Evaluating Optimal Stopping problems under Multivariate Settings and Model Uncertainty *†

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Abstract

In this article we study the evaluation of American options with stochastic volatility models and
the optimal fish harvesting decision with stochastic convenience yield models, in the presence of drift
ambiguity. From the perspective of an ambiguity averse agent, we transfer the problem to the solution
of a reflected backward stochastic differential equations (RBSDE) and prove the uniform Lipschitz
continuity of the generator. We then propose a numerical algorithm with the theory of RBSDEs and a
general stratification technique, and an alternative algorithm without using the theory of RBSDEs. We
test the accuracy and convergence of the numerical schemes. By comparing to the one dimensional case,
we highlight the importance of the dynamic structure of the agent’s worst case belief. Results also show
that the numerical RBSDE algorithm with stratification is more efficient when the optimal generator is
attainable.

Keywords: American option; optimal fish harvesting; stochastic volatility; stochastic convenience
yield; model uncertainty; reflected backward stochastic differential equation; Monte Carlo; stratification

1 Introduction

Classic articles on derivatives pricing often assume that a pricing measure exists and specific events occur
with a certain probability distribution under that measure. However, the probability distribution is not
usually known a priori in reality, which is named uncertainty or ambiguity. We are interested in numerically
evaluating financial American options and real life investments i.e. real options of American style under
specific models that accommodate ambiguity, especially in multivariate settings.

There are two main directions of theoretical settings for describing ambiguity and corresponding optimal
strategies. One is to postulate a set of equivalent probability measures, which is known as multiple priors or
drift ambiguity, given the existence of a dominating probability measure. Gilboa & Schmeidler (1989) ax-
iomatize the agent’s robust decision and introduce the maximin method under drift ambiguity. It describes
that the mother nature drives the agent to evaluate the payoff or utility function with the worst belief,
and then seek for the optimal strategy to maximize it. The other direction is to assume the absence of a
dominating probability measure, which means that all priors are not necessarily absolutely continuous to a
reference measure. This leads to the capability of involving sources of uncertainty on the standard deviation
of a distribution, i.e. volatility ambiguity. One can find the first attempts to attack the pricing problem
in such a context in Avellaneda et al. (1995) and Lyons (1995). Specifically, this paper considers valuation
of American options under stochastic volatility model and optimal fish harvesting decision under stochastic
convenience yield model, both within drift ambiguity framework.

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The models used in this paper are related to two main streams of previous literature. First, the feature of stochastic volatility is a generalization of the famous constant-volatility model in Black & Scholes (1973). As the stochastic volatility models are capable of better capturing the implied volatility smile and the leverage effect, there are substantial classes of models that explicitly parameterize the stochastic volatility, for example, models introduced by Stein & Stein (1991), Heston (1993), Bates (1996), etc. Nevertheless, all of them concentrate on European options. It is natural but challenging to extend to the American case, because there is no analytical solution for the prices of American options, even under the simplest Black-Scholes model. Second, the advantages of stochastic convenience yield model over the Black-Scholes model in pricing commodity derivatives are initially indicated by Gibson & Schwartz (1990) and now well known (refer to Schwartz & Smith (2000), Ewald et al. (2019) and Moreno et al. (2019) for instance). It is widely applied to the field of real options, which can be understood as an adaption of methods of traditional financial options to real life investments (see e.g. Dixit & Pindyck (1994), Cortazar et al. (2008) numerically value the copper mine by using a stochastic convenience yield model. Ewald et al. (2016) investigate the optimal fish harvesting problem and evaluate the fish farm under stochastic convenience yield model numerically.

As for the specifications of drift ambiguity, we employ the \( \kappa \)-ignorance introduced by Chen & Epstein (2002) in one dimensional case, so that the density generator  does lie in an interval that centers at origin with radius \( \kappa \). Nishimura & Ozaki (2007) and Trojanowska & Kort (2010) adopt such a setting in optimal investment decision in real options. They highlight that an increase of drift ambiguity leads to a lower value of investment, which is in sharp contrast of the effect of an increase in risk. Cheng & Riedel (2013) and Vorbrink (2011) address the optimal stopping problem of exotic options under the Black-Scholes model. For the multivariate case, we consider an ellipsoid shape of uncertainty set, which is initially proposed by Goldfarb & Iyengar (2003) on the portfolio optimization problems. Similar approaches have been taken by Cohen & Tegnér (2017) for European options pricing and Balter & Pelsser (2020) for hedging strategy. Note that the ellipsoid uncertainty set reduces to an ellipse in two dimensions. Thus, we name it the elliptical ambiguity. To the best of our knowledge, our work is the first attempt to valuing American options under stochastic volatility or real options of American style under stochastic convenience yield with drift ambiguity.

We investigate the formulation of the optimal value of American option and American real option, for which we take a single-rotation fish farm valuation as an example. Without ambiguity, it is well known that the value is essentially a supermartingale and has a dual representation of a reflected backward stochastic differential equation(RBSDE), according to El Karoui, Pardoux, & Quenez (1997) and El Karoui, Kapoudjian, Pardoux, Peng, & Quenez (1997). We link the multivariate case with the elliptical ambiguity to the solutions of RBSDEs and prove the existence and uniqueness, provided the foundations of one dimensional case by Cheng & Riedel (2013) and Vorbrink (2011).

The question remains how we numerically evaluate the optimization function given the correct formulation. We propose two possible algorithms to conduct numerical implementations. The first one, Stratified Regression One-step Forward Dynamic Programming(SRODP), is constructed through the "Max method" for RBSDEs by Gobet & Lemor (2008), joint with a general stratification approach by Gobet et al. (2016). Algorithms for RBSDEs are based on the convergence results by Ma & Zhang (2005) and Bouchard & Chassagneux (2008). The general stratification approach is a sampling method to approximate the object conditional expectation function by local polynomials, which differs from the conventional Monte Carlo methods (see Tsitsiklis & Van Roy (2001), Longstaff & Schwartz (2001) and Glasserman (2013) to name a few) in not depending on the starting point and starting value. Gobet et al. (2016) implement it in multivariate European option case and underline its superiority in conserving computational memory and efficiency, especially when enabling GPU computing. The idea behind the general stratification approach also contributes to the second algorithm, Stratified Least Square Monte Carlo(SLSM), which is a combination of the classic dynamic programming principle and that approach.

We conduct numerical experiments in non-ambiguous cases to show the convergence of algorithms to the exact value of European and American options. In cases with ambiguity, results by the two proposed algorithms are close to the benchmark by Least Square Monte Carlo (LSM) algorithm (Longstaff & Schwartz 2001) under Black-Scholes model. Moreover, it is observed that the American option value increases when shrinking the uncertainty interval. This is in accordance with theoretical analysis in Cheng & Riedel (2013) and Vorbrink (2011). However, we do not have an exact value under multivariate settings as we argue

\[ \text{Gibson & Schwartz (1990) describe the convenience yield as "the flow of services accruing to the holder of the spot commodity but not to the owner of a futures contract".} \]
in Section 3.4 Results for values of American options in that case show that the SRODP algorithm has superior efficiency than the SLSM algorithm, which is confirmed by the fish farm valuation case. In spite of it, the SLSM algorithm still provides possible solution in scenarios when the optimal generator of the RBSDE cannot be solved explicitly.

The remainder of this paper is constructed as follows. Section 2 formulate the American option value under Heston’s model to the solution of an RBSDE, obtain the optimal driver within the elliptical ambiguity framework and prove the existence and uniqueness of the solution. Section 3 introduces the two algorithms. Section 4 presents numerical results of financial options. Section 5 demonstrates the application of theoretical results and algorithms in optimal fish harvesting problem. The last section concludes.

2 Theoretical Framework

2.1 American Option Pricing

We characterize the probability space by the triplet \((\Omega, \mathcal{F}, P)\), and let \((\mathcal{F}_t)_{0 \leq t \leq T}\) be the augmented filtration generated by a standard two dimensional Brownian motion \(W = (W^1, W^2)^*\). Without uncertainty in probability measures, the dynamics of the price process \(S_t\) and the variance process \(V_t\) in Heston (1993) under the objective measure \(P\) are denoted as,

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{V_t} (\rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t), \\
\frac{dV_t}{V_t} &= \alpha (\beta - V_t) dt + \sigma \sqrt{V_t} dW^1_t.
\end{align*}
\]

There exists a money market account \(\gamma\) with a risk-free rate \(r\), and \(\gamma\) evolves according to,

\[
d\gamma_t = r\gamma_t dt, \quad \gamma_0 = 1.
\]

The payoff function of an option is

\[
H_t := H(t, X_t) = \Phi(X_t),
\]

if we denotes \(X_t = (S_t, V_t)^*\). To price an option, we need to choose an equivalent martingale measure, under which the discounted option price \(J_t\) is the smallest supermartingale dominating the discounted payoff.(see for example Chapter 5 of Pham (2009)) Denote a stopping time \(\tau\) as the exercise time, the option price \(J_\tau\) is denoted as,

\[
J_t := J(t, X_t) = \text{ess sup}_{\tau \in \mathcal{S}} \mathbb{E}^Q[H_{\tau \wedge \gamma^{-1}}],
\]

where \(\mathcal{S}\) is the set of all stopping times dominated by \(T\), and \(\mathcal{S} = \{\tau \in \mathcal{T}; t \leq \tau \leq T\}\). The conditional Radon-Nikodym derivative required for the change of measure is defined as,

\[
\frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} = \mathcal{E}\left(-\int_0^t \lambda_s^* dW_s\right)_t,
\]

where the stochastic exponential \(\mathcal{E}(\cdot)_t\) of \(-\int_0^t \lambda_s^* dW_s\) is a \(P\)-martingale, given by,

\[
\mathcal{E}\left(-\int_0^t \lambda_s^* dW_s\right)_t = \exp\left(-\int_0^t \lambda_s^* dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds\right).
\]

In addition, we still need the stochastic exponential to satisfy technical conditions such as progressively measurability and the Novikov condition to have \(Q\) as an equivalent measure to \(P\) (see for example Chapter 3 of Karatzas & Shreve (1998)).

