac{1}{2}u_m + \frac{1}{2}\frac{k}{m}u_m \text{ for all } k, 0 \leq k \leq m \tag{13}$$

If u is convex $\frac{u_k}{k}$ increases weakly in k, so the inequalities (13) boil down to $\partial_1 u = u_1 \geq \frac{1}{2}\frac{u_m}{m}$; similarly if u is concave (13) reduces to $\partial_m u = u_m - u_{m-1} \geq \frac{1}{2}\frac{u_m}{m}$.

We turn to the opposite effect already illustrated in the Introduction for the very simple step functions: when it does not coincide with the Familiar Share, the Γ^{SB} guarantee is more sensitive than Γ^{DC} to the convexity/concavity of the utility. One formal way to say this is that the S&B guarantee diverges more from the FS benchmark than the D&C guarantee does.

Proposition 7

For all n and all $u \in \mathcal{M}^+$ we have

$$\frac{n}{(n-1)m+1} \leq \frac{\Gamma_n^{SB}(u)}{\frac{1}{n}u(A)} \leq \frac{n \times m}{n+m-1}$$

$$\frac{1}{n} \leq \frac{\Gamma_n^{DC}(u)}{\frac{1}{n}u(A)} \leq \frac{\min\{m,n\}+n-1}{n}$$

In both cases the bounds are achieved at u_G and u_F respectively.

Proof It is enough to show that u_G *and* u_F achieve those bounds: indeed every utility u in $\mathcal{M}^+$ s. t. $u(A)=1$ is between these two: $u_G \leq u \leq u_F$ and the two guarantees $\Gamma_n^{SB}, \Gamma_n^{DC}$ increase weakly in u and are scale invariant (Proposition 5).

Example 1 in section 7 gives $\Gamma_n^{SB}(u_F)$ for which the upper bound is an equality, and $\Gamma_n^{SB}(u_G)$ for which the lower bound is. The same situation applies to $\Gamma_n^{DC}(u_F)$ and $\Gamma_n^{DC}(u_G)$ after checking $Maxmin_n(u_F) = \min\{1, \frac{m}{n}\}$. ■

We see that the ratio $\frac{\Gamma_n^{DC}}{\Gamma_n^{FS}}$ is always below 2, while $\frac{\Gamma_n^{SB}}{\Gamma_n^{FS}}$ can be arbitrarily large.[7]

In the last example of this section the D&C guarantee is unpalatable because it does not take into account important aspects of the externalities across objects. This critique is more subtle than – but similar to – that of the Familiar Share by the way it treats Greedy and Frugal.

*Example 4***:** The goods are partitioned as $A = R \cup R^* \cup L \cup L^*$, where each subset contains ℓ objects so $m = 4\ell$. Think of 2ℓ right gloves and 2ℓ left gloves.

Abstemious is happy with any pair of one right and one left glove: her utility is the cover of $\{((r,\ell),1)\}$ over the whole set $(R \cup R^*) \times (L \cup L^*)$. Choosy wants no less than all gloves in $R \cup L$ or all in $R^* \cup L^*$: his utility is the cover of $\{(R \cup L, 1), (R^* \cup L^*, 1)\}$.

For both agents $minMax_2(u) = 0$, $Maxmin_2(u) = 1$, so $\Gamma_2^{DC}(u_A) = \Gamma_2^{DC}(u_C) = \frac{1}{4}$: the D&C guarantee is shockingly coarse, the more so as ℓ grows. By contrast one checks that the optimal bid in the S&B rule is $x^* = \frac{2\ell}{\ell+1}$ for *both* agents (almost twice larger than $u(A)$): to Abstemious this guarantees $\Gamma_2^{SB}(u_A) = \frac{\ell}{\ell+1}$,[8] but to Choosy only $\frac{1}{\ell+1}$.[9]

Subsection 11.6 gives three more examples contrasting the guarantees of our two rules.

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[7] The upper bound of $\frac{\Gamma_n^{SB}}{\Gamma_n^{FS}}$ is strictly larger than that of $\frac{\Gamma_n^{DC}}{\Gamma_n^{FS}}$, with a single exception at $n = m = 2$. The lower bound of $\frac{\Gamma_n^{SB}}{\Gamma_n^{FS}}$ is strictly lower than that of $\frac{\Gamma_n^{DC}}{\Gamma_n^{FS}}$ if $m \geq n+2$, strictly larger if $m \leq n$, and equal if $m = n+1$.

[8] because her worst case as Seller is to sell exactly $R \cup R^*$ or exactly $S \cup S^*$ for a net utility $\frac{x}{2}$; and as Buyer there will be at least one pair costing at most $\frac{2x}{m}$.

[9] because his worst case as Seller is to sell exactly one glove in $R \cup R^*$ and one in $S \cup S^*$ for a net utility $\frac{2x}{m}$; and as Buyer he will have to pay $\frac{x}{2}$ to get any benefit.

9 The efficiency of guarantees and safe play

Before running extensive numerical experiments, we provide only two systematic results, one about the efficiency loss measured from the profile of guarantees, the other about the outcome of safe play when one player's utility dominates that of all the others.

