Country portfolios with habit persistence

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Abstract

This article studies the composition of country portfolios when consumers’ utility function features habit persistence.

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1 Optimality conditions

1.1 External habit

The lagrangian function related to the representative home household $z$’s maximazing problem is given by

$$L = E_t \sum \theta_t U(C_t(z) - hC_{t-1})$$

$$- \sum \lambda_t [(\alpha_{1t} + \alpha_{2t}) - (\alpha_{1t-1}r_{1t} + \alpha_{2t-1}r_{2t}) - Y_t + C_t]$$

where the habit-adjusted consumption is defined as the deviation of the level of individual consumption in this period to the level of average aggregate consumption in the last period. Similar to the analysis as in the case with internal habit (below), the orthogonality condition will be

$$E_t \left\{ \hat{\delta}_{zt+1} \hat{d}_{t+1} \right\} = 0 \quad (*)$$

where the consumption differential is given by $\hat{d}_{t+1} = \hat{C}_{zt+1} - \hat{C}^*_{zt+1} - \frac{1}{p} \hat{Q}_{t+1}$.

1.2 Internal habit

The lagrangian function corresponding to the home households’ maximazing problem is given by

$$L = E_t \sum \theta_t U(C_t - hC_{t-1})$$

$$- \sum \lambda_t [(\alpha_{1t} + \alpha_{2t}) - (\alpha_{1t-1}r_{1t} + \alpha_{2t-1}r_{2t}) - Y_t + C_t]$$

The first order condition for the optimal $C_t$ is

$$E_t \left[ \theta_t U'(C_{zt}) - \theta_{t+1}hU'(C_{zt+1}) \right] = \lambda_t$$

Condition for the optimal $\alpha_1$
\[-\lambda_t + \lambda_{t+1}r_{1t+1} = 0\]

Condition for the optimal \(\alpha_2\)

\[-\lambda_t + \lambda_{t+1}r_{2t+1} = 0\]

Combining above three conditions to get

\[E_t \left[ \left( \theta_{t+1}U' (C_{xt+1}) - \theta_{t+2}hU' (C_{xt+2}) \right) r_{1t+1} \right]\]

\[= E_t \left[ \left( \theta_{t+1}U' (C_{xt+1}) - \theta_{t+2}hU' (C_{xt+2}) \right) r_{2t+1} \right]\] (1)

Similarly for the foreign country, the related lagrangian function is

\[L^* = E_t \sum \theta_t U (C_t^* - h^*C_{t-1}^*) - \sum \lambda_t^* \left[ \frac{1}{Q_t} (\alpha_{1t}^* + \alpha_{2t}^*) - \frac{1}{Q_{t+1}} (\alpha_{1t-1}^* r_{1t} + \alpha_{2t-1}^* r_{2t}) - Y_t^* + C_t^* \right]\]

Conditions for the optimal \(C_t^*\), \(\alpha_{1t}^*\) and \(\alpha_{2t}^*\)

\[E_t \left[ \theta_t U' (C_{xt}^*) - \theta_{t+1}h^*U' (C_{xt+1}^*) \right] = \lambda_t^*\]

\[-\frac{1}{Q_t} \lambda_t^* + \frac{1}{Q_{t+1}} \lambda_{t+1}^* r_{1t+1} = 0\]

\[-\frac{1}{Q_t} \lambda_t^* + \frac{1}{Q_{t+1}} \lambda_{t+1}^* r_{2t+1} = 0\]

Combining above three conditions to get

\[E_t \left[ \left( \theta_{t+1}U' (C_{xt+1}^*) - \theta_{t+2}h^*U' (C_{xt+2}^*) \right) \frac{1}{Q_{t+1}} r_{1t+1} \right]\]

\[= E_t \left[ \left( \theta_{t+1}U' (C_{xt+1}^*) - \theta_{t+2}h^*U' (C_{xt+2}^*) \right) \frac{1}{Q_{t+1}} r_{2t+1} \right]\] (2)
The conditions (1) and (2) are key Euler equations.