Therefore, the price process and the variance process under the risk-neutral measure \(Q\) are given as,

\[
\begin{align*}
\frac{dS_t}{S_t} &= r dt + \sqrt{V_t} (\rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t), \\
\frac{dV_t}{V_t} &= \hat{\alpha} (\hat{\beta} - V_t) dt + \sigma \sqrt{V_t} d\tilde{W}^1_t.
\end{align*}
\]
By the Girsanov theorem, a vector of market price of risk process \( \lambda_t = (\lambda^1_t, \lambda^2_t)^* \) is given as,

\[
\begin{align*}
\mathrm{d} \tilde{W}^1_t &= \lambda^1_t \mathrm{d} t + \mathrm{d} W^1_t, \\
\mathrm{d} \tilde{W}^2_t &= \lambda^2_t \mathrm{d} t + \mathrm{d} W^2_t,
\end{align*}
\]

\[
\frac{\mu_t - r}{\sqrt{\lambda_t^1}} = \rho \lambda^1_t + \sqrt{1 - \rho^2} \lambda^2_t.
\]

Note that here the Feller condition should be satisfied to guarantee the strict positivity of variance, that is, \( 2\alpha \beta > \sigma^2 \). In order to keep the variance process within the affine class under \( Q \) measure, we employ an essential affine structure of market price specification \( \lambda^2_t = a \sqrt{\lambda^1_t} + b \) for some constant \( a \), which is introduced by Duffee (2002). Thus, we have,

\[
\tilde{\alpha} = \alpha + a \sigma, \\
\tilde{\beta} = \frac{\alpha \beta - b \sigma}{\alpha + a \sigma}.
\]

In the Markovian framework of the underlying processes, the American option price \( J_t \) can be viewed as a unique viscosity solution to the following Hamilton-Jacobi-Bellman (hereafter HJB) variational inequality:

\[
\min \left \{ r J(t, x) - \frac{\partial J}{\partial t} - \mathcal{L} J(t, x), J(t, x) - H(t, x) \right \} = 0, \quad \forall t \in [0, T), \quad J(T, x) = H(T, x),
\]

where \( \mathcal{L} \) is the infinitesimal generator of \( X \) with,

\[
\mathcal{L} J(t, x) = \frac{1}{2} (S_t^2 \frac{\partial^2 J}{\partial S^2} + \sigma^2 S_t \frac{\partial J}{\partial S} + \rho \sigma \lambda_t V_t \frac{\partial^2 J}{\partial S \partial V} + r S_t \frac{\partial J}{\partial V} + \tilde{\alpha} (\tilde{\beta} - V_t) \frac{\partial J}{\partial V}.
\]

This is a well-known result by general Itô’s lemma and Feynman-Kac representation formula. (see for example Theorem 9.4.7 and 9.4.8 of Pascucci (2011))

2.2 Set of Multiple Priors

We use the previous setting of probability space, in which \( P \) is a reference measure here. This means that it functions only to fix sets with measure zero (\( P \)-null sets) in order to define a set of equivalent probability measures, and it does not have to be the real world objective measure.

Following Chen & Epstein (2002), we start constructing a set of multiple priors \( \mathcal{P} \) by defining a density generator (Girsanov kernel) \( \theta_t : [0, T] \times \Omega \to \mathbb{R}^d \) (for Heston’s model \( d = 2 \)). Note that \( \theta_t \) is allowed to be dynamic and stochastic by definition. \( \theta_t \) is defined in such a way that the conditional Radon-Nikodym derivative \( M_t \) is a \( P \)-martingale with \( M_0 = 1 \), where

\[
\frac{\mathrm{d} Q^\theta}{\mathrm{d} P} \big{|}_{F_t} = M_t = \mathcal{E} \left( - \int_0^t \theta^*_s \mathrm{d} W_s \right) t,
\]

\[
\mathcal{E} \left( - \int_0^t \theta^*_s \mathrm{d} W_s \right) = \exp \left( - \int_0^t \theta^*_s \mathrm{d} W_s - \frac{1}{2} \int_0^t \theta^*_s ^2 \mathrm{d} s \right).
\]

The probability measure \( Q^\theta \) of certain set in the filtration \( \mathcal{F}_T \) is given by,

\[
Q^\theta(A) = \mathbb{E}^P \left[ \mathbb{1}_A \mathcal{E} \left( - \int_0^T \theta^*_s \mathrm{d} W_s \right) \right], \quad \forall A \in \mathcal{F}_T.
\]

The set of multiple priors is defined by,

\[
\mathcal{P}^\theta := \{ Q^\theta : \theta_t \in \Theta \text{ and } Q^\theta \text{ is defined in (5)} \}.
\]

It is straightforward that \( Q^\theta \) will be identical to be reference measure \( P \) if we let \( \theta = 0 \). Additionally, we will have \( Q = Q^\theta \) if \( \theta_t = \lambda_t \) for all \( t \in [0, T] \). The last step is to define a set of density generators \( \Theta \). For now we have two options:

- **\( \kappa \)-ignorance:** \( \Theta = \{(\theta_t) : |\theta_t| \leq \kappa_t \text{ for } t \in [0, T] \text{ and } i = 1, ..., d \} \), for a fixed positive constant \( \kappa = (\kappa_1, ..., \kappa_d) \text{ in } \mathbb{R}^d_+ \). This is introduced by Chen & Epstein (2002). In addition, it can be naturally extended to a case where \( \kappa \) is time-varying, which is called IDD ambiguity.
• **elliptical ambiguity:** $\Theta = \{ (\theta_t) : \theta_t^* \Sigma^{-1} \theta_t \leq \chi, t \in [0, T] \}$ for some fixed positive semi-definite matrix $\Sigma$ and positive constant $\chi$. This is similar to the elliptical uncertainty sets used in [Cohen & Tegnér (2017)]. Note that elliptical ambiguity set implies and elliptical set of priors, while $\kappa$-ignorance implies a rectangular set of priors. Elliptical ambiguity set nests circular ambiguity set that restricts the Euclidean norm of density generators within a compact and convex set.

2.3 Financial Markets with Multiple Priors

We characterize that the agent in the real world is facing various probabilities with respect to specific events, which is known as *Knightian uncertainty*. That means the agent cannot be absolutely certain about the drift coefficients $\mu_t$, $\alpha$ and $\beta$, in Heston’s stochastic volatility model [1-2]. With partial information of specified events, the agent may act with a prior probability measure, and this can be seen as ambiguity. We define a set of equivalent probability measures $\mathcal{P}^\theta$ that the agent may refer to by a set of density generators $\Theta$.

Then we proceed to the evaluation rule of financial claims and the dynamics of underlying processes when there is ambiguity. According to [Gilboa & Schmeidler (1989)], the behaviour agent with multiple priors will be uncertainty-averse, or ambiguity-averse if the agent acts in accordance with some certain sensible axioms. That means the agent will use a probability measure corresponding to the "worst" case scenario when evaluating a claim. In the American option case, the desired price is denoted as,

$$v_t := v(t, X_t) = \text{ess sup}_{\tau \in \mathcal{G}_t} \text{ess inf}_{Q^\theta \in \mathcal{P}^\theta} \mathbb{E}^{Q^\theta}[H_t \gamma_{T-t}^{-1} | F_t], t \in [0, T], \quad (7)$$

and we will use $v_t := \text{ess sup}_{\tau \in \mathcal{G}_t} \text{ess inf}_{Q^\theta \in \mathcal{P}^\theta} \mathbb{E}^{Q^\theta}[H_t \gamma_{T-t}^{-1} | F_t]$ for simplicity.

A direct result by the Girsanov theorem reveals the dynamics of underlying processes with ambiguity,

$$dS_t/S_t = \mu_t dt - \sqrt{V_t}(\rho \theta_t^1 + \sqrt{1 - \rho^2} \theta_t^2) dt + \sqrt{\frac{1}{V_t}} (\rho dW^{q_1}_t + \sqrt{1 - \rho^2} dW^{q_2}_t),$$

$$dV_t = \alpha(\beta - V_t) dt - \sigma \theta_t^1 \sqrt{V_t} dt + \sigma \sqrt{V_t} dW^{q_1}_t,$$

where $W^\theta = (W^{q_1}, W^{q_2})^*$ is a standard two dimensional Brownian motion under measure $Q^\theta$, and

$$dW^{q_1} = \theta_t^1 dt + dW^1_t,$$

$$dW^{q_2} = \theta_t^2 dt + dW^2_t.$$

For the evaluation purpose, we change the reference measure directly to the risk-neutral measure $Q$, as the agent does not know the real world probability measure and it is equivalent to $P$. This approach is also adopted by [Cheng & Riedel (2013), Vorbrink (2011) and Cohen & Tegnér (2017)]. Thus we are considering under probability space $(\Omega, \mathcal{F}, Q)$, and the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is generated by $W = (W^1, W^2)^*$. To avoid confusion and stay in the previous multiple priors framework, we let $Q \in \mathcal{P}^\theta$, and we have equivalently,

$$dS_t/S_t = r dt - \sqrt{V_t}(\rho \theta_t^1 + \sqrt{1 - \rho^2} \theta_t^2) dt + \sqrt{\frac{1}{V_t}} (\rho d\tilde{W}^{q_1}_t + \sqrt{1 - \rho^2} d\tilde{W}^{q_2}_t),$$

$$dV_t = \alpha(\beta - V_t) dt - \sigma \theta_t^1 \sqrt{V_t} dt + \sigma \sqrt{V_t} d\tilde{W}^{q_1}_t,$$

where

$$d\tilde{W}^{q_1}_t = \theta_t^1 dt + d\tilde{W}^1_t,$$

$$d\tilde{W}^{q_2}_t = \theta_t^2 dt + d\tilde{W}^2_t. \quad (8)$$

2.4 Relation to Reflected Backward Stochastic Differential Equations

Following [El Karoui, Kapoudjian, et al. (1997)], we introduce some notations for a proper definition of the reflected backward stochastic differential equation.

$$\mathbb{L}^2 = \{ \xi \text{ is an } \mathcal{F}_T \text{-measurable random variable s.t. } \mathbb{E}[|\xi|^2] < +\infty \},$$

$$\mathbb{H}^2 = \{ \phi_t, \ 0 < t < T \} \text{ is a predictable process s.t. } \mathbb{E} \left[ \int_0^T |\phi_t|^2 dt \right] < +\infty \},$$

$$\mathbb{S}^2 = \{ \phi_t, \ 0 < t < T \} \text{ is a predictable process s.t. } \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^2 \right] < +\infty \}. $$

5
Definition 2.1. Assume conditions such as the terminal condition $\xi \in \mathbb{L}^2$, the continuous reflection bound $L \in \mathbb{S}^2$, a uniform Lipschitz generator $f(t, y, z)$ and $f(., y, z) \in \mathbb{H}^2$, $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^2$ are met, a triplet \((Y_t, Z_t, K_t), t \in [0, T]\) of \(\mathcal{F}_t\) progressively measurable processes taking values in \(\mathbb{R}, \mathbb{R}^2\) and \(\mathbb{R}_+\) is the unique solution of the following reflected backward stochastic differential equation (RBSDE) satisfying:

\begin{enumerate}[(i)]
    \item \(Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s^*dB_s,\)
    \item \(Y_t \geq L_t\) and \((Y, Z, K) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2,\)
    \item \(K_t\) is a continuous and increasing process, \(\int_0^T (Y_s - L_t)dK_t = 0\) and \(K_0 = 0,\)
\end{enumerate}

where \(B\) is a \(d\)-dimensional standard Brownian motion under reference measure \(Q\), \(L_t\) is called an "obstacle" or reflection bound, and \(K_t\) is a process that "pushes \(Y_t\) upwards" minimally in the sense of condition (iii).

The existence and uniqueness of the solution of above RBSDE are proved in Section 6 of [El Karoui, Kapoudjian, et al. (1997)] followed by Proposition 2.3 of [El Karoui, Kapoudjian, et al. (1997)].

\[Y_t = \text{ess sup}_{\tau \in \mathcal{F}_t} E^\theta [\int_\tau^T f(s, Y_s, Z_s)ds + L_T 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_\tau].\]

where \(\mathcal{F}\) is the set of all stopping times dominated by \(T\), and \(\mathcal{F}_t = \{\tau \in \mathcal{F} : t \leq \tau \leq T\}\).

Without uncertainty, which means we take \(\theta_t = 0\) for all \(t \in [0, T]\), the American option price \(J_t\) with payoff \(H_t\) at time \(t\), defined in equation (11), will have a dual representation of the solution of an RBSDE \((Y_t)\).

Hence \(J_t = Y_t\), and the RBSDE is denoted in differential form as,

\[dY_t = -rY_t dt + dK_t - Z_t^* d\tilde{W}_t,\]

with obstacle \(L_t = H_t \forall t \in [0, T]\). This is obtained by taking a linear generator \(f(t, y, z) = -ry\) and let \(B_t = \tilde{W}_t\), according to Proposition 7.1 of [El Karoui, Kapoudjian, et al. (1997)]. Further, this result can be extended to the case of multiple priors.