9.1 guarantees and the bargaining gap

Given a profile of utilities $\overrightarrow{u} \in (\mathcal{M}^+)^N$ we call the interval $G(\overrightarrow{u}) = [\min_{\mathcal{P}(N;A)} \overrightarrow{u}(\pi), \max_{\mathcal{P}(N;A)} \overrightarrow{u}(\pi)]$ the *bargaining gap*. It is the worst case efficiency loss resulting from a misallocation of the objects.

We say that the guarantee Γ_n reduces the bargaining gap if

$$\max_{\pi \in \mathcal{P}(N;A)} \overrightarrow{u}(\pi) \geq \sum_N \Gamma_n(u_i) \geq \min_{\pi \in \mathcal{P}(N;A)} \overrightarrow{u}(\pi) \text{ for all } \overrightarrow{u} \in (\mathcal{M}^+)^N \quad (14)$$

where the first inequality is just the definition (1) of a guarantee, so only the second inequality has bite.

For instance the FS $\frac{1}{n}u(A)$ meets (14) because $u_i(A) \geq \min_{\mathcal{P}(N;A)} \overrightarrow{u}(\pi)$ for each i.

Recall that Γ^{SB}, Γ^{DC} and Γ^{FS} meet the lower bound $\sum_N \Gamma_n(u_i) \geq \sum_N minMax_n(u_i)$ implied in the duality interval: this lower bound and (14) are not logically related. On the one hand if $A = \{a,b,c,d\}$, u_1 is the cover of $\{ab, bc, cd, ad\}$ and u_2 is the cover of $\{ac, ad, bc, bd\}$ (all with value 1), then $minMax_2(u_i) = 0$ for $i = 1,2$, but $\min_\pi \overrightarrow{u}(\pi)$ for all π; on the other hand if $A = \{a, a', b, b'\}$, u_1 is the cover of $\{a, a'\}$ and u_2 is the cover of $\{b, b'\}$, (all with value 1) then $\overrightarrow{u}(\pi) = 0$ if each agent gets useless goods, but $minMax_2(u_i) = \frac{1}{2}$ for $i = 1,2$.

Whether a guarantee systematically reduces the bargaining gap, and if so by how much, is one way to measure its efficiency performance.

Lemma 10
i) With two agents, $n = 2$, the S&B and D&C guarantees reduce the bargaining gap:
ii) With three or more agents, the D&C guarantee may not reduce the bargaining gap.

Proof *Statement i)*
Step 1: for S&B. Fix a profile u, v where u's optimal bid x^* wins against v's larger optimal bid y^*. Let $p \in \Delta(x^*)$ be such that $\Gamma_2^{SB}(u) = L(u; x^*) = (u-p)^+$. We increase p to some $q \in \Delta(y^*)$ so that $\Gamma_2^{SB}(u) \geq (u-q)^+$.

Then $\Gamma_2^{SB}(v) = W(v; y^*) \geq (v-q)^- + y^* = v(S) - q_S + y^*$ for some $S, \varnothing \subseteq S \subseteq A$. Also $\Gamma_2^{SB}(u) \geq u(S^c) - q_{S^c}$ so $\Gamma_2^{SB}(u) + \Gamma_2^{SB}(v) \geq u(S^c) + v(S)$.

Step 2: for D&C. Fix a profile u, v and let S, T be such that $\min_\pi [u](\pi) = u(S) + u(S^c)$ and $\min_\pi [v](\pi) = v(T) + v(T^c)$. The definition of Γ_2^{DC} implies

$$\Gamma_2^{DC}(u) \geq \frac{1}{4}(u(T) + u(T^c) + u(S) + u(S^c))$$

and a similar lower bound for $\Gamma_2^{DC}(v)$. Summing up these inequalities and rearranging gives the desired inequality (14).

Statement ii) Recall the step functions u^θ in Example 1 (section 7.2): the m goods are identical and $u^\theta(S) = 1$ if $|S| \geq \theta$, $= 0$ if $|S| < \theta$ (so u^1 is the frugal utility). The simplest[10] profile violating (14) for Γ_2^{DC} has three goods and the profile $\overrightarrow{u} = (u^2, u^2, u^1)$:

$$\Gamma_2^{DC}(u^2) = \frac{1}{9}\ ,\ \Gamma_2^{DC}(u^1) = \frac{5}{9} \text{ but } G(\overrightarrow{u}) = [1, 2]$$

■

We **conjecture** that the S&B guarantee reduces the bargaining gap for any n. Our intuition comes from the profiles $\overrightarrow{u} = (u^{\theta_i})_{i=1}^n$ of step functions in Example 1 over m identical goods, just used in the proof above. From the earlier computation $\Gamma^{SB}(u^\theta) = \frac{m+1-\theta}{m+1+(n-2)\theta}$ and the fact that $\min_\pi \overrightarrow{u}(\pi) = 1$ if and only if $\sum_{i=1}^n \theta_i \leq m + n - 1$ it is easy to deduce that the inequality in (14) follows from the convexity of $\theta \to \Gamma^{SB}(u^\theta)$ and is tight.

