Escaping Expectation Traps: How Much Commitment is Required?*

Christoph Himmels† University of Exeter

Tatiana Kirsanova‡ University of Glasgow

September 26, 2012

Abstract

In this paper we study the degree of precommitment that is required to eliminate multiplicity of policy equilibria, which arise if the policy maker acts under pure discretion. We apply a framework developed by Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010) to a standard New Keynesian model with government debt. We demonstrate the existence of expectation traps under limited commitment and identify the minimum degree of commitment which is needed to escape from these traps. We find that the degree of precommitment which is sufficient to generate uniqueness of the Pareto-preferred equilibrium requires the policy maker to stay in office for a period of two to five years. This is consistent with monetary policy arrangements in many developed countries.

Key Words: Limited Commitment, Commitment, Discretion, Multiple Equilibria, Monetary and Fiscal Policy Interactions

JEL References: E31, E52, E58, E61, C61

---

*We thank Andy Blake, Xiaoshan Chen, Richard Dennis, Campbell Leith, Charles Nolan and all seminar participants in Exeter, St.Andrews, ANU, Bank of England and at the Bundesbank. All errors are ours.

†Address: Department of Economics, Business School, University of Exeter, Streatham Court, Rennes Drive, Exeter EX4 4PU; e-mail ch308@exeter.ac.uk.

‡Address: Economics, Adam Smith Building, University of Glasgow, Glasgow G12 8RT; e-mail tatiana.kirsanova@glasgow.ac.uk

1 Introduction

In this paper we study the existence and uniqueness properties of monetary policy in a limited commitment framework in the Blanchard and Kahn (1980) class of linear quadratic rational expectation models (LQ RE). This class of models is typically used to study aggregate fluctuations in macroeconomics. Building on research in Schaumburg and Tambalotti (2007) and Debertoli and Nunes (2010) we show the existence of multiple equilibria under limited commitment policy.[1] Similar to the case of pure discretion, under limited commitment policy makers cannot manage private sector expectations which can lead to expectation traps and coordination failures. We investigate the question of how much precommitment is needed to escape such expectation traps and to coordinate on the Pareto-preferred equilibrium. We find that the necessary degree of precommitment to eliminate multiplicity is relatively small – from two to five years - which is consistent with tenure terms of monetary policy makers in many countries.

It is well known that in LQ models with rational expectations policies under commitment and discretion may imply very different dynamics for the economy. With full commitment the policy maker has complete control over the private sector's expectations about future policy and steers them in a way that furthers his stabilization goals. The policy maker can coordinate all future actions of consequent policy makers, which allows him to choose once, and apply indefinitely, an intertemporal contingency plan (Kydland and Prescott (1977)). In linear quadratic models a commitment policy, if it exists, is always unique (Kwakernaak and Sivan (1972), Backus and Driffill (1986)).

With no commitment at all, i.e. under pure discretion, the policy maker does not control the expectations of the private sector and fails to coordinate the actions of consequent policy makers. Under discretion the policy maker optimizes in each period of time and the private sector knows that future policy makers will implement the same decision process in subsequent periods (see e.g. Oudiz and Sachs (1985), Backus and Driffill (1986), Currie and Levine (1993)). However, under pure discretionary policy expectation traps and multiple equilibria can arise because the expectations of the private sector are shaped by anticipations about future policy actions. Since the policy maker cannot fully control private sector expectations, those expectations may trap the policy maker into implementing a policy that validates them. The trap is closed if it is less costly for the policy maker to validate the private sector beliefs about future policy than to ignore

[1] Originally, their framework is based on Roberds (1987). Lohmann (1992) studied limited commitment policies in a one-period setting.

those expectations, see King and Wolman (2004).[2]

Under limited commitment a new policy maker arrives in office with an exogenous probability α every period, reneges on the past policy plan of its predecessor and credibly commits to a new policy plan that is optimal at this point in time. Clearly, this framework has elements of both discretion and commitment. However, the policy maker can neither completely control the expectations of the private sector, nor can he coordinate the actions of all future policy makers. Therefore coordination failures between the sequence of policy makers and the private sector can occur and may result in multiple equilibria and expectation traps. Models with expectation traps can help us to explain the observed excess volatility of macroeconomic data.[3] These models should also be used to improve macroeconomic policy to avoid such traps.

Our contribution is twofold. First, we demonstrate, by example, that similar to discretion expectation traps also exist under limited commitment.[4] We use a simple New Keynesian (NK) model with government debt accumulation which describes an economic behavior that is familiar from the literature on the fiscal theory of the price level (see e.g. Leeper (1991)). Second, we obtain the minimum degree of policy precommitment that is required to select the best equilibrium. We demonstrate that a small degree of precommitment is enough to select the best equilibrium; a tenure of about 2-5 years is sufficient to eliminate all equilibria except the Pareto-preferred.

The paper is organized as follows. In Section 2 we introduce the NK model with debt accumulation. We first review properties of discretion and commitment policies for this model and demonstrate the existence of expectation traps under quasi-commitment. Then we find the minimum length of precommitment that is required to select the best equilibrium in our model. Section 5 concludes. Finally, the Appendix presents a numerical algorithm to find policy with limited commitment.

2 The Model with Government Debt

This section demonstrates the existence of multiple equilibria under limited commitment by example. We present a simple NK model with government debt accumulation in the spirit of

[2] Dynamic RE models with multiple discretionary equilibria are presented in King and Wolman (2004) and Blake and Kirsanova (2012). Lockwood and Philippopoulos (1994), Albanesi et al. (2003) give examples of multiplicity in models with static expectations.

[3] Discretionary policy with multiple equilibria generates data series which can be observed as satisfying a Markov-switching regime (Blake and Kirsanova (2012)). There is much empirical evidence on such regimes; for one example which uses a similar model as we study here see Davig and Leeper (2006).

[4] Schaumburg and Tambalotti (2007) term limited commitment 'quasi-commitment' and Debertoli and Nunes (2010) use 'loose commitment'. In this paper we use these terms interchangeably.