Suppose conditions needed for the existence and uniqueness are satisfied, \((Y^\theta_t, Z^\theta_t, K^\theta_t), t \in [0, T]\) are the solution to the following linear RBSDE

\[dY^\theta_t = f(t, Y^\theta_t, Z^\theta_t)dt + dK^\theta_t - Z^\theta_t^* d\tilde{W}^\theta_t,\]

with generator \(f(t, Y^\theta_t, Z^\theta_t) = -rY^\theta_t\) and obstacle \(L_t = H_t, \forall t \in [0, T]\), where \(\tilde{W}^\theta = (\tilde{W}^{\theta_1}, \tilde{W}^{\theta_2})^*\) defined in (11-12) is a standard two dimensional Brownian motion under probability measure \(Q^{\theta}\).

Proposition 2.1. Let \(u_t\) be defined as,

\[u_t := \text{ess sup}_{\tau \in \mathcal{F}_t} E^\theta [H_{\gamma_t^{-1}}^{-1} | \mathcal{F}_t],\]

where \(\gamma_t\) is defined in (3). Then \(u_t\) has a dual representation of the solution of an RBSDE in equation (11), which means \(u_t = Y^\theta_t\) for every \(t \in [0, T]\). Additionally, \(u_t\) is the unique viscosity solution of the following variational inequality:

\[\min \left\{ -f(t, Y^\theta_t, Z^\theta_t) - \frac{\partial u}{\partial t} - L^\theta(t, x), u(t, x) - H(t, x) \right\} = 0, (t, x) \in [0, T] \times \mathbb{R}^2,\]

\[u(T, x) = H(T, x),\]

with the generator explicitly denoted by

\[f(t, Y^\theta_t, Z^\theta_t) = -rY^\theta_t - \theta^1_t Z^\theta_{t - 1} - \theta^2_t Z^\theta_t.\]

Proof. By the Girsanov theorem, (11) is equivalent to

\[-dY^\theta_t = -(rY^\theta_t + \theta^1_t Z^\theta_t^* )dt + dK^\theta_t - Z^\theta_t^* d\tilde{W}_t,\]
then by Proposition 7.1 of El Karoui, Kapoudjian, et al. (1997), \( \{Y^\theta_t, Z^\theta_t, K^\theta_t\}, t \in [0,T] \) satisfies,
\[
\Gamma_t Y^\theta_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^\mathbb{Q}[\Gamma_{\tau} H_T \mathbb{1}_{\{\tau = T\}} + \Gamma_{\tau} H_T \mathbb{1}_{\{\tau < T\}}],
\]
\[
d\Gamma_t = \Gamma_t (rdt + \theta_t^z d\tilde{W}_t), \quad \Gamma_0 = 1.
\]

We know \( \Gamma_{t-} = \gamma_{t-}^{-1} M_{t-} \), where \( M \) is defined in (5), and we have (12) by definition of conditional expectation under measure \( \mathbb{Q}^\theta \). The second part is a direct computation from Theorem 8.6 of El Karoui, Kapoudjian, et al. (1997).

**Proposition 2.2.** Under the \( \kappa \)-ignorance and the elliptical ambiguity framework as in the section 2.2, the American option price \( v_t \) under the worst case belief as defined in (7) is the value of a minimax (optimal) control problem such that,
\[
v_t = \text{ess inf}_{\theta \in \Theta} Y^\theta_t,
\]
and we have,
\[
\text{ess sup}_{\tau \in \mathcal{T}_t} \text{ess inf}_{\theta \in \Theta} \mathbb{E}^\mathbb{Q}[\Gamma_{\tau} H_T \mathbb{1}_{\{\tau = T\}} + \Gamma_{\tau} H_T \mathbb{1}_{\{\tau < T\}}], \quad \forall t \in [0,T],
\]
where \( Y^\theta_t \) is an element of the solution to the RBSDE (11), and \( H_t \) is the payoff function defined in (4). Moreover, there exists a pair \( (\tilde{r}, \tilde{b}) \), \( \tilde{r} \in [0,T], \tilde{b} \in \Theta \) such that it reaches the optimal (saddle) point \( (\tilde{r}, \tilde{b}) \) when the generator given by (13) satisfies \( f(t, Y^\tilde{t}, Z^\tilde{t}) = \text{ess inf}_{\theta \in \Theta} f(t, Y^\theta_t, Z^\theta_t), \) \( \forall t \in [0,T] \).

**Proof.** Equation (15) holds if we take the essential infimum of the generator of (14) while remain other parameters unchanged, according to the comparison theorem, Theorem 4.1 of El Karoui, Kapoudjian, et al. (1997). Then equation (16) follows directly from Theorem 7.2 of El Karoui, Kapoudjian, et al. (1997). The only thing we need is the existence of the minimum of the generator \( f(t, Y^\theta_t, Z^\theta_t) \).

From Proposition 2.1 we know that \( f(t, Y^\theta_t, Z^\theta_t) = -rY^\theta_t - \theta^*_t Z^\theta_t \), then the minimum of \( f(t, Y^\theta_t, Z^\theta_t) \) corresponds to the maximum of \( \theta^*_t Z^\theta_t \). The existence of \( \max_{\theta \in \Theta} \theta^*_t Z^\theta_t \) under \( \kappa \)-ignorance and IID ambiguity is proved in Section 2.4 of Chen & Epstein (2002). The existence of maximum point of generator under elliptical ambiguity follows similarly because the set of generators \( \Theta \) is also compact and convex valued in \( \mathbb{R}^2 \).

Let us consider the case of elliptical ambiguity \( \Theta = \{\theta : \theta \Sigma^{-1} \theta^* \leq \chi\} \), as defined in section 2.2. To obtain the minimum of the generator, we can solve the following optimization problem,
\[
f(t, Y^\theta_t, Z^\theta_t) = \text{min}_{\theta \in \Theta} (-rY^\theta_t - \theta^*_t Z^\theta_t),
\]
subject to \( \theta \Sigma^{-1} \theta^* = \chi \),

where we have the equality in the constraint, since there is no internal stationary points of the elliptical set due to the affine structure inside the minimum, this argument is similar to that in Cohen & Tegnér (2017).

Solving the Lagrangian, the solution of the above optimization problem is given by,
\[
f(t, Y^\theta_t, Z^\theta_t) = -rY^\theta_t - \sqrt{Z^\theta_t \Sigma^* Z^\theta_t / \chi},
\]
with \( \tilde{\theta} = -\frac{Z^\theta_t \Sigma^*}{\sqrt{Z^\theta_t \Sigma^* Z^\theta_t / \chi}} \).

It is notable that in the case of \( \kappa \)-ignorance, in which the uncertainty set is rectangular (convex and compact), the optimal solution is located in one of four vertexes of the rectangle.

**Remark 2.1.** Given the above optimal generator, we still need to check if the uniqueness of the solution of the RBSDE (14) still holds. To prove that, it is sufficient to show the generator is uniform Lipschitz, i.e. \( |f(t, y, z) - f(t, y', z')| \leq \text{Constant} \cdot (|y - y'| + ||z - z'||) \) \( \forall y, y', z, z' \in \mathbb{R} \). Note that
\[
|f(t, y, z) - f(t, y', z')| = ||r(y - y') + (\sqrt{z^\Sigma z} - \sqrt{z'^\Sigma z'})||,
\]
\[
\leq r|y - y'| + \sqrt{\chi}(|z^\Sigma z| - ||z'^\Sigma z'||),
\]
\[
\leq r|y - y'| + \sqrt{\chi} ||z^\Sigma z|| ||z'^\Sigma z'||,
\]
where the first equality holds by the definition of the optimal generator; the second inequality holds by the triangle inequality and the last inequality holds by the reverse triangle inequality. Then the optional generator is uniform Lipschitz because $\Sigma$ is a positive semi-definite matrix and $\chi$ is a positive constant by definition.

3 Numerical Methods

In this section we briefly introduce the conventional numerical methods for reflected backward stochastic differential equations (RBSDEs) and our proposed amended method for RBSDEs, and investigate the feasibility of a new numerical dynamic programming method without using the theory of RBSDE.

Firstly we split the time interval $[0, T]$ in to $N$ equal parts with each part to be $\Delta t$. Hence we have a time grid $\pi : 0 = t_0 < ... < t_N = T$ with $(i + 1)$-th time step $t_{i+1} - t_i$ denoted as $\Delta t$, and the $(i + 1)$-th Brownian motion increment under measure $Q$: $W_{t_{i+1}} - W_{t_i}$ is defined by $\Delta W_i$. The conditional expectation $\mathbb{E}^Q[\cdot | F_{t_i}]$ under measure $Q$ is denoted as $\mathbb{E}^Q[\cdot]$. Then we simulate the dynamics of $X_t$ under $Q$ measure in $M$ different paths by only one time step $\Delta t$, using the Euler scheme,

$$X_{t_{i+1}}^\pi = X_{t_i}^\pi + b(t_i, X_{t_i}^\pi) \Delta t + \sigma(t_i, X_{t_i}^\pi) \Delta W_i.$$  

3.1 Numerical Methods for BSDEs

We begin with numerical methods for solutions of discrete backward stochastic differential equations (BSDEs), since methods for RBSDEs are based on those for BSDEs.

- **One-step Forward Dynamic Programming (ODP) scheme**: Based on the one-step scheme introduced by Bouchard & Touzi [2004], Lemor et al. [2006] propose a modified scheme as such,

$$Z_i^\pi = \frac{1}{\Delta t} \mathbb{E}^Q_i[Y_{t_{i+1}}^\pi \Delta W_i], \quad Y_N^\pi = H(X_N^\pi), \quad (17)$$

$$Y_i^\pi = \mathbb{E}^Q_i[Y_{t_{i+1}}^\pi + f(X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi, Z_{t_{i+1}}^\pi) \Delta i]. \quad (18)$$

This Markovian representation of a BSDE is obtained by taking conditional expectation on both sides of the following discrete BSDE:

$$Y_{t_{i+1}}^\pi - Y_i^\pi = - f(X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi, Z_{t_{i+1}}^\pi) \Delta i + Z_{t_{i+1}}^\pi \Delta W_i.$$  

- **Multi-step Forward Dynamic Programming (MDP) scheme**: Bender & Denk [2007] introduced a more stable multi-step scheme:

$$Z_i^\pi = \frac{1}{\Delta t} \mathbb{E}^Q_i[(Y_{t_{i+1}}^\pi + \sum_{k=i}^{N-1} f(X_{t_k}^\pi, Y_{t_{k+1}}^\pi, Z_{t_k}^\pi) \Delta W_k)], \quad Y_N^\pi = H(X_N^\pi), \quad (19)$$

$$Y_i^\pi = \mathbb{E}^Q_i[Y_N^\pi + \sum_{k=i}^{N-1} f(X_{t_k}^\pi, Y_{t_{k+1}}^\pi, Z_{t_k}^\pi) \Delta W_k], \text{ for } i = 0, 1, ..., N - 1. \quad (20)$$

The backward induction $(19)-(20)$ are obtained by substituting subsequent terms of $Y_i^\pi$ into $(17)-(18)$ and applying the tower law of conditional expectation. According to Gobet & Turkedjiev [2016], the advantage of the MDP over the ODP is that the error of approximating conditional expectation is the average rather than the sum of local error terms, thus the result is tighter in a sense.