9.2 safe play when an agent dominates

For any two $u, v \in \mathcal{M}^+$ we say that u *dominates* v (resp. dominates strictly) if we have

$$\max_{\varnothing \subseteq S \subseteq A} \partial_a v(S) \leq \min_{\varnothing \subseteq S \subseteq A} \partial_a u(S) \text{ (resp. a strict inequality) for all } a \in A$$

If in the profile $\overrightarrow{u} = (u_i)_{i=1}^n$ utility u_1 dominates all u_i-s, $i \geq 2$, it is efficient to give all the goods to agent 1, strictly so if each domination is strict. This follows by repeated application of the inequality $u_1(S) + u_i(T) \leq u_1(S \cup a) + u_i(T \diagdown a)$ when S, T are disjoint and $a \in T$.

Our last result reveals another serious advantage of the Sell&Buy rule over the Divide&Choose rules.

Proposition 8 *Fix a profile $\overrightarrow{u} = (u_i)_{i=1}^n$ where utility u_1 dominates strictly all u_i-s, $i \geq 2$.*
i) the S&B division rule where all agents play safely implements the efficient outcome where agent 1 eats all the goods.
ii) the outcome of safe play in the D&C rules may only collect $\frac{1}{n}$-th of the efficient surplus

Proof of statement *i*) in subsection 11.8.

For statement *ii*) suppose that all agents have strictly subadditive utilities: nobody proposes the trivial partition with all goods in a single share, so efficiency is lost. In the worst case in each partition the shares are of equal value to the proposer, so we can lose all but $\frac{1}{n}$-th of the efficient surplus. With the D&C[1] rule, there is the additional possibility that the utility-dominant agent does not have the largest duality gap.

[10] A similar example applies for any $m = 2m' + 1$ at the profile $(u^{m'+1}, u^{m'+1}, u^1)$.

10 Future research

We have assumed throughout that objects are goods – freely disposable objects –. Other types of division problems include the division of bads (aka chores), of a mixture of goods and bads, even of objects of which the marginal utilities $\partial_a(S)$ can change sign for different S-s, as in the familiar example of utilities single-peaked or single-dipped over identical objects.

The definition of the two Divide&Choose rules is unchanged in these more complex domains. That of the Sell&Buy rule requires some adjustments. The fairness and efficiency performance of these rules will be the object of future research.

11 Appendix: missing proofs and more

11.1 the Multi Auction rule

Recall the definition of MA: each agent i bids on each good and his profile of bids is $\beta^i \in \mathbb{R}^A_+$; (one of) the highest bidders on a, agent i^*, gets it and pays $\frac{1}{n}\beta_{i^*a}$ to every other agent. Given the utility $u \in \mathcal{M}^+$ the safe vector of bids solves the program:

$$\Gamma_n^{MA}(u) = \max_{\beta\in\mathbb{R}^A_+}\{\min_{\varnothing\subseteq S\subseteq A}(u(S)-\frac{n-1}{n}\beta_S+\frac{1}{n}\beta_{S^c}\} = \max_{\beta\in\mathbb{R}^A_+}\min_{\varnothing\subseteq S\subseteq A}(u(S)-\beta_S)+\frac{1}{n}\beta_A \tag{15}$$

If our agent wins the auctions for the goods in S and those only, she pays $\frac{n-1}{n}\beta_a$ for each a in S, and gets in the worst case $\frac{1}{n}\beta_a$ for each a outside S.

The guarantee Γ_n^{MA} is Responsive but Not Positive: $\Gamma_n^{MA}(u_G) = 0 < \frac{1}{n} = \Gamma_n^{MA}(u_F)$. Indeed if Greedy's bid β is not zero, pick a such that $\beta_a = \min_{b\in A}\beta_b$, suppose Greedy wins all auctions except a and check that his worst utility is negative or zero. And Frugal's safe bid is $\frac{1}{n}$ on every good, securing $\frac{1}{n}$ in the worst cases where she wins all auctions or none of them.

Moreover Γ_n^{MA} is dominated by the *minMax* guarantee, often strictly so. To check the first claim pick any $u \in \mathcal{M}^+$ any partition $\pi = \{S_k\}_{k\in[n]}$ of A and any bid β: (15) implies $\Gamma_n^{MA}(u) \leq (u(S_k) - \beta_{S_k}) + \frac{1}{n}\beta_A$ for all k and the sum of these inequalities is $\Gamma_n^{MA}(u) \leq [u](\pi)$.

An example where domination is strict is the utility $u = u_F + u_G$. We let the reader check that $Maxmin_2(u) = minMax_2(u) = 1$ but $\Gamma_n^{MA}(u) = \frac{m}{2(m-1)}$.

11.2 proof of Lemma 6 and Proposition 3

Fixing A and a single utility $u \in \mathcal{M}^+$, the first step is to rewrite the programs (6) (7) in a more compact though less transparent format using a well known combinatorial concept.

A vector $\delta = (\delta_S) \in \mathbb{R}^{2^A}_+$ is a *balanced (set of) weights* if for all $a \in A$ we have $\sum_{S:S\ni a}\delta_S = 1$. We call δ *minimal* if it is an extreme point of the convex compact set of balanced weights, and write $\mathcal{B}_m$ the set of minimal balanced weights for m goods.[11] The simplest elements of $\mathcal{B}_m$ come from the *true partitions* $\{S_k\}$ of A, i.e. each S_k is non empty: set $\delta_{S_k} = 1$ for each k, and all other weights to 0. Let $\mathcal{B}^*_m$ be $\mathcal{B}_m$ minus the balanced weights $\delta_A = 1$ coming from the trivial partition $\{A\}$.