Leeper (1991). This model is well suited to use as an example to demonstrate the existence of expectation traps and to study the dynamic properties of an economy under monetary policy with limited commitment. First, unlike the model in Schaumburg and Tambalotti (2007) this model has an endogenous predetermined state variable, government debt, which is affected by policy. The presence of such a variable is crucial to generate multiple equilibria under discretionary policy in LQ RE models (Blake and Kirsanova (2012)). A necessary condition for multiplicity is the existence of strategic complementarities between the decisions of agents. An endogenous state variable ensures that the current policy maker reacts (indirectly) to the past actions of the private sector and his predecessors. Therefore the policy maker can be trapped into implementing an undesired policy, if it is less costly to validate the expectations formed in the past, than sticking to his initial policy plan. Second, the model is simple enough to derive most of our results analytically.[5]

We adopt the model from Benigno and Woodford (2003).[6] The economy consists of a representative household, a representative firm that produces the final good, a continuum of intermediate goods producing firms and a monetary and fiscal authority. The intermediate goods producing firms act under monopolistic competition and produce according to a production function that depends only on labor. Goods are combined via a Dixit and Stiglitz (1977) technology to produce aggregate output. Firms set their prices subject to a Calvo (1983) price rigidity. Households choose consumption and leisure and can transfer income through time through their holdings of government bonds. All agents can observe and affect the accumulation of real government debt. The accumulation of government debt must depend on a fiscal stance. Hence, there is a non-optimizing fiscal authority facing a stream of exogenous public consumption. These expenditures are financed by levying income taxes and by issuing one-period risk-free nominal bonds. We assume that the fiscal authority imposes a simple proportional rule for the tax rate: if real debt is higher (lower) than in the steady state the tax rate rises (falls). We shall refer to the tax rate as 'taxes' and to the parameter of the proportional rule as the 'fiscal feedback'. The size of this fiscal feedback measures the strength of the fiscal stabilization of debt and, as we shall show, plays an important role in the model. The presence of the non-optimizing fiscal authority in the economy is captured by this single feedback parameter μ.

We assume that all public debt consist of riskless one-period bonds. Accordingly, the nominal

[5] Debertoli and Nunes (2010) use a non-linear model to illustrate a generalization of the quasi-commitment equilibrium concept to a non-linear setting. Their model is not suitable for our analysis because of the assumption of non-linearity.

[6] It was also used in Blake and Kirsanova (2012) to demonstrate existence and investigate the properties of multiple equilibria under discretionary policy.

value of end-of-period public debt $\mathcal{B}_t$ evolves according to the following law of motion:

$$\mathcal{B}_t = (1+i_{t-1})\,\mathcal{B}_{t-1} + P_t G_t - \Upsilon_t P_t Y_t, \tag{1}$$

where Υ_t represents the share of nominal income that the government taxes in period t. G_t denotes government purchases which are exogenously given. The aggregate price level is denoted by P_t and the nominal interest rate of government bonds is denoted by i_t. The national income identity yields

$$Y_t = C_t + G_t, \tag{2}$$

where C_t is private consumption. For analytical convenience, we define $B_t = (1+i_{t-1})\mathcal{B}_{t-1}/P_{t-1}$ as a measure of real government debt. Because B_t is observed at the beginning of period t, (1) can be rewritten as

$$B_{t+1} = (1+i_t)\left(B_t\frac{P_{t-1}}{P_t} - \Upsilon_t Y_t + G_t\right). \tag{3}$$

We assume that fiscal policy is conducted according to a simple mechanistic feedback rule that relates the tax rate, Υ_t, to the stock of real debt, B_t

$$\Upsilon_t = \tilde{\Upsilon}\left(\frac{B_t}{\tilde{B}}\right)^{\mu\frac{\tilde{B}}{\tilde{Y}}}. \tag{4}$$

Here and below the tilde denotes the steady-state value of the corresponding variable in the model's zero-inflation non-stochastic steady state.

Log-linearizing (3) and (4) yields

$$b_{t+1} = \frac{\tilde{B}}{\tilde{Y}}\iota_t + \frac{1}{\beta}\left(\left(1-\mu\tilde{\Upsilon}\right)b_t - \frac{\tilde{C}}{\tilde{Y}}\tilde{\Upsilon}c_t - \frac{\tilde{B}}{\tilde{Y}}\pi_t\right), \tag{5}$$

where $b_t = \frac{\tilde{B}}{\tilde{Y}}\ln\left(\frac{B_t}{\tilde{B}}\right)$, $c_t = \ln\left(\frac{C_t}{\tilde{C}}\right)$ and $\iota_t = \ln\frac{(1+i_t)}{(1+\tilde{\imath})}$. The private sector's discount factor, β, satisfies $\beta = 1/(1+\tilde{\imath})$. To make the model particularly simple we assume $\tilde{B} = 0$, which eliminates the first-order effect of the interest rate and inflation on debt, and obtain the final version of linearized debt accumulation equation

$$b_{t+1} = \rho b_t - \eta c_t, \tag{6}$$

where the parameter $\rho = \left(1-\mu\tilde{\Upsilon}\right)/\beta$ is a function of the tax rate, implying that with stronger fiscal feedback μ the stock of real debt is stabilized more rapidly, and where the parameter $\eta = \tilde{C}\tilde{\Upsilon}/\left(\beta\tilde{Y}\right)$ describes the sensitivity of debt to the tax base.

The derivation of the appropriate Phillips curve is standard (Benigno and Woodford (2003), Sec. A.5) and real marginal cost is a function of output and taxes. Log-linearizing the price-setting-firms' pricing decision around the zero-inflation non-stochastic steady state yields the following New Keynesian Phillips curve

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \delta \left(\left(\frac{1}{\sigma} + \frac{\theta}{\psi} \right) c_t + \frac{\tilde{\Upsilon}}{\left(1 - \tilde{\Upsilon}\right)} \tau_t \right) + u_t,$$

where $\delta = \frac{(1-\gamma\beta)(1-\gamma)\psi}{\gamma(\psi+\epsilon)}$ is the slope of Phillips curve, $\tau_t = \ln\left(\frac{\Upsilon_t}{\tilde{\Upsilon}}\right)$, σ is the inverse of the intertemporal elasticity of substitution, ψ is the elasticity of labour supply, $\theta = \tilde{C}/\tilde{Y}$ is the steady state consumption to output ratio and u_t is an AR(1) cost push shock with persistence parameter ρ_u. $\mathbb{E}_t$ is the expectation operator conditional on information available at time t. Substituting the log-linearized equations (2) and (4) into the Phillips curve yields

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa c_t + \nu b_t + u_t, \tag{7}$$

where $\nu = \mu\kappa\tilde{\Upsilon}/\left(1 - \tilde{\Upsilon}\right)$ and $\kappa = \delta\left(1/\sigma + \theta/\psi\right)$.