The utility function has the form of \( U(C) = \frac{C^{1-\rho}}{1-\rho} \). The marginal utility is \( U(C) = C^{-\rho} \). The discount factor follows \( \theta_{t+1} = \theta_1 \beta \). Using these facts and approximating the condition (1) gives (I omit the time subscript for \( r_1 \) and \( r_2 \) here)

\[
\hat{\epsilon}_1 + \frac{1}{2} \hat{r}_1^2 - \rho \hat{C}_{xt+1} \hat{r}_1 - \beta h \left( \hat{\epsilon}_1 + \frac{1}{2} \hat{r}_1^2 - \rho \hat{C}_{xt+2} \hat{r}_1 \right) = 0
\]

\[
\hat{\epsilon}_2 + \frac{1}{2} \hat{r}_2^2 - \rho \hat{C}_{xt+1} \hat{r}_2 - \beta h \left( \hat{\epsilon}_2 + \frac{1}{2} \hat{r}_2^2 - \rho \hat{C}_{xt+2} \hat{r}_2 \right) = 0
\]

\[
E \left\{ (1 - \beta h) \left( \hat{r}_x + \frac{1}{2} \hat{r}_x^2 \right) - \rho \hat{r}_x \left( \hat{C}_{xt+1} - \beta h \hat{C}_{xt+2} \right) \right\} = 0
\]

Without the habit persistence, the condition is

\[
E \left\{ \hat{r}_x + \frac{1}{2} \hat{r}_x^2 - \rho \hat{r}_x \hat{C}_{t+1} \right\} = 0
\]

which corresponds to equation (12) in Devereux and Sutherland (DS) (2008).

Similarly to approximate the condition (2) note that \( Q^{-1} C_{xt+1} r_1 \) on the left hand side (LHS) can be approximated as

\[
\begin{align*}
\tilde{Q}^{-1} \tilde{C}^{-\rho \tilde{r}} - (Q - \tilde{Q}) \tilde{C}^{-\rho \tilde{r}} - \rho \tilde{C}^{-\rho - 1 \tilde{r}} (C - \tilde{C}) \tilde{Q}^{-1} \\
+ [\tilde{Q}^{-1} \tilde{C}^{-\rho} (r_1 - \tilde{r})] + \frac{1}{2} \tilde{Q}^{-1} \rho (\rho + 1) \tilde{C}^{-\rho - 2 \tilde{r}} \tilde{C}^2 \hat{C} \\
+ \left[ (-\rho) \tilde{C}^{-\rho - 1} \tilde{C} \tilde{r} \hat{C} \tilde{r}_1 \tilde{Q}^{-1} + \tilde{C}^{-\rho \tilde{r}_1} (-1) \tilde{Q} \tilde{Q} \frac{1}{\tilde{Q}^2} \right] \\
+ (-\rho) \tilde{C}^{-\rho - 1} \tilde{C} \tilde{C} (-1) \frac{1}{\tilde{Q}^2} \tilde{Q} \tilde{r} 
\end{align*}
\]

All the terms other than the three terms in squared brackets can be concealed out with the counterparts on the right hand side (RHS), which leaves the following if (i)
without habit persistence

\[ \tilde{Q}^{-1} \tilde{C}^{-\rho} (r_1 - \tilde{r}) - \rho \tilde{C}^{-\rho-1} \tilde{C} \tilde{Q}^{-1} \tilde{r}_1 \tilde{Q}^{-1} \tilde{Q} = RHS \]

Divided by \( \tilde{Q}^{-1} \tilde{C}^{-\rho} \) the LHS leaves

\[ (r_1 - \tilde{r}) - \rho \tilde{C} \tilde{r}_1 - \tilde{r} \tilde{r}_1 \hat{Q} \]

Can be approximated as

\[ \tilde{r} \left( \tilde{r}_1 + \frac{1}{2} \tilde{r}_1^2 \right) - \rho \tilde{C} \tilde{r}_1 - \tilde{r} \tilde{r}_1 \hat{Q} \]

\[ = \tilde{r} \left( \tilde{r}_1 + \frac{1}{2} \tilde{r}_1^2 - \rho \tilde{C} \tilde{r}_1 - \tilde{r}_1 \hat{Q} \right) \]

Combined with RHS it follows

\[ E_t \left\{ \hat{r}_x + \frac{1}{2} \hat{r}_x^2 - \rho \hat{r}_x \hat{C}_{t+1} - \hat{r}_x \hat{Q} \right\} = 0 \]

which corresponds to equation (36) in DS (2008).