The total weight of δ is $\overline{\delta} = \sum_{S \in 2^A \smallsetminus \varnothing} \delta_S$. Then $\overline{\delta} > 1$ for each δ in $\mathcal{B}^*_m$. The smallest of these sums is $\overline{\delta} = \frac{m}{m-1}$ when $\delta_{A \smallsetminus a} = \frac{1}{m-1}$ for all a, and the largest one is $\overline{\delta} = m$ when $\delta_a = 1$ for all a. Both claims follow from the identity $\sum_{S \subsetneq A} |S| \cdot \delta_S = m$.

Lemma 11

The programs (6) (7) can be rewritten as follows:

$$W_2(u;x) = \min\{x, u(A), \min_{\mathcal{B}^*_m} \frac{1}{\overline{\delta}}(\delta \cdot u - x) + x\} \tag{16}$$

$$L_2(u;x) = \max\{0, u(A) - x, \max_{\mathcal{B}^*_m} \frac{1}{\overline{\delta}}(\delta \cdot u - x)\} \tag{17}$$

Proof.

We write $\nabla(Z)$ for the set of convex weights on Z. We apply the minmax theorem to the maxmin expression in (6):

$$W_2(u;x) - x = \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (u(T) - p_T) = \max_{p \in \Delta(x)} \min_{\xi \in \nabla(2^A)} \sum_{T \in 2^A} \xi_T (u(T) - p_T)$$

where ξ has two coordinates ξ_A and $\xi_\varnothing$.

Note that the mapping $\nabla(2^A) \ni \xi \rightarrow \zeta \in \mathbb{R}^A : \zeta_a = \sum_{T: a \in T} \xi_T$ is onto $[0,1]^A$, and apply the minimax theorem to rewrite the last maxmin term above as

$$\min_{\xi \in \nabla(2^A)} \max_{a \in A} \sum_{T \in 2^A} \xi_T u(T) - x\zeta_a \Longrightarrow W_2(u;x) = \min_{\xi \in \nabla(2^A)} \sum_{T \in 2^A} \xi_T u(T) + x(1 - \min_a \varsigma_a)$$

We check now that for some optimal ξ in the minimisation program above ζ_a is independent of a. Assume $\zeta_a > \min_b \zeta_b$ where the minimum is achieved by some good b^*. We can choose S containing a but not b^* and such that $\xi_S > 0$: if this was impossible $\zeta_a \leq \zeta_{b^*}$ would follow. For ε small enough we construct $\xi' \in \nabla(2^A)$ identical to ξ except for $\xi'_S = \xi_S - \varepsilon$, $\xi'_{S \smallsetminus \{a,b^*\}} = \xi'_{S \smallsetminus \{a,b^*\}} + \varepsilon$. By construction $\zeta'_{b^*} = \zeta_{b^*}$ and the same holds true for any b minimising ζ: thus $\min \zeta' = \min \zeta$ for ε small enough and the net change on the objective is $-\varepsilon u(S) + \varepsilon u(S \smallsetminus \{a, b^*\}) \leq 0$.

If ξ is deterministic on $\varnothing$ or on A, we get the first two terms in (16). For another ξ we can assume ξ puts no weight on $\varnothing$ or on A, and write $\zeta \in [0,1]$ for the common value ζ_a. Setting $\delta = \frac{1}{\zeta}\xi$ defines a balanced set of weights and $\sum_T \xi_T u(T) + x(1-\zeta) = \zeta(\delta \cdot u) + (1-\zeta)x$. Without loss we can minimise over minimal balanced weights. Finally $\overline{\delta} = \frac{1}{\zeta}$ and the proof of (16) is complete.

[11] The size of $\mathcal{B}$ grows astronomically fast with m: $|\mathcal{B}| = 2$ for $m = 2$, $= 6$ for $m = 3$, $= 27$ for $m = 4$ and more than $15,000$ for $m = 5$: see [16].

The similar argument for (17) starts with

$$L_2(u,x) = \min_{p\in\Delta(x)} \max_{\xi\in\nabla(2^A)} \sum_{T\in 2^A} \xi_T(u(T) - p_T) = \max_{\xi\in\nabla(2^A)} \{\sum_{T\in 2^A} \xi_T u(T) - x(\max_{a\in A} \zeta_a)\}$$

The critical argument that we can take ζ_a independent of a assumes $\zeta_a < \max \zeta = \zeta$, picks S s. t. $\xi_S > 0$ and containing b^* but not a and changes ξ by $\xi'_S = \xi_S - \varepsilon$, $\xi'_{S\cup a} = \xi_{S\cup a} + \varepsilon$: for ε small enough the $\max \zeta$ does not change and the net change on the objective is at least$-\varepsilon u_S + \varepsilon u_{S\cup a} \geq 0$. ■

Equation (16) defines a concave function. Each term in x increases strictly because $\overline{\delta} > 1$ and reaches $u(A)$ for x large enough, therefore $W_2(u;x)$ increases strictly up to $u(A)$. Similarly in (17) $L_2(u;x)$ is convex and strictly decreasing as long as all terms in x are positive, which terminates for x large enough. So the intersection of $W_2(u;\cdot)$ and $L_2(u;\cdot)$ as $\Gamma_2^{SB}(u|A)$ is well defined.