3.2 Numerical Methods for RBSDEs

"Max method": Introduced in Gobet & Lemor [2008], it is a method to approximate $(X_t, Y_t)$ by a discrete-time doublet $(X^\pi, Y^\pi)$ for a discrete RBSDE:

$$Z_i^\pi = \frac{1}{\Delta t} \mathbb{E}^Q_i[Y_{t_{i+1}}^\pi \Delta W_i], \quad Y_N^\pi = H(X_N^\pi), \quad (21)$$

$$\hat{Y}_i^\pi = \mathbb{E}^Q_i[Y_{t_{i+1}}^\pi + f(X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi, Z_{t_{i+1}}^\pi) \Delta i], \quad (22)$$

$$Y_i^\pi = \hat{Y}_i^\pi \lor H(X_{t_i}^\pi). \quad (23)$$
As for practically operating the backward induction, one can choose to approximate the conditional expectation by stratified regression. In this article we use the "Max method", along with the stratification regression method introduced by Gobet et al. (2016) to approximate solutions of discrete RBSDEs. We use the name Stratified Regression One-step Forward Dynamic Programming (SRODP) for this scheme hereafter.

### 3.3 Approximating Conditional Expectation by Stratified Regression

As for practically operating the backward induction, one can choose to approximate the conditional expectation $\mathbb{E}_t^Q[\cdot]$ with $\mathbb{E}_t^Q[X_i]$ by solving a least-square optimization problem

$$
\hat{\mathbb{E}}_t^Q[X_i] := \arg\inf_{\phi} \frac{1}{M} \sum_{m=1}^M |\phi(X_i^{(m)}) - \mathbb{E}_t^Q[X_i^{(m)}]|^2,
$$

where generally $\phi(\cdot)$ is defined in $L^2(\Omega, \mathcal{F}, Q)$, and $X_i^{(m)}$ is the $(m)$-th path of all the paths by Monte Carlo simulation. To make it suitable for a search policy, one can fix $\phi(\cdot)$ in a finite linear space $\mathcal{X}$. Then it is possible to choose a polynomial basis $p_1(x), p_2(x), \ldots, p_K(x)$, (possibly using the Legendre, Laguerre or simple power polynomials, according to Longstaff & Schwartz (2001)) and let $y := [\mathbb{E}_t^Q[X_i^{(m)}]]_m$, $P := [p_k(X_i^{(m)})]_{m,k}$. The general least-square optimization (24) becomes solving a linear regression:

$$
\inf_{\mathcal{X}} \frac{1}{M} \sum_{m=1}^M |\phi(X_i^{(m)}) - \mathbb{E}_t^Q[X_i^{(m)}]|^2 = \inf_{\beta \in \mathbb{R}^K} \frac{1}{M} |P\beta - y|^2.
$$

Conventional methods of using all the simulated paths (Tsitsiklis & Van Roy 2001) or the in-the-money paths (Longstaff & Schwartz 2001) to run the regression and globally approximate the objective function are popular in finance. Yet it yields a higher number of polynomials when dealing with multi-dimensional cases. Gobet et al. (2016) propose a general stratified regression method to approximate conditional expectations locally in partitions (hypercubes) of simulated sample paths. The main benefits of it is one can minimize the memory consumed by simulations and regression coefficients, since one only needs to generate samples on one hypercube at a time and the number of polynomials required is much less than the traditional method. We briefly state it here, as it is also an essential part of our proposed numerical method without using the theory of RBSDEs.

Firstly we choose a domain of discretized state space $D \subset \mathbb{R}^d$ centered on $\Theta$ ($D = \prod_{k=1}^d [\Theta_k - R, \Theta_k + R]$) with radius $R$, and partition it into small hypercubes with the same length of edge $\delta$. Then we generate certain randomly distributed points of $X^\pi$ and simulate the dynamics. Approximation of conditional expectation by least-square regression is implemented independently on different hypercubes. The general stratification in Gobet et al. (2016) differs from the traditional stratified sampling in that it does not need the explicit distribution of $X_i^\pi$, since the conditional expectation is determined by the transition equation of $X$ after $t_i$, and simulations of $X^\pi$ can start from an random variable at arbitrary time rather than a fixed point at time 0. This feature enables us to resimulate at each time point and save memory for storing simulated paths. It also sheds some light to another method without using the theory of RBSDEs. We present it in the next subsection.
Z_t is dynamic and stochastic. The only case this method will work is when the sign of Z_t is certain, and Z_t in (13) is directly related to the Greeks Delta and Vega.

A possible alternative will be to simulate the density generator processes \(M_i^\theta\) and \(X_t\) simultaneously. The difficulty is that we will have \(L\) different choices of \(\theta\) at each discretized time point, which makes the simulation of forward process \(M_i^\theta\) to be a \(L\)-nominal tree. This will finally consume explosive memory of storing simulated paths when increasing the number of discretization time points \(N\), for example, it will have to store \(L^N\) simulations for the last step. Thus, this alternative is not realistic.

We raise another potential way to solve the minimax problem and, in the mean time, avoid using the theory of RBSDEs. We take the name Stratified Least-square Monte Carlo (SLSM) method hereafter. The idea, using stratification and resimulation at each time point after discretization, can be seen as an extension of regression based method. However, the method we propose is very different from the LSM method, in which the underlying process \(X_t\) and the stochastic exponential \(M_t\) defined in (3) must be simulated from a fixed point at time 0. Instead, we simulate the underlying processes at time \(t_i\) by only one step forward to \(t_{i+1}\), assuming that \(X_{t_i}\) and \(M_{t_i}\) follow some certain distributions, for example, logistic distribution (Gobet et al., 2016) or Laplace distribution. This means we take \(M_t\) as an extra dimension of underlying state variables.

Then we evaluate the cash flow at time \(t\) by comparing the immediate payoff with the continuation value of American option, which is approximated by basis regression method. This implies that we only need to store \(L^N\) simulations for the last step. Thus, this alternative is not realistic.

To solve this problem, we propose the Stratified Least-squares Monte Carlo (SLSM) method, where we will use \(L\) different realizations of \(M_t\) and \(X_t\) at each time point to make \(L\) disjoint strata (hypercubes) in order to approximate the objective function locally. Stratification also enables us to use highly parallelized computation to decrease computation time.

(i) Firstly we discretize the stochastic exponential \(M_t\) defined in (3), and select \(L\) discrete points from the set of density generator \(\Theta\). Assuming \(X_{t_i}\) and \(M_{t_i}\) are i.i.d. conditional logistic random variables, we simulate the dynamics of \(X_t\) and \(M_t\) under \(Q\) measure in \(M\) different paths by only one time step \(\Delta t\), either by the Euler dynamics,

\[
X_{t_{i+1}} = X_{t_i} + b(t_i, X_{t_i})\Delta t + \sigma(t_i, X_{t_i})^\top \Delta \tilde{W}_i, \\
M_{t_{i+1}}^\pi,\theta_j = M_{t_i}^\pi \exp \left\{ -\frac{1}{2} \theta_j^\top \Delta t - \theta_j^\top \Delta \tilde{W}_i \right\}, \quad j = 1, 2, \ldots, L.
\]

(ii) We define the one-step back continuation value of the American option under measure \(Q^\theta_j\) at time \(t_i\),

\[
h(S_i^\pi, M_i^\pi, \theta_j) := \mathbb{E}_t^Q \left[ \frac{M_{i+1}^\pi,\theta_j Y_{i+1}^\pi,\theta_j^{-1}}{M_i^\pi} \right] = \mathbb{E}_t^Q \left[ Y_{i+1}^\pi,\theta_j \right], \quad Y_i^\pi,\theta_j = h(S_i^\pi, M_i^\pi),
\]

\[
h(S_i^\pi, M_i^\pi, \bar{\theta}) = \text{ess inf}_{\theta_j \in \Theta} h(S_i^\pi, M_i^\pi, \theta_j) \wedge H(S_i^\pi).
\]

It should be noted that we will have \(L\) different realizations of \(M_{i+1}^\pi,\theta_j\) as for each \(M_i^\pi\), since we split \(\Theta\) into \(L\) discrete points. This should be differentiated from \(M\) paths of underlying processes.

(iii) The approximation of \(h(S_i^\pi, M_i^\pi, \theta_j)\) is then obtained by solving the linear least-square regression (25),

\[
\hat{h}(S_i^\pi, M_i^\pi, \theta_j) = \sum_{k=1}^K c_k^j p_k(X_i^\pi, M_i^\pi),
\]

\[
\hat{h}(S_i^\pi, M_i^\pi, \bar{\theta}) = \text{ess inf}_{\theta_j \in \Theta} \hat{h}(S_i^\pi, M_i^\pi, \theta_j),
\]

where \(p_k(X_i^\pi, M_i^\pi)\) is the basis functions defined in (25) such as Hermite or Legendre polynomials, and \(c_k^j\) is the constant coefficient of each polynomial. In practice, we take the approximated American option value \(\hat{Y}_i^\pi,\theta_j\) as \(\hat{Y}_i^\pi,\theta = \hat{h}(S_i^\pi, M_i^\pi, \bar{\theta}) \wedge H(S_i^\pi)\).

(iv) We take the above steps from time \(t_{N-1}\) to \(t_1\) recursively, and re-simulate the trajectories of \(X_t^\pi\) and \(M_t^\pi\) at each time point in order to conserve memory. The difference of our method to the LSM algorithm is that we store the regression coefficients \(c_k\), \(k = 1, 2, \ldots, K\) rather than the continuation value \(\hat{h}(S_i^\pi, M_i^\pi, \theta_j)\) at each time point, which means we will use new sample paths to evaluate the continuation value. Stratification method is also employed, i.e. to stratify \(X_t^\pi\) and \(M_t^\pi\) to several different strata (hypercubes) in order to approximate the objective function locally. Stratification also enables us to use highly parallelized computation to decrease computation time.
4 Numerical Results for Financial Options

We will start this part by presenting results for European and American put options using SRODP within the Black-Scholes (hereafter BS) and Heston’s framework \([1973, 1993]\). The main purpose is to show that our proposed methods converge and results are close enough to the exact value, since we have closed form European option prices and American option prices (by numerical PDE methods).

4.1 European Put Option Prices

We use Laguerre local polynomials up to first order (LP1) as the basis functions when implementing SRODP scheme. We fix the space domain for the logarithmic stock price to be \(D = [-6.5, 6.5]\). We use 1000 hypercubes and 2000 simulations for each hypercube. Furthermore, we launch 50 times the SRODP algorithm, and collect each time the result, denoted as \((Y_0^i)_{1 \leq i \leq 50}\). The SRODP European put option price is then the empirical mean as following,

\[
\bar{Y}_0 = \frac{1}{50} \sum_{i=1}^{50} Y_0^i,
\]

we also calculate the empirical standard deviation,

\[
\sigma_0 = \frac{1}{49} \sqrt{\sum_{i=1}^{50} |Y_0^i - \bar{Y}_0|^2},
\]

then the standard error of the mean is \(\sigma_0/\sqrt{50}\) and the 95 percent confidence interval is \([\bar{Y}_0 - 1.96\sigma_0, \bar{Y}_0 + 1.96\sigma_0]\). We compare the SRODP and the BS results, and present the standard deviation and the 95% confidence interval of SRODP in the following Table 1.