In summary, the model is described by the debt accumulation equation (6) and the Phillips curve (7). The aggregate agents' decision variable is inflation, π_t, and the initial state, $b_0 = \bar{b}$, is known to all agents. We assume that the policy maker chooses consumption c_t. In contrast to the standard NK model (used in Schaumburg and Tambalotti (2007)) the next-period predetermined state variable, b_{t+1}, is affected by policy, c_t.

The intertemporal welfare criterion of the policymaker is defined by the following quadratic objective

$$L = \frac{1}{2} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \lambda c_t^2\right). \tag{8}$$

This criterion is microfounded and derived under the assumption of a steady state labour subsidy, in the absence of technology and taste shocks.[7] Parameter λ is a function of model parameters, $\lambda = \theta\kappa/\epsilon$, and ϵ is the elasticity of substitution between any pair of monopolistically produced goods.

The policy maker knows the laws of motion (6)-(7) of the aggregate economy and takes them into account when formulating policy.

[7] For a derivation see Kirsanova and Wren-Lewis (2011). Of course we could get technology and taste shocks and reinterpret the log-linearized variables as 'gap' variables.

3 Preliminaries: Discretion and Commitment

We shall compare the dynamics of the model under quasi-commitment policy with dynamics under the two limiting cases, discretion and commitment.[8] This Section gives all necessary definitions and presents solutions to these two limiting cases in a comparable form using the model above as an example.

3.1 Discretionary Policy

Under discretion there is a sequence of policy makers: each period a new policy maker arrives in office. The new policy maker chooses the best policy knowing that he stays in office for only one period and the next-period's policy maker will re-optimize again.[9] The law of motion of the aggregate economy (6)-(7) is known by the policy maker and taken into account when he formulates the optimal policy. Furthermore, the policy maker finds the best action every period and knows that future policy makers have the freedom to change policy, but will apply the same decision process. At every point in time t the decision rules of each agent are linear functions of the current state

$$c_t = c_u u_t + c_b b_t, \tag{9}$$

$$\pi_t = \pi_u u_t + \pi_b b_t. \tag{10}$$

Note that from

$$\begin{aligned}&\mathbb{E}_t \pi_{t+1} \overset{eq.(10)}{=} \pi_u \mathbb{E}_t u_{t+1} + \pi_b b_{t+1} \overset{eq.(6)}{=} \pi_u \rho_u u_t + \pi_b \left(\rho b_t - \eta c_t\right) \\ &\overset{eq.(7)}{=} \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta} c_t - \frac{\nu}{\beta} b_t - \frac{1}{\beta} u_t,\end{aligned}$$

it follows that the private sector's decision can also be written as

$$\pi_t = \left(\beta \pi_u \rho_u + 1\right) u_t + \left(\beta \rho \pi_b + \nu\right) b_t + \left(\kappa - \beta \eta \pi_b\right) c_t. \tag{11}$$

The policy maker moves first within each period and the private sector observes the action of the policy maker. Thus, the private sector takes into account the 'instantaneous' influence of the policy choice measured by $\left(\kappa - \beta \eta \pi_b\right)$.

We can give now a more precise definition of discretionary policy: A policy determined by (9) is *discretionary* if the policy maker *finds it optimal* to follow it in every period $s > t$, given

[8] In this section we largely follow the approach and results in Blake and Kirsanova (2012) and in Kirsanova and Wren-Lewis (2011), but present the results in a form that is most convenient for our purposes.

[9] Our definition of discretionary policy is standard and follows Oudiz and Sachs (1985), Backus and Driffill (1986) and Clarida et al. (1999).

the private sector (i) observes the current policy, (ii) knows that future policy makers re-optimize and use the same decision process, and (iii) expects policy (9) will be implemented in all future periods.

We can write the criterion for optimality as

$$S_{uu}u_t^2 + 2S_{ub}u_t b_t + S_{bb}b_t^2 = \min_{c_t}\left(\left(\pi_t^2 + \lambda c_t^2\right) + \beta\left(S_{uu}u_{t+1}^2 + 2S_{ub}u_{t+1}b_{t+1} + S_{bb}b_{t+1}^2\right)\right), \quad (12)$$

subject to constraints (6) and (11).

One can solve the problem using Lagrange multipliers. The expected Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_t^d &= \frac{1}{2}\left(\pi_t^2 + \lambda c_t^2\right) + \beta\frac{1}{2}\left(S_{uu}\rho_u^2 u_t^2 + 2S_{ub}\rho_u u_t b_{t+1} + S_{bb}b_{t+1}^2\right) \\ &\quad + \xi_{t+1}\left(\rho b_t - \eta c_t - b_{t+1}\right) \\ &\quad + \phi_{t+1}\left(\pi_t - \kappa c_t - \nu b_t - u_t - \beta\left(\pi_u\rho_u u_t + \pi_b b_{t+1}\right)\right).\end{aligned} \quad (13)$$

This approach exploits the intertemporal representation (6)-(7) together with the underlying assumption that the private sector's expectations about its own future decisions will be necessarily a function of the future state.