We proceed now by induction after checking that the function $S \to \Gamma_2^{SB}(u|S)$ is in $\mathcal{M}^+(A)$. Going back to the definition (6) we see that $W_2(u;x|S)$ increases weakly in S because agent u can choose in the problem augmented to $S\cup a$ a price s. t. $p_a = 0$; and so does $L_2(u;x|S)$ by (7) because in the augmented problem the agent can choose only among subsets not containing a. Both $W_2(u;x|S)$ and $L_2(u;x|S)$ increase weakly in S, so their intersection in x increases too.

The induction step applies Lemma 11 to $\Gamma_{n-1}^{SB}(u|\cdot) \in \mathcal{M}^+(A)$ and gives $W_n(u;x|A), L_n(u;x|A)$ by the two programs

$$W_n(u;x|A) = \min\{x, u(A), \min_{\mathcal{B}_m} \frac{1}{\overline{\delta}}(\delta \cdot \Gamma_{n-1}^{SB}(u|\cdot) - x) + x\} \tag{18}$$

$$L_n(u;x|A) = \max\{0, u(A) - (n-1)x, \max_{\mathcal{B}_m} \frac{1}{\overline{\delta}}(\delta \cdot u - (n-1)x)\} \tag{19}$$

with the properties announced in Lemma 6, and their intersection $\Gamma_n^{SB}(u|\cdot)$ as a function in $\mathcal{M}^+(A)$.

Turning to Proposition 3. If $u(A) > 0$ both functions $W_n(u;x), L_n(u;x)$ are strictly positive for x small enough, proving Positivity. For Responsiveness we compute formally $\Gamma_2^{SB}(u_F)$ (more rigorously than in the Introduction). First (16) gives $W_2(u_F;x) = \min\{x, 1\}$ because the smallest $\overline{\delta}$ in $\mathcal{B}_m^*$ is $\frac{m}{m-1}$ and $L_2(u_F;x) = \max\{0, 1 - \frac{1}{m}x\}$ because the largest $\overline{\delta}$ in $\mathcal{B}_m^*$ is m. This shows $\Gamma_2^{SB}(u_F|S) = \frac{|S|}{|S|+1}$

We omit the straightforward induction argument giving $\Gamma_n^{SB}(u_F|S) = \frac{|S|}{|S|+n-1}$.

It remains to check $\Gamma_n^{SB}(u) \geq minMax_n(u)$ for all u and n. This is true for $n = 1$. Assume next it holds for Γ_{n-1}^{SB} and pick any $u \in \mathcal{M}^+(A)$ with optimal bid x^* where W_n and intersect. Choose $p \in \Delta(x^*)$ optimal in program (9) so that $L_n(u;x^*|A) = \max_{\varnothing\subseteq S\subseteq A}(u(S) - (n-1)p_S)$. Then (8) and the inductive argument imply

$$W_n(u;x^*) \geq \min_{\varnothing\subseteq S\subseteq A}(\Gamma_{n-1}^{SB}(u|S) - p_S) + x^* = \Gamma_{n-1}^{SB}(u|T) - p_T + x^* \text{ for some } T$$

$$\Longrightarrow W_n(u;x^*) \geq \frac{1}{n-1}[u](\pi) - p_T + x^* \text{ where } \pi \text{ is some } (n-1)\text{-partition of } T$$

We can now combine this lower bound for $(n-1)W_n(u;x^*)$ with $L_n(u;x^*) \geq u(T^c) - (n-1)p_{T^c}$ to get $n\Gamma_n^{SB}(u) \geq [u](\pi) + u(T^c)$ which completes the proof.

11.3 proof of Lemma 7

For brevity we give the argument for $n = 2$ and omit the obvious induction argument.

Fix $u \in \mathcal{M}^+$ and assume u is symmetric in the goods a, b. In the program (*7*) defining $L_2(u;x)$ assume the worst price p has $p_a < p_b$. Let q obtains from p by averaging p_a and p_b and changing nothing else. Then $(u-p)^+$ differs from $(u-q)^+$ only in pairs of terms of the form $u(S) - p_S$, $u(S-b+a) - p_{S-b+a}$. Replacing p by q lowers the largest of these two terms, so q is still optimal in the program (*7*). The argument for $W_2(x;u)$ is identical.

The group version of the Lemma is very useful and almost as easy to prove. *Suppose two subsets T, T^* of equal size are treated symmetrically by u: for any permutation σ of A exchanging T and T^* and leaving other goods unchanged we have $u(\sigma(S)) = u(S)$ for all S. Then to solve $W_2(x;u)$ and $L_2(x;u)$ we can take identical prices for all goods in $T \cup T^*$.*

11.4 proof of Proposition 4

By the symmetry Lemma the programs (6) and (7) simplify to

$$W_2(u;x) = \min_{0\leq k\leq m}\{u_k + \frac{m-k}{m}x\}\ ;\ L_2(u;x) = \max_{0\leq k\leq m}\{u_k - \frac{k}{m}x\}$$

The optimal bid x^* solves $W_2(u;x^*) = L_2(u;x^*)$. Because W_2 increases and L_2 decreases, both strictly, the inequality $x \geq x^*$ is equivalent to $W_2(u;x) \geq L_2(u;x)$. If $k' \leq k$ the inequality $u_k + \frac{m-k}{m}x \geq u_{k'} - \frac{k'}{m}x$ is automatic, therefore $x \geq x^*$ amounts to

$$u_\ell + \frac{m-\ell}{m}x \geq u_{\ell+k} - \frac{\ell+k}{m}x \text{ for all } k, \ell \geq 0 \text{ s. t. } \ell + k \leq m$$

$$\Longleftrightarrow x \geq \max_{0\leq \ell+k\leq m} \frac{m}{m+k}(u_{\ell+k} - u_\ell)$$

which proves (10) and in turn (11).