If (\( ii \)) with habit persistence, the LHS is

\[ \left( \tilde{r}_1 + \frac{1}{2} \tilde{r}_1^2 - \rho \tilde{C}^{*}_{t+1} \tilde{r}_1 - \tilde{r}_1 \hat{Q} \right) - \beta h^* \left( \tilde{r}_1 + \frac{1}{2} \tilde{r}_1^2 - \rho \tilde{C}^{*}_{t+2} \tilde{r}_1 - \tilde{r}_1 \hat{Q} \right) \]

Combining with RHS gives

\[ E_t \left\{ (1 - \beta h^*) \left( \hat{r}_x + \frac{1}{2} \hat{r}_x^2 - \hat{r}_x \hat{Q} \right) - \rho \hat{r}_x \left( \hat{C}^{*}_{t+1} - \beta h^* \hat{C}^{*}_{t+2} \right) \right\} = 0 \]

\[ E_t \left\{ \hat{r}_x + \frac{1}{2} \hat{r}_x^2 - \hat{r}_x \hat{Q} - \frac{\rho \hat{r}_x}{(1 - \beta h^*)} \left( \hat{C}^{*}_{t+1} - \beta h^* \hat{C}^{*}_{t+2} \right) \right\} = 0 \]

Equation (3) and (4) can be used to get the orthogonality condition. Subtracting (4) from (3) we have the key orthogonality condition

\[ E_t \left\{ \hat{r}_{xt+1} c \hat{d}_{t+1} \right\} = 0 \]

(\( ** \))
if denote

\[ cd_{t+1} = \frac{(\hat{C}_{xt+1} - \beta h \hat{C}_{xt+2})}{(1 - \beta h)} - \frac{(\hat{C}_{xt+1}^* - \beta h^* \hat{C}_{xt+2}^*)}{(1 - \beta h^*)} - \frac{1}{\rho} \hat{Q}_{t+1} \]

Notice from equation (**) if without habit persistence \((h = h^* = 0)\), the consumption differential is the same as in the equation (37) of DS (2008) and equation (18) of DS (2012).

2 Budget constraint

To get the behaviour of consumption differential \(cd\), start from the home budget constraint

\[ W_t = \alpha_1 r_{xt} + r_2 W_{t-1} + Y_{ct} - C_t \]

where the \(Y_c\) denotes the disposable income and \(\bar{Y}_c = \bar{C}\). Approximating gives

\[ W_t - \bar{W} = (\alpha - \bar{\alpha}) \bar{r}_x + \bar{\alpha} \bar{r}_x + \bar{r} (W_{t-1} - \bar{W}) + (r_2 - \bar{r}) \bar{W} + (Y_{ct} - \bar{Y}_{ct}) - (C_t - \bar{C}) \]

The first and forth terms on the RHS are zero so

\[ W_t - \bar{W} = \bar{\alpha} \bar{r}_x + \bar{r} (W_{t-1} - \bar{W}) + (Y_{ct} - \bar{Y}_{ct}) - (C_t - \bar{C}) \]

Divided by the home country GDP, the equation becomes

\[ \frac{W_t - \bar{W}}{Y} = \bar{\alpha} \bar{r}_x + \bar{r} \frac{(W_{t-1} - \bar{W})}{Y} + \frac{\bar{Y}_c}{Y} \frac{(Y_{ct} - \bar{Y}_{ct})}{Y_c} - \frac{\bar{C}}{Y} (C_t - \bar{C}) \]

With new definition of variables, it becomes

\[ \hat{W}_t = \bar{\alpha} \bar{r}_x + \frac{1}{\beta} \hat{W}_{t-1} + \bar{c} (\hat{Y}_c - \bar{C}) \]

where \(\bar{c} = \frac{\bar{C}}{\bar{r}}\) denotes the home ratio of steady-state consumption to GDP. Rearrange the equation to get
\[
\hat{C} = \hat{C}_t + \frac{1}{\hat{c}} \hat{a} \hat{r}_x + \frac{1}{\hat{c}} \hat{a} \hat{r}_{t-1} - \frac{1}{\hat{c}} \hat{W}_t
\]  

(5)

Similarly, for the foreign budget constraint

\[
\frac{W^*_t}{Q_t} = \frac{1}{Q_t} \left( \alpha_t^* r_{xt} + r_{2t} W^*_{t-1} \right) + Y^*_c - C^*_t
\]

Approximating the equation gives (omitting stars)

\[
\frac{1}{Q} (W - \bar{W}) - \frac{\bar{W}}{Q^2} (Q - \bar{Q}) = \frac{1}{Q} \left[ (\alpha - \bar{a}) \bar{r}_x + \bar{a} (r_x - \bar{r}_x) + \bar{r} (W - \bar{W}) + (r - \bar{r}) \bar{W} \right] - \frac{[\bar{a} \bar{r}_x + \bar{r} \bar{W}]}{Q^2} (Q - \bar{Q}) + (Y - \bar{Y}) - (C - \bar{C})
\]