Only current period constraints matter for the policy maker and the first order conditions can be written as

$$0 = \beta S_{bb}b_{t+1} + \beta S_{ub}\rho_u u_t - \xi_{t+1} - \beta\pi_b\phi_{t+1}, \quad (14)$$

$$0 = \pi_t + \phi_{t+1}, \quad (15)$$

$$0 = \lambda c_t - \eta\xi_{t+1} - \kappa\phi_{t+1}, \quad (16)$$

$$0 = \rho b_t - \eta c_t - b_{t+1}, \quad (17)$$

$$0 = \beta\pi_b b_{t+1} + \kappa c_t + \nu b_t - \pi_t + \left(1 + \beta\pi_u\rho_u\right)u_t. \quad (18)$$

The optimal policy response can be written in the form of (9) with

$$c_u = -\frac{\left(\left(\kappa - \beta\pi_b\eta\right)\left(\beta\pi_u\rho_u + 1\right) - \eta\beta S_{ub}\rho_u\right)}{\left(\beta\eta^2 S_{bb} + \left(\kappa - \beta\eta\pi_b\right)^2 + \lambda\right)} \quad (19)$$

$$c_b = -\frac{\left(\left(\kappa - \beta\pi_b\eta\right)\left(\beta\pi_b\rho + \nu\right) - \eta\beta S_{bb}\rho\right)}{\left(\beta\eta^2 S_{bb} + \left(\kappa - \beta\eta\pi_b\right)^2 + \lambda\right)} \quad (20)$$

and so the components of the value function satisfy the following equations

$$
\begin{aligned}
S_{uu} &= ((\beta\pi_u\rho_u + 1) + (\kappa - \beta\pi_b\eta)\, c_u)^2 + \lambda c_u^2 \\
&\quad +\beta\left(\rho^2 S_{uu} - 2\rho\eta S_{ub}c_u + \eta^2 S_{bb}c_u^2\right), && (21)\\
S_{\eta b} &= ((\beta\pi_u\rho_u + 1) + (\kappa - \beta\pi_b\eta)\, c_u)\,((\beta\rho\pi_b + \nu) + (\kappa - \beta\eta\pi_b)\, c_b) && (22)\\
&\quad +\lambda c_u c_b + \beta S_{ub}\rho_u(\rho - \eta c_b) - \beta S_{bb}\eta c_u(\rho - \eta c_b),\\
S_{bb} &= ((\beta\rho\pi_b + \nu) + (\kappa - \beta\eta\pi_b)\, c_b)^2 + \beta S_{bb}(\rho - \eta c_b)^2 + \lambda c_b^2. && (23)
\end{aligned}
$$

This yields the following coefficients in (10)

$$\pi_u = \beta\pi_u\rho_u + 1 + (\kappa - \beta\eta\pi_b)\, c_u, \tag{24}$$

$$\pi_b = \beta\rho\pi_b + \nu + (\kappa - \beta\eta\pi_b)\, c_b. \tag{25}$$

The coefficients $\{c_u, c_b, \pi_u, \pi_b,\ S_{uu}, S_{ub}, S_{bb}\}$ describe the solution to the discretionary optimization problem outlined above. They uniquely define the trajectories $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ for any given $b_0 = \bar{b}$. Conversely, if the sequence $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ solves the discretionary policy outlined above, then there is a unique set of coefficients $\{c_u, c_b, \pi_u, \pi_b,\ S_{uu}, S_{ub}, S_{bb}\}$ that satisfies equations (19)-(25). We call the set of coefficients $\{c_u, c_b, \pi_u, \pi_b,\ S_{uu}, S_{ub}, S_{bb}\}$ a discretionary equilibrium.

Note that the discretionary equilibrium is fully characterized by the deterministic component of the solution, $\{\pi_b, c_b, S_{bb}\}$. Indeed, we can solve system (19)-(25) in a recursive way. We first solve (20), (23) and (25) for $\{c_b, \pi_b, S_{bb}\}$ and then solve the rest of the system for the stochastic component of the solution. We use this well known fact to find all discretionary equilibria in the following simple and illustrative way.[10]

Suppose the policy maker guesses the response of the private sector to the state, π_b. Then the optimal discretionary policy is given by the pair (20) and (23). We find c_b and therefore the optimal response π_b^* of the private sector is given by (25). Then, for every - not necessarily optimal - π_b we can compute a unique π_b^* and plot the dependence $\pi_b^*(\pi_b)$, see the first panel in Figure 1, Panel I. Clearly, if $\pi_b = \pi_b^*$ we have a solution to the discretionary problem.

Our benchmark calibration is standard and follows Schaumburg and Tambalotti (2007) and Blake and Kirsanova (2012). The model's frequency is quarterly. The subjective discount rate β is set to 0.99, the government share of total output $1-\rho$ is 0.25. The elasticity of intertemporal substitution σ is 1/2, the Frisch elasticity of labor supply $\varphi = 1/2$, and the elasticity of demand $\epsilon = 5$. The Calvo parameter $\gamma = 0.75$ and the cost-push shock is an exogenous process with

[10] See Anderson et al. (1996) on certainty equivalence in this class of models and Blake and Kirsanova (2012) for explicit formulae for stochastic components as functions of deterministic components for discretionary models.