<table>
<thead>
<tr>
<th>(S_0)</th>
<th>BS</th>
<th>SRODP</th>
<th>std</th>
<th>95% C.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>3.844</td>
<td>3.8379</td>
<td>0.0412</td>
<td>[3.8264, 3.8493]</td>
</tr>
<tr>
<td>38</td>
<td>2.852</td>
<td>2.8447</td>
<td>0.0349</td>
<td>[2.8351, 2.8544]</td>
</tr>
<tr>
<td>40</td>
<td>2.066</td>
<td>2.0675</td>
<td>0.0331</td>
<td>[2.0583, 2.0767]</td>
</tr>
<tr>
<td>42</td>
<td>1.465</td>
<td>1.4625</td>
<td>0.0254</td>
<td>[1.4554, 1.4695]</td>
</tr>
<tr>
<td>44</td>
<td>1.017</td>
<td>1.0205</td>
<td>0.0220</td>
<td>[1.0144, 1.0266]</td>
</tr>
</tbody>
</table>

Note: The strike price is 40, volatility is 0.2, risk free rate is 0.06, and time to maturity is 1. We use 5 time steps, 1000 hypercubes and 2000 simulations for each hypercube. Each 50 times SRODP algorithm takes about 120 seconds.

4.2 American Put Option Prices without Ambiguity

4.2.1 One Dimensional Case

We use LP1 as the basis functions when implementing SRODP scheme. We fix the space domain for the logarithmic stock price to be \(D = [-6.5, 6.5]\). We use 1000 hypercubes and 2000 simulations for each hypercube, and we launch 50 times the SRODP algorithm. The PDE and LSM results are extracted from \([2001]\), and PDE results are taken as the exact value. The option is exercisable 50 times per year, so it is of American style. The results are in Table 2.

\[^{3}\text{It should be noted that the proposed SLSM method is identical to the SRODP method in cases without ambiguity.}\]
Table 2: One Dimensional American Put Option Prices

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>BS</th>
<th>PDE</th>
<th>LSM (std)</th>
<th>SRODP (std)</th>
<th>95% C.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>3.844</td>
<td>4.478</td>
<td>4.472 (0.010)</td>
<td>4.4825</td>
<td>0.0193</td>
</tr>
<tr>
<td>38</td>
<td>2.852</td>
<td>3.250</td>
<td>3.244 (0.009)</td>
<td>3.2570</td>
<td>0.0169</td>
</tr>
<tr>
<td>40</td>
<td>2.066</td>
<td>2.314</td>
<td>2.313 (0.009)</td>
<td>2.3203</td>
<td>0.0158</td>
</tr>
<tr>
<td>42</td>
<td>1.465</td>
<td>1.617</td>
<td>1.617 (0.007)</td>
<td>1.6224</td>
<td>0.0123</td>
</tr>
<tr>
<td>44</td>
<td>1.017</td>
<td>1.110</td>
<td>1.118 (0.007)</td>
<td>1.1225</td>
<td>0.0108</td>
</tr>
</tbody>
</table>

Note: The strike price is 40, volatility is 0.2, risk free rate is 0.06, and time to maturity is 1. We use 50 time steps, 1000 hypercubes and 2000 simulations for each hypercube. The LSM results are from Longstaff & Schwartz (2001). Each 50 times SRODP algorithm takes about 2400 seconds.

4.2.2 Two Dimensional Case

We extend the one dimensional American option prices to the case under Heston’s model. The American option price under Heston’s model can be obtained via standard numerical PDE methods and are exploited by previous researches (see for example Ikonen & Toivanen (2008)). Thus, the PDE results from Ikonen & Toivanen (2008) are taken as the exact value. Meanwhile, we employ the same parameters for the dynamics of stock price and volatility as in Ikonen & Toivanen (2008) for simplicity. We fix the space domain for the logarithmic stock price to be $[-6.5, 6.5]$ and the space domain for the volatility to be $[0, 10]$. We launch 10 times for the LSM algorithm and the SRODP algorithm. We also implement the standard "Max method" for RBSDE for 10 times, i.e. without stratification. We conduct several experiments to show the convergence of the SRODP method with the number of hypercubes and simulations increasing. LP2 stands for regressions with local polynomials up to the second order.

It can be observed that both LP1 and LP2 regression SRODP scheme converges when increase the number of hypercubes (from Table 3-7) or the number of simulated paths per hypercube (from Table 5-6). However, the number of hypercubes required for the LP2 scheme to converge is much less than that for the LP1 scheme, although higher order of local polynomials come with more computational burden. It should be noted that the standard RBSDE algorithm converge as well, since the standard deviations shrink when the number of simulation paths increase from 10,000 to 80,000. The standard deviations are as small as that for the LSM algorithm. The reason is when applying the stopping rule and approximating the stopping time in the backward induction, both the standard RBSDE and LSM algorithms use the original paths of state variables for simulation (Glasserman, 2013). One can certainly resimulate the whole sample paths when applying the stopping rule to evaluate the option as this is the more realistic case, yet it will come with larger standard deviations. Moreover in Table 7, either the SRODP algorithm with LP1 or LP2 scheme is close enough to the exact value and the LSM results, whereas the pricing bias by the standard RBSDE algorithm is not negligible, especially for at-the-money and out-of-money options.

Table 3: Two Dimensional American Put Option Prices

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>PDE</th>
<th>LSM (std)</th>
<th>RBSDE (std)</th>
<th>LP1-SRODP (std)</th>
<th>Time(s)</th>
<th>LP2-SRODP (std)</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.19929 (0.0027)</td>
<td>1.9909 (0.0019)</td>
<td>2.5287 (0.0163)</td>
<td>21.81</td>
<td>2.1777 (0.0299)</td>
<td>51.63</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.1076</td>
<td>1.1125 (0.0074)</td>
<td>1.1334 (0.0058)</td>
<td>1.9240 (0.0198)</td>
<td>22.17</td>
<td>1.4866 (0.0412)</td>
<td>63.38</td>
</tr>
<tr>
<td>10</td>
<td>0.52</td>
<td>0.5376 (0.0067)</td>
<td>0.5660 (0.0063)</td>
<td>1.3622 (0.0379)</td>
<td>23.34</td>
<td>0.9549 (0.0355)</td>
<td>52.39</td>
</tr>
<tr>
<td>11</td>
<td>0.2137</td>
<td>0.2252 (0.0040)</td>
<td>0.2675 (0.0050)</td>
<td>0.8788 (0.0276)</td>
<td>22.54</td>
<td>0.5523 (0.0283)</td>
<td>53.03</td>
</tr>
<tr>
<td>12</td>
<td>0.082</td>
<td>0.0843 (0.0029)</td>
<td>0.1059 (0.0038)</td>
<td>0.4247 (0.0320)</td>
<td>22.15</td>
<td>0.2406 (0.0336)</td>
<td>52.90</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $\sigma_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\alpha = 5$, $\beta = 0.16, \sigma = 0.9$ and $\rho = 0.1$. We use 25 time steps, 10 hypercubes and 3000 simulations for each hypercube. For each single run of the LSM algorithm and standard RBSDE algorithm, we use 10,000 simulations.
Table 4: Two Dimensional American Put Option Prices

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>PDE</th>
<th>LSM (std)</th>
<th>RBSDE (std)</th>
<th>LP1:SRODP(std)</th>
<th>Time(s)</th>
<th>LP2:SRODP(std)</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>1.9910 (0.0013)</td>
<td>1.9909 (0.0013)</td>
<td>2.1090 (0.0155)</td>
<td>213.90</td>
<td>1.9966 (0.0087)</td>
<td>485.55</td>
</tr>
<tr>
<td>9</td>
<td>1.1076</td>
<td>1.1071 (0.0040)</td>
<td>1.1331 (0.0035)</td>
<td>1.3375 (0.0127)</td>
<td>215.86</td>
<td>1.1257 (0.0277)</td>
<td>488.99</td>
</tr>
<tr>
<td>10</td>
<td>0.52</td>
<td>0.5326 (0.0036)</td>
<td>0.5680 (0.0048)</td>
<td>0.9581 (0.0139)</td>
<td>218.83</td>
<td>0.5441 (0.0126)</td>
<td>457.90</td>
</tr>
<tr>
<td>11</td>
<td>0.2137</td>
<td>0.2227 (0.0021)</td>
<td>0.2665 (0.0028)</td>
<td>0.9581 (0.0139)</td>
<td>218.83</td>
<td>0.5441 (0.0126)</td>
<td>457.90</td>
</tr>
<tr>
<td>12</td>
<td>0.082</td>
<td>0.0816 (0.0014)</td>
<td>0.1059 (0.0026)</td>
<td>0.3393 (0.0054)</td>
<td>230.50</td>
<td>0.0778 (0.0141)</td>
<td>451.28</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $v_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\bar{\alpha} = 5$, $\bar{\beta} = 0.16$, $\sigma = 0.9$ and $\rho = 0.1$. We use 25 time steps, 30 hypercubes and 3000 simulations for each hypercube. For each single run of the LSM algorithm and standard RBSDE algorithm, we use 30,000 simulations.

Table 5: Two Dimensional American Put Option Prices

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>PDE</th>
<th>LSM (std)</th>
<th>RBSDE (std)</th>
<th>LP1:SRODP(std)</th>
<th>Time(s)</th>
<th>LP2:SRODP(std)</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>1.9909 (0.0009)</td>
<td>1.9909 (0.0010)</td>
<td>2.0142 (0.0116)</td>
<td>165.05</td>
<td>2.0052 (0.0121)</td>
<td>500.76</td>
</tr>
<tr>
<td>9</td>
<td>1.1076</td>
<td>1.1055 (0.0032)</td>
<td>1.1332 (0.0029)</td>
<td>1.2060 (0.0219)</td>
<td>167.71</td>
<td>1.1226 (0.0362)</td>
<td>500.71</td>
</tr>
<tr>
<td>10</td>
<td>0.52</td>
<td>0.5326 (0.0027)</td>
<td>0.5682 (0.0036)</td>
<td>0.5237 (0.0307)</td>
<td>169.05</td>
<td>0.5501 (0.0499)</td>
<td>499.92</td>
</tr>
<tr>
<td>11</td>
<td>0.2137</td>
<td>0.2215 (0.0019)</td>
<td>0.2672 (0.0024)</td>
<td>0.2201 (0.0143)</td>
<td>168.17</td>
<td>0.2273 (0.0106)</td>
<td>500.16</td>
</tr>
<tr>
<td>12</td>
<td>0.082</td>
<td>0.0808 (0.00012)</td>
<td>0.1054 (0.0015)</td>
<td>0.1254 (0.0071)</td>
<td>167.26</td>
<td>0.0828 (0.0036)</td>
<td>500.31</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $v_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\bar{\alpha} = 5$, $\bar{\beta} = 0.16$, $\sigma = 0.9$ and $\rho = 0.1$. We use 25 time steps, 50 hypercubes and 1000 simulations for each hypercube. For each single run of the LSM algorithm and standard RBSDE algorithm, we use 50,000 simulations.