Remark: a similar computation using the recursive definition of S&B for three agents gives the optimal bid $x^* = \max_{0\leq k+k'\leq m} \frac{m}{m+2k'-k}(u_{k'} - \Gamma_2^{SB}(u|k))$, where $\Gamma_2^{SB}(u|k)$ refers to the restriction of the problem to just k goods.

11.5 proof of Proposition 5

Proof. Statement $i)$ is clear for Γ_n^{DC} and Γ_n^{FS}; for Γ_n^{SB} both functions $W_n(u;\cdot)$ and $L_n(u;\cdot)$ increase weakly in u, so their intersection does too.

Statement ii). For Γ_n^{DC} scale invariance is clear. For the first inequality one checks easily that both $Maxmin_n(u|A)$ and $minMax_n(u|A)$ increase weakly in A. For the second inequality observe that for any $\pi^* \in \mathcal{P}(n+1;A)$ we can find a share S in π^* such that $[u](\pi^*) - u(S) \geq \frac{n}{n+1}[u](\pi^*)$; as $[u](\pi^*) - u(S) \leq \max_{\pi \in \mathcal{P}(n;A)}[u](\pi)$ we see that $Maxmin_n(u)$ decreases weakly in n. This is clearly true for $minMax_n(u)$ as well, and in turn for Γ^{DC}.

For Γ^{SB} scale invariance is routinely checked. The monotonicity in A was proven in subsection 11.2, shortly after the proof of Lemma 11.

For the monotonicity in n: taking $T = A$ in the minimisation part of program (8) gives $W_{n+1}(u;x) \leq \Gamma_n^{SB}(u)$ for all x, in particular at the x^* optimal in the problem with $n+1$ agents. ■

11.6 three more examples

11.6.1 *two agents, three identical goods:* $n = 2, A = \{a, a, a\}$

The utilities (u_1, u_2, u_3) cover the cone $0 \leq u_1 \leq u_2 \leq u_3$. The D&C guarantee is simply additive: $\Gamma_3^{DC}(u) = \frac{1}{4}(u_1 + u_2 + u_3)$ because $[u](\pi)$ takes only two values; it coincides with $\Gamma_2^{FS}(u) = \frac{1}{2}u_3$ if and only if u is additive.

Contrast this with $\Gamma_2^{SB}(u)$ which is piecewise linear – linear on 6 subcones of utilities – and coincides with the familiar share first in a thick cone where the ratios $\frac{u_1}{u_3}$ and $\frac{u_2}{u_3}$ are both not far from the additive benchmark, but also in a slice of additive utilities where these ratios are "extreme":

$$\Gamma_2^{SB}(u) = \frac{1}{2}u_3 \Longleftrightarrow \{\frac{1}{6}u_3 \leq u_1 \leq \frac{2}{3}u_3 \text{ and } \frac{1}{3}u_3 \leq u_2 \leq \frac{5}{6}u_3\} \text{ or } \{u_1 + u_2 = u_3 \text{ and } u_1 \leq \frac{1}{6}u_3\}$$

The thick subcone occupies 39% of the entire cone of utilities (computed in any slice where u_3 is constant).

11.6.2 a one-dimensional path from u_F to u_G

We have two agents and identical goods sufficiently numerous to justify a smooth approximation. we construct a parametrised family of utilities connecting Frugal to Greedy.

Part 1. Utilities with decreasing concavity from Additive at $\lambda = 0$ to Frugal as λ goes to 1

$$u_k = \frac{k}{(1-\lambda)m + \lambda k} \text{ for } 0 \leq k \leq m \text{ so } u(A) = u_m = 1$$

The Corollary of Proposition 4 shows that Γ_2^{SB} is the FS $\frac{1}{2}u_m$ iff $\lambda \leq \frac{1}{2}$:

$$\Gamma_2^{DC}(\lambda) = \frac{4-\lambda}{4(2-\lambda)} \text{ ; } \Gamma_2^{SB}(\lambda) = \max\{\frac{\sqrt{\lambda}}{\sqrt{\lambda} + \sqrt{1-\lambda}}, \frac{1}{2}\}$$

Part 2.: Utilities with increasing convexity from Additive at $\mu = 0$ to Greedy for μ going to 1

$$u_k = \frac{(1-\mu)k}{m - \mu k} \text{ for } 0 \leq k \leq m$$

Here too $\Gamma_2^{SB} = \Gamma_2^{FS} = \frac{1}{2}$ iff $\mu \leq \frac{1}{2}$

$$\Gamma_2^{DC}(\mu) = \frac{4-3\mu}{4(2-\mu)} \ ; \ \Gamma_2^{SB}(\mu) = \frac{1}{2} \text{ in } [0, \frac{1}{2}] \ ; \ = \frac{2\sqrt{\mu(1-\mu)}}{(\sqrt{\mu}+\sqrt{1-\mu})^2} \text{ in } [\frac{1}{2}, 1]$$

11.6.3 a versatile dichotomous example

We have $m = h \times \ell$ goods and $A = \cup_{k=1}^{h} B_k$ is partitioned into $h \geq 2$ subsets B_k each of size $\ell \geq 2$. The utility u is the cover of the sequence $\{(B_k, 1)\}$.