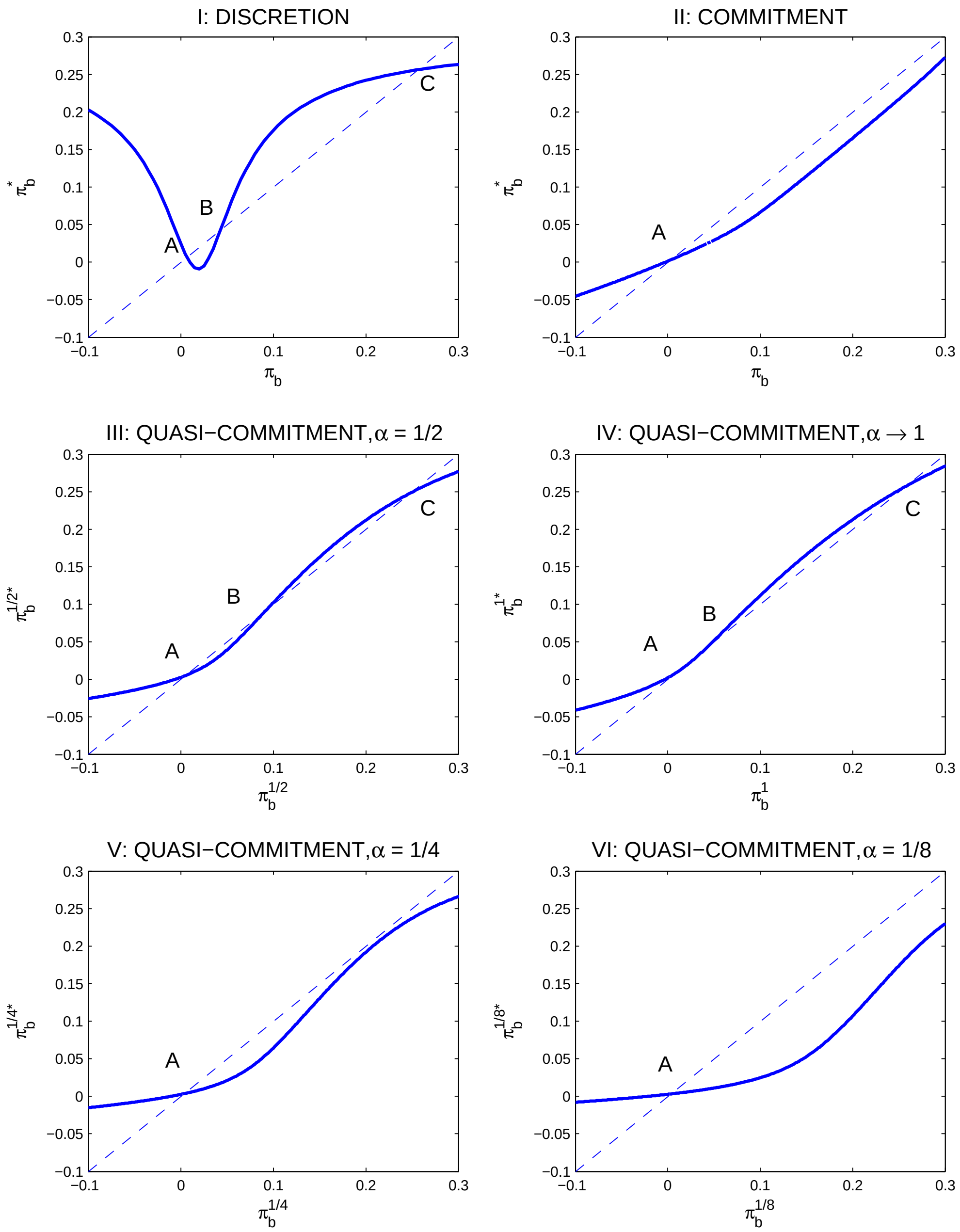

Figure 1: Multiple policy equilibria for different degrees of precommitment

standard deviation 0.005 and $\rho_u = 0$. Finally, the most crucial parameter for our results is the fiscal feedback, μ. The recent empirical evidence suggests that, although the strength of fiscal feedback varies across countries and with time, the chosen value of $\mu = 0.05$ is realistic. See e.g. Leeper et al. (2010) who find a reaction of labour taxes to debt of about 0.05 percentage points for the post-1960 period in the US; see also Coenen and Straub (2005) and Forni et al. (2009) who estimate the response of taxes to debt for the Euro Area.

For our baseline calibration the graph of $\pi_b^*(\pi_b)$ intersects the 45^o degree line in three points labelled A, B and C, so we have three discretionary policy equilibria.[11] A moderate inflation, set by the firms in response to a given debt level, π_b, increases the marginal return to a policy decision that increases consumption in response to this level of debt, c_b. Higher consumption raises demand and firms will increase their response to debt, π_b^*. This complementarity ensures the steepness of $\pi_b^*(\pi_b)$ and three equilibria arise.

The three equilibria, whose characteristics are presented in Table 1 result in qualitatively and quantitatively different dynamics of the economy. Figure 2, which shows the responses of key variables to a unit markup shock for equilibria A and C (as equilibrium B is similar to equilibrium A for the benchmark calibration) using dotted lines with markers.[12] Focusing first on equilibrium A, inflation rises following the markup shock and the policy response is to defer consumption (by raising the nominal interest rate sufficiently high, this is implicit in our model). The decline in consumption lowers output and government tax revenues, which leads to a rise in government debt. In subsequent periods, although interest rates are lowered to stimulate the economy and bring it out of recession, government debt is brought back to baseline predominantly through (primary) fiscal surpluses, rather than through a decline in the cost of financing government debt.

In equilibrium C monetary policy responds to the markup shock by stimulating consumption and output, raises real marginal costs, and causes inflation to rise by more than it otherwise would. This monetary policy causes tax revenues to rise and leads to a decline in government debt. To stabilize government debt, future policy makers raise the cost of financing government debt, which causes consumption, output, and real marginal costs to decline and places downward pressure on inflation. In the spirit of Leeper (1991) monetary policy can be thought of as being active in equilibria A and B and passive in equilibrium C. Table 1 reveals this trade-off between

[11] The graph is continuous. This is because the denominator of (20) is always positive: $S_{bb} > 0$, and $\lambda > 0$ (see Blake and Kirsanova (2012)). In order to find points of intersection we note that $\pi_b^*(\pi_b) - \pi_b$ changes sign in these points. We obtain the solutions with the tolerance level 1e-12.

[12] These impulse responses are identical in each panel.