Divided by foreign country GDP, the equation becomes

\[
\frac{1}{Q} \hat{W}^*_t - 0 = \frac{1}{Q} \left[ 0 + \hat{a}^* \hat{r}_x + \frac{1}{\beta} \hat{W}^*_{t-1} + 0 \right] + \hat{c}^* \hat{Q} \left( \hat{Y}^* - \hat{C}^* \right)
\]

or

\[
\hat{C}^* = \hat{Y}^*_c + \frac{1}{\hat{c}^* Q} \hat{a}^* \hat{r}_x + \frac{1}{\hat{c}^* Q} \hat{W}^*_{t-1} - \frac{1}{\hat{c}^* Q} \hat{W}^*_t
\]

Notice that \( \hat{W}^*_t = \hat{W}_t \) and

\[
\hat{c}^* = \frac{\hat{a}^*}{\hat{Y}^*} = \frac{-\hat{a}}{\hat{Y}^*} = \frac{-\hat{Y}}{\hat{Y}^* \hat{a}}
\]

so above equation can be written as

\[
\hat{C}^* = \hat{Y}^*_c - \frac{\hat{Y}}{\hat{Y}^* \hat{c}^* Q} \hat{a}^* \hat{r}_x - \frac{1}{\hat{c}^* Q} \hat{W}^*_{t-1} + \frac{1}{\hat{c}^* Q} \hat{W}_t
\]  

(6)
3 Portfolios by variance-covariance decomposition approach

It is ready now to represent the optimal portfolios by combining the first order dynamics of excess return $\hat{r}_{xt+1}$ and that of consumption differential $c\hat{d}_{xt+1}$. For convenience, I list the home and foreign budget constraint (equation (5) and (6)) as below

$$\hat{C}_t = \hat{Y}_{et} + \frac{1}{c} \hat{\alpha} \hat{r}_{xt} + \frac{1}{c} \beta \hat{W}_{t-1} - \frac{1}{c} \hat{W}_t$$ (5)

$$\hat{C}^*_t = \hat{Y}_{et} - \frac{\bar{Y}}{c} \hat{r}_{xt} - \frac{1}{\bar{Q} \beta} \hat{W}_{t-1} + \frac{1}{\bar{Q}} \hat{W}_t$$ (6)

3.1 External habit

For the case with external habit formation, the first order dynamics of $c\hat{d}_{xt+1}$ is described by

$$c\hat{d}_{t+1} = \hat{C}_{xt+1} - \hat{C}^*_{xt+1} - \frac{1}{\rho} \hat{Q}_{t+1}$$

which corresponds to variable $cd$ in the code. Because $C_{xt+1} = C_{t+1} - h C_t$, approximation gives

$$(1 - h) \hat{C}_{xt+1} = \hat{C}_{t+1} - h \hat{C}_t$$

Substituting into the consumption differential we get

$$c\hat{d}_{t+1} = \frac{1}{1 - h} \left( \hat{C}_{t+1} - h \hat{C}_t \right) - \frac{1}{1 - h^*} \left( \hat{C}^*_{t+1} - h \hat{C}^*_t \right) - \frac{1}{\rho} \hat{Q}_{t+1}$$

So

$$\sum \beta^i c\hat{d}_{t+1+i} = \sum \beta^i \left[ \frac{1}{1 - h} \left( \hat{C}_{t+1+i} - h \hat{C}_{t+i} \right) - \frac{1}{1 - h^*} \left( \hat{C}^*_{t+1+i} - h \hat{C}^*_t \right) - \frac{1}{\rho} \hat{Q}_{t+1+i} \right]$$

By equation (5) and (6), above equation equals to
\[ \sum \beta^i \left[ \frac{1}{1-h} (\hat{Y}_{t+1+i} - h\hat{Y}_{t+i}) - \frac{1}{1-h^*} (\hat{Y}^*_{t+1+i} - h\hat{Y}^*_{t+i}) \right] \\
+ \sum \beta^i \left[ +\omega_1 \cdot 2\hat{\alpha}r_{xt+1+i} - \omega_2 \cdot 2\hat{\alpha}\hat{r}_{xt+i} - \frac{1}{\rho} \hat{Q}_{t+1+i} \right] \quad + t.i. \]

where

\[
\omega_1 = \left[ \frac{1}{1-h} \frac{1}{2\bar{c}} + \frac{1}{1-h^*} \frac{\bar{Y}}{\bar{c}^*Q} \right]
\]