Computing Γ_n^{DC} is easy: $minMax_n(u) = 0$, $Maxmin_n(u) = \min\{1, \frac{h}{n}\}$ therefore

$$\Gamma_n^{DC}(u) = \frac{\min\{h, n\}}{n^2}$$

in particular $\Gamma_n^{DC}(u) = \Gamma_n^{FS}(u)$ if and only if $n \leq h$.

For S&B we compute first $W_2(u; x)$ by the group version of the symmetry Lemma 7 (subsection 11.3): each object is priced $\frac{x}{h \times \ell}$ then $W_2(u; x) = \frac{x}{\ell}$ because in the worst case the Seller sells one object in each B_k; and $L_2(u; x) = 1 - \frac{x}{h}$ because each set B_k costs $\frac{x}{h}$.

So $\Gamma_2^{SB}(u) = \frac{h}{h+\ell}$ and a routine induction argument shows

$$\Gamma_n^{SB}(u) = \frac{h}{h + (n-1)\ell}$$

in particular $\Gamma_n^{SB}(u) = \Gamma_n^{FS}(u)$ if and only if $h = \ell$.

So for this family of utilities Γ_n^{DC} equals the Familiar Share much more frequently than Γ_n^{SB}.

11.7 proof of Proposition 6 and Lemma 9

11.7.1 Lemma 9

For statement *i)* suppose u and the price p meet the inequalities (12). The left one implies $\min_{\varnothing \subseteq S \subseteq A}(u(S) - p_S) = 0$ hence $W_2(u; \frac{1}{2}u(A)) \geq \frac{1}{2}u(A)$ and in fact $W_2(u; \frac{1}{2}u(A)) = \frac{1}{2}u(A)$ because $W_2(u; x) \leq x$ is always true.

Next the right hand inequality gives $L_2(u; \frac{1}{2}u(A)) \leq \max_{\varnothing \subseteq S \subseteq A}(u(S) - p_S) \leq \frac{1}{2}u(A)$, but $L_2(u; x) \geq u(A) - x$ is always true therefore $L_2(u; \frac{1}{2}u(A)) = \frac{1}{2}u(A)$ and the desired conclusion $\Gamma_2^{SB}(u) = \frac{1}{2}u(A)$ holds. Checking that p is the optimal selling price is now easy.

Statement *ii)* then follows from Proposition 4 and the fact that if goods are identical the inequalities (12), if they hold at all, must hold for the symmetric price $p_a = \frac{1}{2m}u(A)$ for all a.

11.7.2 Proposition 6

Step 1. Fixing n and A we show that $\Gamma_n^{SB}(u|A) = \frac{1}{n}u(A)$ if the utility u satisfies the following system of (many) inequalities. For all $n', 2 \leq n' \leq n$, and all S,

$\varnothing \neq S \subseteq A$ there exists a price $p \in \mathbb{R}^S_+$ s. t.

$$p_S = \frac{1}{n'}u(S) \text{ and } (n'-1)p_T \leq u(T) \leq (n'-1)p_T + \frac{1}{n'}u(S) \text{ for all } T \subsetneq S \quad (20)$$

We prove the claim by induction on n'. For $n' = 2$ the system above is exactly (12) when we replace A by S, so by Lemma 9 $\Gamma_2^{SB}(u|S) = \frac{1}{2}u(S)$ for all S, and for $S = A$ this is our claim.

For $n' = 3$ we fix again S, $\varnothing \neq S \subseteq A$, and the previous step allows us to simplify $W_3(u;x|S)$ in (**??**):

$$W_3(u;x|S) = \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq S} (\frac{1}{2}u(T) - p_T) + x$$

so that (20) applied to $n' = 3$ and the corresponding price p implies $W_3(u; \frac{1}{3}u(S)|S) \geq \frac{1}{3}u(S)$ hence an equality because $W_3(u;x|S) \leq x$ for all x. We use next (**??**) to majorise $L_3(u; \frac{1}{3}u(S)|S)$ by $\max_{\varnothing \subseteq T \subseteq S}(u(T) - 2p_T)$, where $2p_S = \frac{2}{3}u(S)$; then (20) to deduce $L_3(u; \frac{1}{3}u(S)|S) \leq \frac{1}{3}u(S)$, which is an equality because $L_3(u;x|S) \geq u(S) - 2x$ is always true. We have just proven $\Gamma_3^{SB}(u|S) = \frac{1}{3}u(S)$ for all S and the claim for $n' = 3$ by taking $S = A$.

The induction argument is now clear.