			Eq. A	Eq. B	Eq. C
Characteristics of Discretionary Policy Equilibria					
(1)	Policy Reaction	$\begin{bmatrix} c_u & c_b \end{bmatrix}$	$\begin{bmatrix} -4.8 & -0.02 \end{bmatrix}$	$\begin{bmatrix} -4.5 & -0.01 \end{bmatrix}$	$\begin{bmatrix} -0.4 & 1.9 \end{bmatrix}$
(2)	Private Sector Reaction	$\begin{bmatrix} \pi_u & \pi_b \end{bmatrix}$	$\begin{bmatrix} 0.7 & 0.01 \end{bmatrix}$	$\begin{bmatrix} 0.8 & 0.02 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.3 \end{bmatrix}$
(3)	Value Function	$\begin{bmatrix} S_{uu} & S_{ub} \\ S_{ub} & S_{bb} \end{bmatrix}$	$\begin{bmatrix} 0.73 & 0.01 \\ 0.01 & 0.0004 \end{bmatrix}$	$\begin{bmatrix} 0.76 & 0.02 \\ 0.02 & 0.01 \end{bmatrix}$	$\begin{bmatrix} 1.00 & 0.28 \\ 0.28 & 0.17 \end{bmatrix}$
(4)	Normalized Loss	L	1.3326	1.3872	1.8283
Characteristics of Commitment Policy Equilibrium					
(5)	Policy Reaction	$\begin{bmatrix} c_u & c_b & c_\phi \end{bmatrix}$	$\begin{bmatrix} -3.6 & -0.01 & 3.6 \end{bmatrix}$	–	–
(6)	Private Sector Reaction	$\begin{bmatrix} \pi_u & \pi_b & \pi_\phi \end{bmatrix}$	$\begin{bmatrix} 0.5 & 0.002 & 0.45 \end{bmatrix}$	–	–
(7)	Normalized Loss	L	1.0	–	–
Degree of Precommitment Required to Select the Best Equilibrium					
(8)	Duration of commitment period to select Eq. A	$1/\alpha$ quarters	7	–	–

Table 1: Properties of Discretionary Equilibria in the NK Model with Debt Accummulation

the response to government debt and the response to the markup shock: The more ‘actively’ the policy maker behaves, the stronger is the policy-induced recession in response to the mark-up shock.

3.2 Commitment Policy

Under the full commitment policy the policy maker optimizes only once, in the initial moment. He chooses a contingency plan, which is than applied indefinitely but can be implemented sequentially. If there is a change of policy makers, the subsequent policy maker continues the policy of its predecessor; therefore we can assume that there is only one policy maker which takes office in period zero and stays infinitely. When optimizing, the policy maker internalizes the effect of its choice on private sector's expectations and solves the following Lagrangian

$$\mathcal{L}^c = \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} \left(\pi_t^2 + \lambda c_t^2 \right) + \xi_{t+1} \left(\rho b_t - \eta c_t - b_{t+1} \right) + \phi_{t+1} \left(\pi_t - \kappa c_t - \nu b_t - u_t - \beta \pi_{t+1} \right) \right).$$

The corresponding first order conditions are:

$$
\begin{aligned}
0 &= -\xi_t + \rho\beta\xi_{t+1} - \nu\beta\phi_{t+1}, && (26)\\
0 &= \pi_t + \phi_{t+1} - \phi_t, && (27)\\
0 &= \lambda c_t - \eta\xi_{t+1} - \kappa\phi_{t+1}, && (28)\\
0 &= \rho b_t - \eta c_t - b_{t+1}, && (29)\\
0 &= \beta\pi_{t+1} + \kappa c_t + \nu b_t + u_t - \pi_t, && (30)
\end{aligned}
$$

for $t \geq 0$; with initial conditions $b_0 = \bar{b}$ and $\phi_0 = 0$, and the transversality condition $\lim_{t\to\infty} b_t < \infty$.

The solution to the commitment problem can be written in the following linear form

$$
\begin{aligned}
\pi_t &= \pi_u u_t + \pi_b b_t + \pi_\phi \phi_t, && (31)\\
c_t &= c_u u_t + c_b b_t + c_\phi \phi_t, && (32)\\
\xi_t &= \xi_u u_t + \xi_b b_t + \xi_\phi \phi_t, && (33)
\end{aligned}
$$

where the coefficients satisfy the following algebraic matrix Riccati equation (see Appendix A for details):

$$
\begin{aligned}
\begin{bmatrix} c_u & c_b & c_\phi \\ \pi_u & \pi_b & \pi_\phi \\ \xi_u & \xi_b & \xi_\phi \end{bmatrix} &= \begin{bmatrix} \lambda + \eta^2\xi_b & \kappa + \eta\xi_\phi & 0 \\ \beta\eta\pi_b - \kappa & \beta\pi_\phi + 1 & 0 \\ \beta\eta\rho\xi_b & -\beta\left(\nu - \rho\xi_\phi\right) & 1 \end{bmatrix}^{-1} \\
&\times \begin{bmatrix} \eta\xi_u\rho_u & \eta\rho\xi_b & \kappa + \eta\xi_\phi \\ \beta\rho_u\pi_u + 1 & \nu + \beta\rho\pi_b & \beta\pi_\phi \\ \beta\rho\xi_u\rho_u & \beta\rho^2\xi_b & -\beta\left(\nu - \rho\xi_\phi\right) \end{bmatrix}
\end{aligned} \tag{34}
$$

and the components of the value function S satisfy the following algebraic matrix Riccati equation:

$$
\begin{aligned}
\begin{bmatrix} S_{uu} & S_{ub} & S_{u\phi} \\ S_{ub} & S_{bb} & S_{b\phi} \\ S_{u\phi} & S_{b\phi} & S_{\phi\phi} \end{bmatrix} &= \begin{bmatrix} \pi_u^2 + \lambda c_u^2 & \pi_b\pi_u + \lambda c_b c_u & \pi_u\pi_\phi + \lambda c_u c_\phi \\ \pi_b\pi_u + \lambda c_b c_u & \pi_b^2 + \lambda c_b^2 & \pi_b\pi_\phi + \lambda c_b c_\phi \\ \pi_u\pi_\phi + \lambda c_u c_\phi & \pi_b\pi_\phi + \lambda c_b c_\phi & \pi_\phi^2 + \lambda c_\phi^2 \end{bmatrix} \\
+\beta \begin{bmatrix} \rho_u & -\eta c_u & -\pi_u \\ 0 & \rho - \eta c_b & -\pi_b \\ 0 & -\eta c_\phi & 1 - \pi_\phi \end{bmatrix} &\begin{bmatrix} S_{uu} & S_{ub} & S_{u\phi} \\ S_{ub} & S_{bb} & S_{b\phi} \\ S_{u\phi} & S_{b\phi} & S_{\phi\phi} \end{bmatrix} \begin{bmatrix} \rho_u & 0 & 0 \\ -\eta c_u & \rho - \eta c_b & -\eta c_\phi \\ -\pi_u & -\pi_b & 1 - \pi_\phi \end{bmatrix}
\end{aligned} \tag{35}
$$