Table 6: Two Dimensional American Put Option Prices

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>PDE</th>
<th>LSM (std)</th>
<th>RBSDE (std)</th>
<th>LP1:SRODP(std)</th>
<th>Time(s)</th>
<th>LP2:SRODP(std)</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>1.9909 (0.0009)</td>
<td>1.9909 (0.0010)</td>
<td>2.0142 (0.0054)</td>
<td>652.96</td>
<td>1.9967 (0.0067)</td>
<td>1299.77</td>
</tr>
<tr>
<td>9</td>
<td>1.1076</td>
<td>1.1055 (0.0032)</td>
<td>1.1332 (0.0029)</td>
<td>1.1945 (0.0081)</td>
<td>669.21</td>
<td>1.1130 (0.0305)</td>
<td>1359.20</td>
</tr>
<tr>
<td>10</td>
<td>0.52</td>
<td>0.5326 (0.0027)</td>
<td>0.5682 (0.0036)</td>
<td>0.5101 (0.0181)</td>
<td>635.60</td>
<td>0.5315 (0.0158)</td>
<td>1296.12</td>
</tr>
<tr>
<td>11</td>
<td>0.2137</td>
<td>0.2215 (0.0019)</td>
<td>0.2672 (0.0024)</td>
<td>0.2201 (0.0143)</td>
<td>638.56</td>
<td>0.2241 (0.0077)</td>
<td>1285.82</td>
</tr>
<tr>
<td>12</td>
<td>0.082</td>
<td>0.0808 (0.00012)</td>
<td>0.1054 (0.0015)</td>
<td>0.1254 (0.0071)</td>
<td>635.32</td>
<td>0.0828 (0.0036)</td>
<td>1286.07</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $v_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\bar{\alpha} = 5$, $\bar{\beta} = 0.16$, $\sigma = 0.9$ and $\rho = 0.1$. We use 25 time steps, 50 hypercubes and 3000 simulations for each hypercube. For each single run of the LSM algorithm and standard RBSDE algorithm, we use 50,000 simulations.

Table 7: Two Dimensional American Put Option Prices

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>PDE</th>
<th>LSM (std)</th>
<th>RBSDE (std)</th>
<th>LP1:SRODP(std)</th>
<th>Time(s)</th>
<th>LP2:SRODP(std)</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>1.9908 (0.0006)</td>
<td>1.9908 (0.0009)</td>
<td>1.9930 (0.0042)</td>
<td>1367.99</td>
<td>1.9932 (0.0047)</td>
<td>3855.30</td>
</tr>
<tr>
<td>9</td>
<td>1.1076</td>
<td>1.1063 (0.0021)</td>
<td>1.1325 (0.0022)</td>
<td>1.1375 (0.0105)</td>
<td>1325.64</td>
<td>1.1172 (0.0136)</td>
<td>3856.99</td>
</tr>
<tr>
<td>10</td>
<td>0.52</td>
<td>0.5320 (0.0021)</td>
<td>0.5692 (0.0030)</td>
<td>0.5387 (0.0129)</td>
<td>1307.86</td>
<td>0.5306 (0.0108)</td>
<td>3866.87</td>
</tr>
<tr>
<td>11</td>
<td>0.2137</td>
<td>0.2210 (0.0016)</td>
<td>0.2670 (0.0019)</td>
<td>0.2393 (0.0084)</td>
<td>1308.15</td>
<td>0.2211 (0.0050)</td>
<td>3849.48</td>
</tr>
<tr>
<td>12</td>
<td>0.082</td>
<td>0.0801 (0.0009)</td>
<td>0.1058 (0.0015)</td>
<td>0.0868 (0.0027)</td>
<td>1307.39</td>
<td>0.0812 (0.0032)</td>
<td>3851.47</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $v_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\bar{\alpha} = 5$, $\bar{\beta} = 0.16$, $\sigma = 0.9$ and $\rho = 0.1$. We use 25 time steps, 80 hypercubes and 3000 simulations for each hypercube. For each single run of the LSM algorithm and standard RBSDE algorithm, we use 80,000 simulations.

4.3 American Put Option Prices with Ambiguity

4.3.1 One Dimensional Case

For the one dimensional case with ambiguity, for example under $\kappa$-ignorance, Cheng & Riedel (2013) and Vorbrink (2011) argue that the worst case evaluation is achieved when $\theta = -\kappa$. This means that the density generator will stay invariant, enabling us to directly simulate the state variables under the worst case measure and using the LSM results as the benchmark to test the accuracy of our proposed algorithms, as
the LSM results are known to be close enough to the exact value for simple one dimensional case.

In this part we initially differentiate the SRODP and SLSM algorithm. The implementation of the SLSM algorithm is introduced in Section 3.4. As the drift ambiguity is introduced in the dynamics of state variables, it adds an extra dimension (the stochastic exponential $M_t$) to the state variables. One should choose the number of hypercubes per dimension as appropriate, since the number of total hypercubes grows geometrically when raising dimensions. As we show in Section 4.2.2, regressions with local polynomials up to the second order (LP2) have superior performance compared to LP1. Hence, we present results with LP2 scheme. $\kappa = 0.3$, so the ambiguity interval, i.e. the set $\Theta$ for the density generator is $[-0.3, 0.3]$. We run the LSM and SLSM algorithm for 10 times. The result in 8 shows that the SLSM algorithm works well in one dimensional case with ambiguity, given the benchmark result by the LSM algorithm. The other parameters are in line with experiments in Section 4.2.1. For the SLSM algorithm, we choose the space domain for the logarithmic stochastic exponential to be $[-6, 6]$.

Table 8: One Dimensional American Put Option Prices with Ambiguity

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>LSM (std)</th>
<th>SLSM (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>3.8243 (0.0134)</td>
<td>3.8214 (0.0531)</td>
<td>3.7886, 3.8543</td>
<td>578.87</td>
</tr>
<tr>
<td>38</td>
<td>2.6001 (0.0160)</td>
<td>2.5700 (0.0336)</td>
<td>2.5492, 2.5908</td>
<td>584.49</td>
</tr>
<tr>
<td>40</td>
<td>1.7188 (0.0199)</td>
<td>1.7179 (0.0366)</td>
<td>1.6952, 1.7406</td>
<td>583.62</td>
</tr>
<tr>
<td>42</td>
<td>1.1693 (0.0101)</td>
<td>1.1718 (0.0273)</td>
<td>1.1099, 1.1347</td>
<td>580.02</td>
</tr>
<tr>
<td>44</td>
<td>0.7137 (0.0051)</td>
<td>0.7158 (0.0162)</td>
<td>0.7058, 0.7259</td>
<td>580.76</td>
</tr>
</tbody>
</table>

Note: The strike price is 40, volatility is 0.2, risk free rate is 0.06, time to maturity is 1, and $\theta_t \in [-0.3, 0.3]$. We use 5 time steps. For the SLSM algorithm, we use 50 hypercubes and 2800 simulations for each hypercube, and 11 uniformly selected points ($L = 11$) for the density generator. For each single run of the LSM algorithm, we use 30,000 simulations.

To make the SRODP and the SLSM algorithms directly comparable, we should obtain results for both algorithms, which are computed at approximate levels of computational cost. We conduct several experiments and find that the SRODP algorithm with LP2 scheme and 4000 hypercubes and 2000 simulations per hypercube will have similar computational cost. Table 9 summarizes the results for one dimensional American put options prices with drift ambiguity by the SRODP algorithm. Compared with those prices by the SLSM algorithm in Table 8, the SRODP algorithm have an evident advantage in reducing standard deviations. Later we will see that advantage is even more remarkable in two dimensional case. Taken the LSM results as our benchmark, we also find that the SRODP results are closer to the benchmark than the standard RBSDE results, which are generally upward biased.

Table 9: One Dimensional American Put Option Prices with Ambiguity

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>RBSDE (std)</th>
<th>SRODP (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>3.8787 (0.0145)</td>
<td>3.8429 (0.0412)</td>
<td>3.8173, 3.8684</td>
<td>583.50</td>
</tr>
<tr>
<td>38</td>
<td>2.6490 (0.0120)</td>
<td>2.6036 (0.0262)</td>
<td>2.5874, 2.6199</td>
<td>582.12</td>
</tr>
<tr>
<td>40</td>
<td>1.7747 (0.0127)</td>
<td>1.7268 (0.0307)</td>
<td>1.7028, 1.7408</td>
<td>583.25</td>
</tr>
<tr>
<td>42</td>
<td>1.1872 (0.0078)</td>
<td>1.1285 (0.0122)</td>
<td>1.1200, 1.1360</td>
<td>583.59</td>
</tr>
<tr>
<td>44</td>
<td>0.7758 (0.0051)</td>
<td>0.7126 (0.0141)</td>
<td>0.7039, 0.7213</td>
<td>589.65</td>
</tr>
</tbody>
</table>

Note: The strike price is 40, volatility is 0.2, risk free rate is 0.06, time to maturity is 1, and $\theta_t \in [-0.3, 0.3]$. We use 5 time steps. For the SRODP algorithm, we use 4000 hypercubes and 3000 simulations for each hypercube. For each single run of the standard RBSDE algorithm, we use 30,000 simulations.

Besides, we find that regressions in the SRODP algorithm with only constant local polynomials (LP0) have decent convergence results in one dimensional cases as well. We present results comparison for LP1 and LP0 in Figure 1. Almost all the LSM results lie in the 95% confidence interval of the corresponding SRODP results. It is notable that put options prices with ambiguity will increase to prices without ambiguity when ambiguity interval shrinks to 0 length. This finding is also in accordance with theoretical arguments in Nishimura & Ozaki (2007).

We have tested several different lengths for the space domain of the logarithmic stochastic exponential, and the results are close to each other. They are available upon request.

The linear shape of these slopes is directly related to the Greek Rho, which is a constant here.
Figure 1: The y-axis stands for American put option prices with strike price 40, volatility 0.2, risk free rate 0.06 and time to maturity 1. The x-axis stands for the radius (κ) of ambiguity interval. We use 5 time steps, 1000 hypercubes and 2000 simulations for each hypercube for the SRODP algorithm.

4.3.2 Two Dimensional Case

As for Heston’s model with ambiguity, the space domain remains identical to the case without ambiguity. We employ the elliptical ambiguity as the form of uncertainty set. The reason is that the elliptical ambiguity resembles the form of statistical uncertainty given by the Wald’s test when implementing maximum likelihood estimator for model calibration, as claimed by Cohen & Tegnér (2017). We run the SLSM algorithm for 10 times, and the space domain for the logarithmic stochastic exponential is $[-6, 6]$. Results in Table 10 are obtained by regressions with LP2 scheme. Σ and χ are given as,

$$\Sigma = \begin{bmatrix} 4/9 & 0 \\ 0 & 0.01 \end{bmatrix}, \chi = 9.$$  

Table 10: Two Dimensional American Put Option Prices with Ambiguity

<table>
<thead>
<tr>
<th>S0</th>
<th>RBSDE (std)</th>
<th>SLSM (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.9041 (0.0062)</td>
<td>1.9323 (0.0179)</td>
<td>[1.9212, 1.9434]</td>
<td>7867.27</td>
</tr>
<tr>
<td>9</td>
<td>0.9977 (0.0064)</td>
<td>1.0697 (0.0107)</td>
<td>[1.0631, 1.0763]</td>
<td>7883.07</td>
</tr>
<tr>
<td>10</td>
<td>0.4268 (0.0062)</td>
<td>0.5173 (0.0144)</td>
<td>[0.5084, 0.5262]</td>
<td>7832.34</td>
</tr>
<tr>
<td>11</td>
<td>0.1621 (0.0042)</td>
<td>0.2004 (0.0194)</td>
<td>[0.1884, 0.2124]</td>
<td>7878.48</td>
</tr>
<tr>
<td>12</td>
<td>0.0521 (0.0023)</td>
<td>0.0445 (0.0105)</td>
<td>[0.0379, 0.0510]</td>
<td>7844.89</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $V_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\tilde{\alpha} = 5$, $\tilde{\beta} = 0.16$, $\sigma = 0.9$ and $\rho = 0.1$. We use 5 time steps. For the SLSM algorithm, we use 20 hypercubes and 3000 simulations for each hypercube, and randomly selected 23 discrete points ($L = 23$) in the ellipse. For the standard RBSDE algorithm, we use 10,000 simulations. We launch 10 times of all the algorithms.