Step 2. Fixing n and m we choose a small positive number ε and consider the subset of utilities in $\mathcal{M}^+$ such that $u(A) = 1$ and

$$\frac{s}{m}(1-\varepsilon) \leq u(S) \leq \frac{s}{m}(1+\varepsilon) \text{ for all } S \subseteq A \text{ with } |S| = s \quad (21)$$

They approximate the additive utility $u_a \equiv 1$. If such a utility satisfies (20) for some price p, the desired equality $\Gamma_n^{SB}(u;A) = \frac{1}{n}u(A)$ follows by step 1. By scale invariance of Γ_n^{SB} it extends to the cone they generate as well, a subset with non empty interior in $\mathcal{M}^+$.

We choose p as $p_T = \frac{\tau}{n's}u(S)$ where $\tau = |T|$ and (20) becomes

$$\frac{n'-1}{n'}\frac{\tau}{s}u(S) \leq u(T) \leq (\frac{n'-1}{n'}\frac{\tau}{s} + \frac{1}{n'})u(S)$$

The left hand inequality holds if we replace $u(S)$ by its upper bound and $u(T)$ by its lower bound in (21). This boils down to $\frac{1+\varepsilon}{1-\varepsilon} \leq 1 + \frac{1}{n'-1}$. A similar evaluation of the right hand inequality gives $\frac{1+\varepsilon}{1-\varepsilon} \leq 1 + \frac{1}{n'}(\frac{s}{t} - 1)$. The worst case happens for $n' = n, s = m, \tau = m-1$ and it is $\frac{1+\varepsilon}{1-\varepsilon} \leq 1 + \frac{1}{(n-1)(m-1)}$. ∎

11.8 proof of Proposition 8 statement $i)$

We assume $n = 2$ and omit for brevity the straightforward induction argument extending the result to any n.

Fix any $u \in \mathcal{M}^+$; from Lemma 6 and Lemma 11(in the proof of Lemma 6) we know that $W_2(u;x)$ reaches $u(A)$ at some finite point denoted $\widetilde{x}(u)$: $W_2(u;\cdot)$

increases strictly up to $\widetilde{x}(u)$ after which it is flat. Agent u's optimal bid $x^*(u)$ in S&B is strictly below $\widetilde{x}(u)$ (because $W_2(u;\widetilde{x}(u)) = u(A) > L_2(u;\widetilde{x}(u))$.

Below we use the shorthands $\partial_a^+ u = \max_{\varnothing\subseteq S\subseteq A} \partial_a u(S)$ and $\partial_a^- u = \min_{\varnothing\subseteq S\subseteq A} \partial_a u(S)$.

Step 1 Fix u and $x \leq \widetilde{x}(u)$ and suppose that in the program (6) an optimal price is $p \in \Delta(x)$. Then $p_a \leq \partial_a^+ u$ for all a.

Proof by contradiction: we assume $p_a > \partial_a^+ u$ for some a and define a new price p' s. t. $p'_a = p_a - \varepsilon$ and $p'_b = p_b$ otherwise; we choose $\varepsilon > 0$ small enough that $p'_a > \partial_a^+ u$ still holds. For some $T \in 2^A$ we have $\min_{\varnothing\subseteq S\subseteq A}(u(S) - p'_S) = u(T) - p'_T$. This implies $a \in T$ otherwise adding a to T would contradict the optimality of T. We compute now

$$W_2(u; x-\varepsilon) \geq u(T) - p'_T + (x-\varepsilon) = u(T) - p_T + x$$

$$\geq \min_{\varnothing\subseteq S\subseteq A}(u(S) - p_S) + x = W_2(u;x)$$

We see that $W_2(u;\cdot)$ is flat before x therefore $x > \widetilde{x}(u)$ contradicting the choice of x.

Step 2 Assume u_1 dominates u_2 strictly.

A first consequence is $L_2(u_1;x) > L_2(u_2;x)$ for all $x \leq x^*(u_2)$. Indeed $u_1(S) - p_S > u_2(S) - p_S$ for all non empty S and $p \in \Delta(x)$, and $L_2(u_2;x)$ is positive therefore for any $p \in \Delta(x)$ the maximum of $u_2(S) - p_S$ is achieved at some non empty S.

Next we pick $p \in \Delta(x^*(u_2))$ optimal in (6) for u_2. By step 1 and inequality $x^*(u_2) < \widetilde{x}(u_2)$ we have $p_a \leq \partial_a^+ u_2 < \partial_a^- u_1$ for all a, implying $u_1(S) > p_S$ for all non empty S. We have now

$$W_2(u_1; x^*(u_2)) \geq \min_{\varnothing\subseteq S\subseteq A}(u(S) - p_S) + x^*(u_2) = x^*(u_2)$$

Because $W_2(u_1;y) \leq y$ for all y we see that $W_2(u_1;y) = y \geq W_2(u_2;y)$ for all $y \leq x^*(u_2)$.

Gathering the first and last statements in this step we conclude that $L_2(u_1;\cdot)$ and $W_2(u_1;\cdot)$ intersect beyond $x^*(u_2)$ so agent u_1's safe bid makes her the Seller in stage 2. We showed a few lines ago $u_1(S) > p_S$ for any S and any possible price charged by agent u_2 therefore agent u_1 will buy all the goods and the proof is complete. ■

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