A set of coefficients $\{\pi_u, \pi_b, \pi_\phi, c_u, c_b, c_\phi, \xi_u, \xi_b, \xi_\phi, S_{uu}, S_{ub}, S_{u\phi}, S_{bb}, S_{b\phi}, S_{\phi\phi}\}$ which solves system (34)-(35) defines a trajectory $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ which solves system (26)-(30) for any given $b_0 = \bar{b}$. Conversely, if a sequence $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ solves system (26)-(30), then its parameters $\{\pi_u$, π_b, π_ϕ, c_u, c_b, c_ϕ, ξ_u, ξ_b, ξ_ϕ, S_{uu}, S_{ub}, $S_{u\phi}$, S_{bb}, $S_{b\phi}$, $S_{\phi\phi}\}$ satisfies equations (34)-(35). We call the set of coefficients $\{\pi_u,\ \pi_b,\ \pi_\phi,\ c_u,\ c_b,\ c_\phi,\ \xi_u,\ \xi_b,\ \xi_\phi,\ S_{uu},\ S_{ub},\ S_{u\phi},\ S_{bb},\ S_{b\phi},\ S_{\phi\phi}\}$ a commitment equilibrium.

Writing the solution in form (31)-(33) allows us to compare it with the discretionary solution. Again, suppose the response of the private sector to debt, π_b, is given. We can guess the other feedback coefficients in the system (31)-(33) and iterate the Riccati equation (34) as suggested in Appendix A, but do not update π_b. If the procedure converges, we have obtained the optimal response of the policy maker to the private sector decision, provided that the private sector responds to the Lagrange multiplier (set by the policy maker) in an optimal way. Then, we iterate the Riccati equation (34) once again to obtain π_b^*. A solution to the commitment problem implies $\pi_b^* = \pi_b$. The graph of $\pi_b^*(\pi_b)$ intersects the 45^o degree line in one point labelled A, see the second panel in Figure 1, and we can verify with standard methods (e.g. Söderlind (1999)) that this point is, indeed, a solution. For the baseline calibration the economy is stabilized by the policy maker in the unique equilibrium A.

Figure 2 reports the responses of all variables to a positive unit cost push shock. Under commitment (the blue dotted line with x-markers) the policy maker engineers a fall in private consumption, which will dampen marginal costs. Although the dynamics of the economy is very similar to the one in discretionary equilibrium A, in contrast to this discretionary equilibrium, the policy maker keeps consumption below the steady state for several periods. Such a policy allows the policy maker to lower expected future inflation and ensures price stability in the long run. Government debt initially increases due to the fall in consumption, but is brought back to the steady state with higher taxes.

4 Quasi-Commitment Policy

This Section studies monetary policy within a limited commitment framework. We discuss the continuum of intermediate cases between commitment and discretion. We want to understand (i) how a 'quasi-commitment bridge' links the economy under commitment and under discretion when multiple equilibria exist, and (ii) how effectively quasi-commitment helps to select the best equilibrium.

4.1 Existence of Multiple Policy Equilibria

A quasi-commitment policy, as introduced in Schaumburg and Tambalotti (2007), also assumes sequential policy making. A new policy maker is appointed with a constant and exogenous probability α every period. When a new policy maker arrives in office, he reneges on the promises of