For a similar computational budget, we implement the SRODP algorithm with 150 hypercubes each dimension and 3000 simulations per hypercube. The standard RBSDE results are also presented in Table 10 but they are generally upward biased, as we see in one dimensional case. We observe that the SRODP results in 11 have much smaller standard deviations than the SLSM results. Further, we implement
the SLSM algorithm with 30 hypercubes and 3000 simulations per hypercube, and present the results in Table 12. Results in Table 11 (SRODP) and in Table 12 (SLSM) have close standard deviations, but the SLSM algorithm takes almost 3 times computational time in order to reach the convergence, implying that the SRODP algorithm is more efficient in evaluating the American options. It should be noted that we do not have an exact value here, as we argue in the Section 3.4. However, results for the SRODP and SLSM algorithm should be closer as we increase the number of hypercubes or the simulated paths per hypercube.

We can obtain more accurate results for the SLSM algorithm when increase the number of selected points in the set of density generator, as this algorithm is essentially an optimization in pointwise sense.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>RBSDE (std)</th>
<th>SRODP (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.9044 (0.0049)</td>
<td>1.8866 (0.0060)</td>
<td>[1.8829, 1.8903]</td>
<td>7944.22</td>
</tr>
<tr>
<td>9</td>
<td>0.9994 (0.0043)</td>
<td>0.9826 (0.0099)</td>
<td>[0.9765, 0.9886]</td>
<td>7990.26</td>
</tr>
<tr>
<td>10</td>
<td>0.4306 (0.0059)</td>
<td>0.4181 (0.0069)</td>
<td>[0.4139, 0.4224]</td>
<td>7959.31</td>
</tr>
<tr>
<td>11</td>
<td>0.1627 (0.0032)</td>
<td>0.1522 (0.0052)</td>
<td>[0.1490, 0.1554]</td>
<td>7969.27</td>
</tr>
<tr>
<td>12</td>
<td>0.0524 (0.0016)</td>
<td>0.0481 (0.0018)</td>
<td>[0.0470, 0.0492]</td>
<td>7981.40</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $V_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\tilde{\alpha} = 5, \tilde{\beta} = 0.16, \sigma = 0.9$ and $\rho = 0.1$. We use 5 time steps. For the SRODP algorithm, we use 150 hypercubes and 3000 simulations for each hypercube. For the standard RBSDE algorithm, we use 15,000 simulations. We launch 10 times of all the algorithms.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>RBSDE (std)</th>
<th>SLSM (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.9074 (0.0047)</td>
<td>1.9267 (0.0114)</td>
<td>[1.9197, 1.9338]</td>
<td>20576.43</td>
</tr>
<tr>
<td>9</td>
<td>0.9974 (0.0040)</td>
<td>1.0394 (0.0083)</td>
<td>[1.0343, 1.0446]</td>
<td>20587.83</td>
</tr>
<tr>
<td>10</td>
<td>0.4316 (0.0037)</td>
<td>0.4774 (0.0083)</td>
<td>[0.4706, 0.4843]</td>
<td>20586.73</td>
</tr>
<tr>
<td>11</td>
<td>0.1651 (0.0029)</td>
<td>0.1667 (0.0052)</td>
<td>[0.1602, 0.1731]</td>
<td>21042.48</td>
</tr>
<tr>
<td>12</td>
<td>0.0531 (0.0014)</td>
<td>0.0422 (0.0047)</td>
<td>[0.0393, 0.0451]</td>
<td>20511.94</td>
</tr>
</tbody>
</table>

Note: The strike price is 10, initial volatility $V_0$ is 0.0625, risk free rate $r$ is 0.1, and time to maturity is 0.25. $\tilde{\alpha} = 5, \tilde{\beta} = 0.16, \sigma = 0.9$ and $\rho = 0.1$. We use 5 time steps. For the SLSM algorithm, we use 30 hypercubes and 3000 simulations for each hypercube, and randomly selected 23 discrete points ($L = 23$) in the ellipse. For the standard RBSDE algorithm, we use 20,000 simulations. We launch 10 times of all the algorithms.

5 An Application in Real Options: Optimal Fish Harvesting Decision

In this section we discuss the optimal fish harvesting and corresponding fish farm evaluation problem. Generally, the fish farmer, or the farm manager, acts rationally to maximize the future benefits by choosing an optimal harvesting time. Meanwhile, the manager evaluates the value of a single lease of farm, which is the case of single rotation, or the value of the farm ownership, i.e. infinite number of rotations. The evaluation of fish harvesting is of some similarities to the American option pricing. For an explicit description of the problem, one can refer to Ewald et al. (2016) and Asche & Bjorndal (2011).

We are interested in evaluating fish farm with ambiguity in the single-rotation scenario. This means that the manager will earn the revenue of harvesting fish while incurring feeding and harvesting cost, but will have to return the fish farm to the original owner when completing a single harvesting cycle. Essentially, the manager decides whether to harvest or postpone it to future, by weighing the instant harvesting benefits against future expected benefits. In the absence of an actively trading market, the probability measure of expectation used for evaluation is subject to the manager’s personal belief. We use the market pricing measure $Q$ as our reference measure, as it has been used by Ewald et al. (2016) for the fish harvesting problem without ambiguity.

6 In this article we only consider the single-rotation case in order to demonstrate the applicability of our approaches to the two-dimensional real option problem. It can be naturally extended to the infinite-rotation case in a similar way.
We start with the problem without ambiguity. Specifically, we use a two-factor model for the dynamics of state variables under \( Q \) measure, following [Ewald et al. (2016)]

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r - \delta_t)dt + \sigma_1 \hat{\rho} d\hat{W}_t^1 + \sqrt{1 - \hat{\rho}^2} d\hat{W}_t^2, \\
\frac{d\delta_t}{\delta_t} &= \hat{\kappa}(\hat{\alpha} - \delta_t - \hat{\lambda})dt + \sigma_2 d\hat{W}_t^1,
\end{align*}
\]

where \( r, \sigma_1, \hat{\rho}, \hat{\kappa}, \hat{\alpha}, \hat{\lambda}, \) and \( \sigma_2 \) are constants.\(^7\) Further, we assume that the average weight \( w_t \) of an individual fish and the total number \( n_t \) of fish in a farm follow,

\[
\begin{align*}
w_t &= w_{\infty}(\hat{a} - \hat{b} \exp(-\hat{c}t))^3, \\
dn_t &= -\hat{m}_tn_t dt,
\end{align*}
\]

where the equation (27) is known as the Von Bertalanffy growth function, which is extensively used in fisheries biology (see for example [Haddon (2010)]), and \( w_{\infty} \) is the asymptotic weight of fish, \( \hat{a}, \hat{b}, \) and \( \hat{c} \) are constants. \( \hat{m}_t \) represents the mortality rate and is assumed to be constant \( \hat{m} \) here for simplicity. By solving an ordinary differential equation (28) we have

\[n_t = n_0 \exp\{-\hat{m}t\}.\]

Therefore, the biomass \( \hat{M}_t \) of the fish farm is given by

\[\hat{M}_t = n_t w_t.\]

We assume the fish farmer begins with a certain amount of infant fish, thus there is no release cost. Then the value function \( J_t \) is naturally defined by maximizing the expected harvesting benefits

\[
J_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^Q[e^{-r(t-\tau)}(S_{\tau} \hat{M}_{\tau} - C_1 \hat{M}_t) - \int_{\tau}^{\tau + \Delta} e^{-r(s-\tau)} C_2 F_s n_s ds | \mathcal{F}_t],
\]

where \( \mathcal{T}_t \) is defined in Section 2.4. \( C_1 \) stands for the harvesting cost per kilogram and \( C_2 \) is the feeding cost per kilogram per year. \( F_t = \int_0^t w_s ds \), where \( w_t \) is the first order derivative of \( w_t \) representing the fish growth rate, and \( f \) is the feed conversion ratio, according to [Asche & Bjorndal (2011)]. Applying the dynamic programming principle, we obtain a discrete-time approximation of the Hamilton-Jacobi-Bellman equation after equally discretize the time horizon \([0, T]\) with each part to be \( \Delta t \).

\[
\hat{J}_t = \max \left\{ S_t \hat{M}_t - C_1 \hat{M}_t, e^{-r \Delta t} \mathbb{E}^Q[J_{\tau + \Delta t} | \mathcal{F}_t] - \Delta t C_2 F_{\tau} n_{\tau} \right\}.
\]

**Proof.** From (29) we have

\[
\hat{J}_t = \max \left\{ S_t \hat{M}_t - C_1 \hat{M}_t, \max_{\tau + \Delta t \leq \tau \leq T} \mathbb{E}^Q[e^{-r(t-\tau)}(S_{\tau} \hat{M}_{\tau} - C_1 \hat{M}_t) - \int_{\tau}^{\tau + \Delta t} e^{-r(s-\tau)} C_2 F_s n_s ds | \mathcal{F}_t] \right\},
\]

where the first equality holds by splitting the investment decision (29) between harvesting now (at time \( t \)) and wait for a short time period and reevaluate the fish farm later (at time \( t + \Delta t \)); the second equality holds by the fact that \( F(\cdot) \) and \( n(\cdot) \) are deterministic functions; the third equality holds by the tower law of the conditional expectation; the last equality holds by the definition of value function \( \hat{J}_{t + \Delta t} \), and approximating \( \int_{\tau}^{\tau + \Delta t} e^{-r(s-\tau)} C_2 F_s n_s ds \) by \( \Delta t C_2 F_{\tau} n_{\tau} \) (This is justified when \( \Delta t \) goes to zero and terms higher than \( \Delta t \) are eliminated). \( \blacksquare \)

\( ^7 \)Some of symbols here are the same to our previous notations for Heston’s model. We hope this will not confuse readers.
Such a discrete-time dynamic programming algorithm \[\text{[30]}\] enables us to use numerical methods to evaluate the fish farm, for example, the LSM algorithm. In the following proposition we show that the value function \(\hat{J}_t\) coincides with the solution of an RBSDE, which is similar to \[\text{[10]}\].

**Proposition 5.1.** Consider the RBSDE

\[-dY_t = g(t, Y_t, Z_t)dt + dK_t - Z^*_t d\tilde{W}_t, \quad Y_T = S_T \hat{M}_T - C_1 \hat{M}_T,\]

with the constraints

\[Y_t \geq L_t, \quad \int_0^T (Y_s - L_s) dK_s = 0 \quad \text{and} \quad K_0 = 0.\]

Assume that the generator \(g(t, Y_t, Z_t) = -rY_t - C_2 F_t n_t\), the obstacle \(L_t = S_t \hat{M}_t - C_1 \hat{M}_t\) and \(K_t\) is a continuous and increasing process.

There exists a unique solution \(\{(Y_t, Z_t, K_t), t \in [0, T]\}\) of the above RBSDE, which are \(\mathcal{F}_t\) progressively measurable processes. The process \(Y_t\) has the dual representation

\[Y_t = \text{ess sup}_{\tau \in \mathcal{F}_t} \mathbb{E}^\theta [e^{-r(\tau - t)}(S_t \hat{M}_t - C_1 \hat{M}_t) - \int_t^\tau e^{-r(s-t)}C_2 F_s n_s ds | \mathcal{F}_t], \quad t \in [0, T].\] \[\text{(31)}\]

**Proof.** The above RBSDE can be taken as a special case in Proposition 2.3 of \[\text{El Karoui, Kapoudjian, et al. [1997]}\]. To show the solution corresponds to the value of an optimal stopping problem, it suffices to show that the RBSDE satisfies the technical conditions in Definition \[\text{2.1}\]. As \(F(\cdot)\) and \(n(\cdot)\) are deterministic and continuous functions, the generator is uniform Lipschitz, and belongs to \(\mathbb{H}^2\). \(M_t\) is deterministic and continuous, meaning that it is bounded in \([0, T]\). Hence, it can be seen that \(L_t\) is in \(S^2\). The terminal value \(Y_T = L_T\) again is in \(S^2\), implying that \(Y_T \in \mathbb{L}^2\). Thus, the solution to the above RBSDE has a probabilistic representation form \[\text{(31)}\]. Then, the uniqueness follows directly from Theorem 5.2 in \[\text{El Karoui, Kapoudjian, et al. [1997]}\].