\[
\omega_2 = \left[ \frac{h}{1-h} \frac{1}{2\bar{c}} + \frac{h^*}{1-h^*} \frac{\bar{Y}}{\bar{c}^*Q} \right]
\]

and \( t.i. \) denotes terms of irrelevance. The summation is equivalent to

\[
\sum \beta^i c\hat{d}_{t+1+i} = \frac{1}{(1-\beta)} \bar{c}\hat{d}_{t+1} \\
= \sum \beta^i \left[ \frac{1-\beta h}{1-h} \hat{Y}_{t+1+i} - \frac{1-\beta h^*}{1-h^*} \hat{Y}^*_{t+1+i} + \omega_2 \cdot 2\hat{\alpha}r_{xt+1+i} - \frac{1}{\rho} \hat{Q}_{t+1+i} \right] \quad + t.i. \]

So

\[
\bar{c}\hat{d}_{t+1} = (1-\beta) \left( \Gamma + \omega \cdot 2\hat{\alpha}r_{xt+1} + t.i. \right) \quad (7)
\]

where

\[
\Gamma = \sum \beta^i \left[ \frac{1-\beta h}{1-h} \hat{Y}_{t+1+i} - \frac{1-\beta h^*}{1-h^*} \hat{Y}^*_{t+1+i} - \frac{1}{\rho} \hat{Q}_{t+1+i} \right] \\
\omega = \omega_1 - \beta \omega_2
\]

Putting equation (7) back into equation (\(*\)) gives

\[
E_t \{ \hat{r}_{xt+1} (\Gamma + \omega \cdot 2\hat{\alpha}r_{xt+1}) \} = 0
\]

Or
\[
\hat{\alpha} = -\frac{1}{2\omega} \frac{\text{cov}(\Gamma, \hat{r}_{x+1})}{\text{var}(\hat{r}_{x+1})}
\]

### 3.2 Internal habit

For the case of internal habit formation, the first order dynamics of \(c\hat{d}_{x+1}\) is described by

\[
c\hat{d}_{t+1} = \left(\frac{\hat{C}_{xt+1} - \beta h \hat{C}_{xt+2}}{(1 - \beta h)}\right) - \left(\frac{\hat{C}^*_{xt+1} - \beta h^* \hat{C}^*_{xt+2}}{(1 - \beta h^*)}\right) - \frac{1}{\rho} \hat{Q}_{t+1}
\]

Again, by \((1 - h) \hat{C}_{xt+1} = \hat{C}_{t+1} - h\hat{C}_t\) above consumption differential equals to

\[
c\hat{d}_{t+1} = \frac{1}{1 - \beta h} \left(\frac{1}{1 - h} \hat{C}_{t+1} - h \hat{C}_t\right) - \frac{\beta h}{1 - \beta h} \left(\frac{1}{1 - h} \hat{C}_{t+2} - h \hat{C}_{t+1}\right)
- \frac{1}{1 - \beta h^*} \left(\frac{1}{1 - h^*} \hat{C}^*_{t+1} - h^* \hat{C}^*_t\right) - \frac{\beta h^*}{1 - \beta h^*} \left(\frac{1}{1 - h^*} \hat{C}^*_{t+2} - h^* \hat{C}^*_t\right)
- \frac{1}{\rho} \hat{Q}_{t+1}
\]

Similarly to the case of external habit formation

\[
c\hat{d}_{t+1} = (1 - \beta) \sum \beta^i \left[\frac{1}{1 - \beta h} \frac{1}{1 - h} (1 - \beta h) \hat{C}_{t+1+i} - \frac{\beta h}{1 - \beta h} \frac{1}{1 - h} \left(\frac{1}{\beta} - h\right) \hat{C}_{t+1+i}\right]
- (1 - \beta) \sum \beta^i \left[\frac{1}{1 - \beta h^*} \frac{1}{1 - h^*} (1 - \beta h^*) \hat{C}^*_{t+1+i} - \frac{\beta h^*}{1 - \beta h^*} \frac{1}{1 - h^*} \left(\frac{1}{\beta} - h^*\right) \hat{C}^*_{t+1+i}\right]
- \frac{1}{\rho} \hat{Q}_{t+1}
\]

\[
c\hat{d}_{t+1} = \hat{C}_{t+1} - \hat{C}^*_t - \frac{1}{\rho} \hat{Q}_{t+1}
\]

which can be used to code \(cd\) in DS solution.

To express the portfolios by decomposition, we have to find \(\Gamma\) and \(\omega\).
REFERENCES