Without ambiguity, the Proposition 5.1 allows us to evaluate a single-rotation fish farm by solving an RBSDE. However, the probability measure used for fish farm evaluation is not restricted to the market pricing measure, as we argue that usually there is no actively trading market for that. There is no need to concern about the arbitrage opportunity. Changing the evaluation measure \(Q\) to \(Q^\theta\) defined in Section \[\text{2.2} \]

the following linear RBSDE

\[-dY^\theta_t = g(t, Y^\theta_t, Z^\theta_t)dt + dK^\theta_t - Z^*_t d\tilde{W}_t, \quad Y^\theta_T = S_T \hat{M}_T - C_1 \hat{M}_T,\]

with generator \(g(t, Y^\theta_t, Z^\theta_t) = -rY^\theta_t - C_2 F_t n_t - \theta^\theta Z^\theta_t\) and obstacle \(L_t = S_t \hat{M}_t - C_1 \hat{M}_t\) has a unique solution \((Y^\theta_t, Z^\theta_t, K^\theta_t), t \in [0, T]\). This claim is analogous to Proposition 2.1 so we skip the proof here. In addition, \(Y^\theta\) has the representation

\[Y^\theta_t = \text{ess sup}_{\tau \in \mathcal{F}_t} \mathbb{E}^\theta [e^{-r(\tau - t)}(S_t \hat{M}_t - C_1 \hat{M}_t) - \int_t^\tau e^{-r(s-t)}C_2 F_s n_s ds | \mathcal{F}_t].\]

When considering drift ambiguity, of which the ambiguity sets \(\mathcal{P}^\theta\) are convex and compact, we can still solve the worst case evaluation problem via solving an RBSDE. The RBSDE and duality arguments are analogous to that in the American option case. We begin with defining the value of a fish farm in the worst case

\[\hat{v}_t := \text{ess sup}_{\tau \in \mathcal{F}_t, Q^\theta \in \mathcal{P}^\theta} \mathbb{E}^Q^\theta [e^{-r(\tau - t)}(S_t \hat{M}_t - C_1 \hat{M}_t) - \int_t^\tau e^{-r(s-t)}C_2 F_s n_s ds | \mathcal{F}_t], \quad t \in [0, T].\] \[\text{(33)}\]

**Proposition 5.2.** Under the \(\kappa\)-ignorance and the elliptical ambiguity framework as in the section \[\text{2.2} \]

the fish farm value \(\hat{v}_t\) under the worst case belief as defined in \[\text{(33)}\] is the value of a minimax (optimal) control problem such that,

\[\hat{v}_t = \text{ess inf}_{\theta \in \Theta} Y^\theta_t,\]
and we have,

\[
\text{ess sup}_{\tau \in \mathcal{T}} \text{ess inf}_{\theta \in \Theta} \mathbb{E}^\theta[H_{t+\tau}|F_t] = \text{ess inf}_{\theta \in \Theta} \text{ess sup}_{\tau \in \mathcal{T}} \mathbb{E}^\theta[H_{t+\tau}|F_t], \quad \forall t \in [0, T],
\]

where \( Y^\theta_t \) is an element of the solution to (32), and \( H_{t+\tau} = e^{-r(\tau-t)}(S^\tau_t \tilde{M}_t - C_1 \tilde{M}_t) - \int_t^\tau e^{-r(s-t)}C_2 F_s n_s \, ds \). Moreover, there exists a pair \((\tilde{\tau}, \tilde{\theta}_t)\), \( \tilde{\tau} \in [0, T] \), \( \tilde{\theta}_t \in \Theta \) such that it reaches the optimal (saddle) point \((\tilde{\tau}, \tilde{\theta}_t)\) when the generator \( g(t, Y_t^\tilde{\theta}, Z_t^\tilde{\theta}) = \text{ess inf}_{\theta \in \Theta} g(t, Y_t^\theta, Z_t^\theta) \), \( \forall t \in [0, T] \).

**Proof.** As the proof is almost identical to that of Proposition 2.2 except for the payoff function and generator, we skip it here. Specifically, under elliptical ambiguity \( \Theta = \{ \theta : \theta \Sigma^{-1} \theta^* \leq \chi \} \), the optimal generator is

\[
g(t, Y_t^\tilde{\theta}, Z_t^\tilde{\theta}) = -r Y_t^\tilde{\theta} - \sqrt{Z_t^\tilde{\theta} \Sigma^* Z_t^\tilde{\theta}} \chi - C_2 F_t n_t,
\]

with \( \tilde{\theta} = -\frac{Z_t^\tilde{\theta} \Sigma^*}{\sqrt{Z_t^\tilde{\theta} \Sigma^* Z_t^\tilde{\theta} / \chi}} \).

Provided the optimal generator (35) and the duality arguments (34), it is direct that the RBSDE (32) with such a generator has a unique solution, as we have an analogous argument in Remark 2.1. Therefore, we can utilize the previous SRODP algorithm in Section 3.2 and SLSM algorithm in Section 3.4 to evaluate the fish farm. Parameters in Table 13 are extracted from Ewald et al. (2016) for the numerical experiments. We start by demonstrating examples without ambiguity in Table 14, in order to show that results are close under different algorithms.

### Table 13: Parameters for Fish Farm

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{m} )</td>
<td>10%</td>
<td>( C_2 )</td>
<td>7</td>
<td>( \hat{\alpha} )</td>
<td>1.135</td>
</tr>
<tr>
<td>( f )</td>
<td>1.1</td>
<td>( \hat{\alpha} )</td>
<td>1.113</td>
<td>( \lambda )</td>
<td>1.142</td>
</tr>
<tr>
<td>( n_0 )</td>
<td>1</td>
<td>( b )</td>
<td>1.097</td>
<td>( \hat{\rho} )</td>
<td>0.736</td>
</tr>
<tr>
<td>( \omega_{\infty} )</td>
<td>6</td>
<td>( \hat{c} )</td>
<td>1.43</td>
<td>( \sigma_1 )</td>
<td>0.153</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>3</td>
<td>( \kappa )</td>
<td>1.012</td>
<td>( \sigma_2 )</td>
<td>0.206</td>
</tr>
</tbody>
</table>

### Table 14: Fish Farm Value without Ambiguity

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>LSM (std)</th>
<th>RBSDE (std)</th>
<th>SRODP (std)</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.3123 (0.0013)</td>
<td>1.3122 (0.0017)</td>
<td>1.3122 (0.0035)</td>
<td>175.26</td>
</tr>
<tr>
<td>20</td>
<td>4.5613 (0.0044)</td>
<td>4.5618 (0.0041)</td>
<td>4.5596 (0.0072)</td>
<td>175.69</td>
</tr>
<tr>
<td>32</td>
<td>7.8061 (0.0076)</td>
<td>7.8071 (0.0060)</td>
<td>7.8161 (0.0135)</td>
<td>176.26</td>
</tr>
</tbody>
</table>

Note: The initial convenience yield \( V_0 \) is -0.3, risk free rate \( r \) is 0.0393, and time to maturity is 0.25. We use 5 time steps. For the SRODP algorithm, we use 50 hypercubes and 3000 simulations for each hypercube. For the LSM and standard RBSDE algorithm, we use 10,000 simulations. We launch 10 times of all the algorithms.

In ambiguous case, we adopt the same parameters for the uncertainty set as we define in (26). Still, we can observe from Table 15 and Table 16 that results by the SRODP algorithm have smaller standard deviations than the SLSM algorithm, given approximate computational budgets. Despite that, the SLSM algorithm can be used in cases when the RBSDE technique is not suitable. For example, the SLSM technique does not apply when the optimal generator cannot be obtained explicitly, as the existence in Proposition 2.2 and Proposition 5.2 relies heavily on the compactness of the uncertainty set. It is not possible to use the RBSDE technique when the value function is not the solution of an RBSDE. Yet these are left for future research.
Table 15: Fish Farm Value with Ambiguity

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>RBSDE (std)</th>
<th>SLSM (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.2129 (0.0090)</td>
<td>1.1832 (0.0216)</td>
<td>[1.1698, 1.1966]</td>
<td>3835.66</td>
</tr>
<tr>
<td>20</td>
<td>4.2897 (0.0280)</td>
<td>4.0111 (0.0687)</td>
<td>[3.9685, 4.0536]</td>
<td>4094.91</td>
</tr>
<tr>
<td>32</td>
<td>7.3635 (0.0535)</td>
<td>6.8086 (0.1221)</td>
<td>[6.7329, 6.8843]</td>
<td>4099.33</td>
</tr>
</tbody>
</table>

Note: The initial convenience yield $V_0$ is -0.3, risk free rate $r$ is 0.0393, and time to maturity is 0.25. We use 5 time steps. For the SLSM algorithm, we use 20 hypercubes and 2000 simulations for each hypercube. For the standard RBSDE algorithm, we use 10,000 simulations. We launch 10 times of all the algorithms.

Table 16: Fish Farm Value with Ambiguity

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>RBSDE (std)</th>
<th>SRODP (std)</th>
<th>95% C.I.</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.2188 (0.0039)</td>
<td>1.1884 (0.0104)</td>
<td>[1.1820, 1.1949]</td>
<td>4731.34</td>
</tr>
<tr>
<td>20</td>
<td>4.3066 (0.0198)</td>
<td>4.1704 (0.0272)</td>
<td>[4.1536, 4.1873]</td>
<td>4682.14</td>
</tr>
<tr>
<td>32</td>
<td>7.4226 (0.0481)</td>
<td>7.1065 (0.0354)</td>
<td>[7.0846, 7.1284]</td>
<td>4725.71</td>
</tr>
</tbody>
</table>

Note: The initial convenience yield $V_0$ is -0.3, risk free rate $r$ is 0.0393, and time to maturity is 0.25. We use 5 time steps. For the SRODP algorithm, we use 70 hypercubes and 5000 simulations for each hypercube. For the standard RBSDE algorithm, we use 20,000 simulations. We launch 10 times of all the algorithms.

6 Concluding Remarks

In this paper, we explore the evaluation of American options with stochastic volatility and single-rotation fish farms with stochastic convenience yield under drift ambiguity framework. We formulate the value function to the solution of a reflected backward differential equations (RBSDEs) and prove the uniqueness of the solution. Moreover, we propose an algorithm (Stratified Regression One-step Forward Dynamic Programming) to numerically solve RBSDEs, combining the traditional numerical RBSDE method by Gobet & Lemor (2008) with a general stratification approach by Gobet et al. (2016). However, the RBSDE approach relies heavily on the explicit formulation of the generators. We also raise another possible numerical algorithm (Stratified Least Square Monte Carlo) without using the theory of RBSDEs, taking advantage of dynamic programming and the general stratification. We conduct numerical experiments to show the convergence of two algorithms. In one dimensional case, our results are in line with the theoretical arguments in Cheng & Riedel (2013) and Vorbrink (2011). Further, the SRODP algorithm exhibits superior efficiency in both one and two dimensional cases.

References