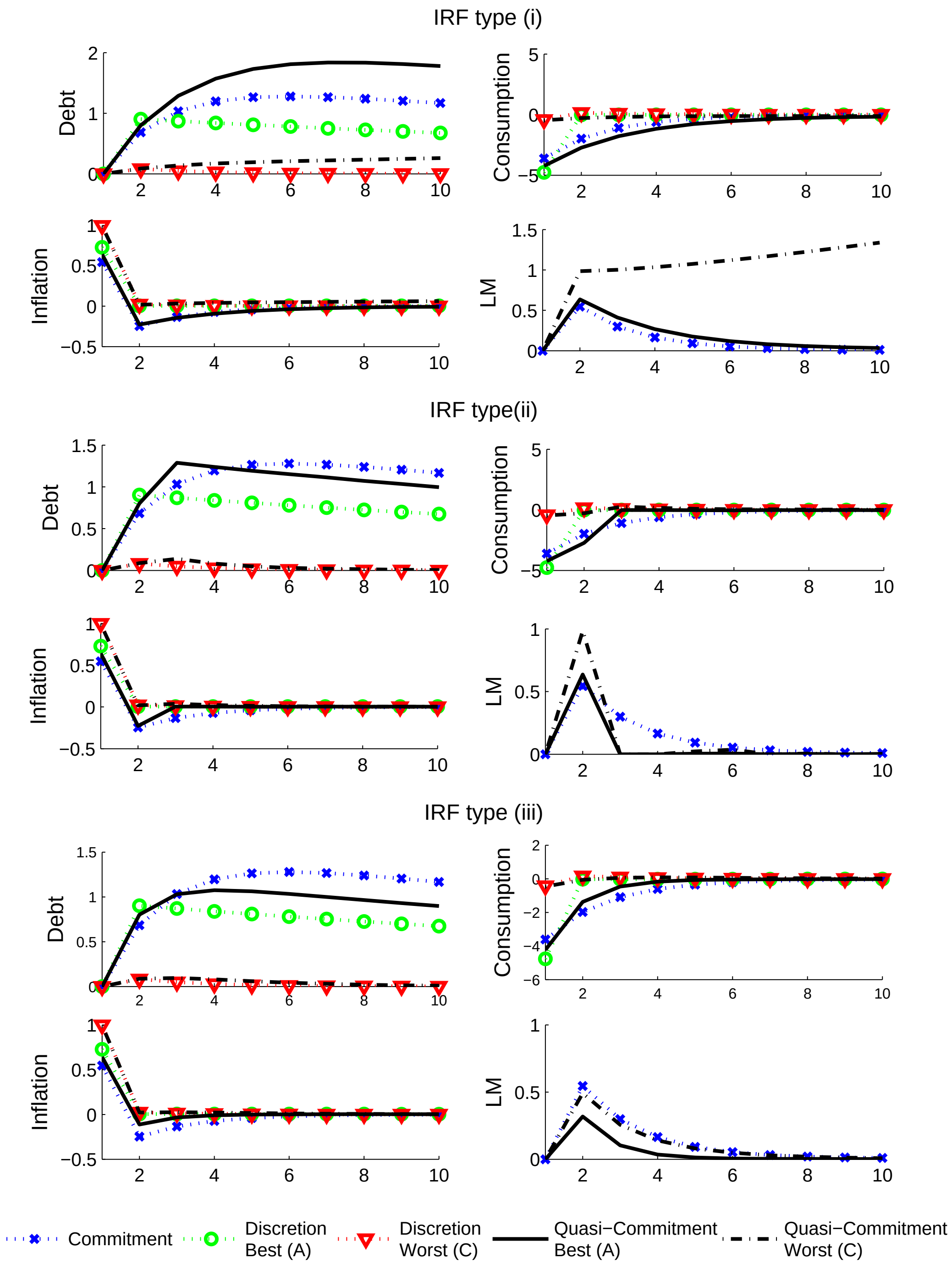

Figure 2: Impulse Responses to a 1% cost push shock in the model with government debt

his predecessor and commits to a new policy plan that is optimal at the time of the change. All agents understand the possibility and the nature of this change and form expectations accordingly. The private sector knows that a new policy maker will re-optimize, therefore it doubts the reliability of outstanding promises.

As in Schaumburg and Tambalotti (2007) and Debertoli and Nunes (2010) we assume that the policy maker's tenure in office depends on a sequence of exogenous i.i.d. Bernoulli signals $\{\Omega_t\}_{t\geq 0}$ with $\mathbb{E}[\Omega_t] = \alpha$. If $\alpha = 1$ the policy authority acts under full discretion and every period a new policy maker arrives in office and re-optimizes the planning problem. If $\alpha = 0$ the policy maker stays in office infinitely long and keeps his promises.

Schaumburg and Tambalotti (2007) and Debertoli and Nunes (2010) demonstrate that the optimization problem under limited commitment can be expressed by the following Lagrangian

$$\begin{aligned}\mathcal{L}^{qc} \;=\; & \sum_{t=0}^{\infty} (\beta(1-\alpha))^t \Big(\frac{1}{2}\left(\pi_t^2 + \lambda c_t^2 + \beta\alpha\left(S_{uu}^{\alpha} u_{t+1}^2 + 2S_{ub}^{\alpha} u_{t+1} b_{t+1} + S_{bb}^{\alpha} b_{t+1}^2\right)\right) \\ & + \phi_{t+1}\left(\pi_t - \kappa c_t - \nu b_t - u_t - \beta(1-\alpha)\pi_{t+1} - \beta\alpha\pi_b^{\alpha} b_{t+1} - \beta\alpha\pi_u^{\alpha} u_{t+1}\right) \\ & + \xi_{t+1}(\rho b_t - \eta c_t - b_{t+1})\Big)\end{aligned} \tag{36}$$

for $0 \leq \alpha < 1$. Here we use superscript α to denote parameters of solution to the limited commitment problem. The first order conditions are

$$0 = \beta\alpha S_{ub}^{\alpha} u_t + \beta\alpha S_{bb}^{\alpha} b_t - \xi_t + \rho\beta(1-\alpha)\xi_{t+1} - \nu\beta(1-\alpha)\phi_{t+1} - \beta\alpha\pi_b^{\alpha}\phi_t, \tag{37}$$

$$0 = \pi_t + \phi_{t+1} - \phi_t, \tag{38}$$

$$0 = \lambda c_t - \eta\xi_{t+1} - \kappa\phi_{t+1}, \tag{39}$$

$$0 = \rho b_t - \eta c_t - b_{t+1}, \tag{40}$$

$$0 = \beta(1-\alpha)\pi_{t+1} + \beta\alpha\pi_b^{\alpha} b_{t+1} + \kappa c_t + \nu b_t + (\beta\alpha\pi_u^{\alpha}\rho_u + 1)u_t - \pi_t, \tag{41}$$

for $t \geq 0$, with initial conditions $b_0 = \bar{b}$ and $\phi_0 = 0$, and the transversality condition $\lim_{t\to\infty} b_t < \infty$. These first order conditions are similar to those for commitment, but depend additionally on the parameters $\{\pi_b^{\alpha}, S_{bb}^{\alpha}, S_{ub}^{\alpha}\}$. These parameters are a part of solution to the limited commitment problem as we explain next.

A solution to system (37)-(41) can be written in the following linear form (see Appendix B for details)

$$\pi_t = \pi_u^{\alpha} u_t + \pi_b^{\alpha} b_t + \pi_{\phi}^{\alpha}\phi_t, \tag{42}$$

$$c_t = c_u^{\alpha} u_t + c_b^{\alpha} b_t + c_{\phi}^{\alpha}\phi_t, \tag{43}$$

$$\xi_t = \xi_u^{\alpha} u_t + \xi_b^{\alpha} b_t + \xi_{\phi}^{\alpha}\phi_t. \tag{44}$$

where coefficients $\left\{\pi_u^\alpha, \pi_b^\alpha, \pi_\phi^\alpha, c_u^\alpha, c_b^\alpha, c_\phi^\alpha\right\}$ solve the following algebraic Riccati equation:

$$\begin{bmatrix} c_u^\alpha & c_b^\alpha & c_\phi^\alpha \\ \pi_u^\alpha & \pi_b^\alpha & \pi_\phi^\alpha \\ \xi_u^\alpha & \xi_b^\alpha & \xi_\phi^\alpha \end{bmatrix} = \begin{bmatrix} \lambda+\eta^2\xi_b^\alpha & \kappa+\eta\xi_\phi^\alpha & 0 \\ \beta\eta\pi_b^\alpha-\kappa & (1-\alpha)\beta\pi_\phi^\alpha+1 & 0 \\ \beta\eta\rho\xi_b^\alpha(1-\alpha) & \beta\left(\rho\xi_\phi^\alpha-\nu\right)(1-\alpha) & 1 \end{bmatrix}^{-1} \tag{45}$$

$$\times \begin{bmatrix} \eta\xi_u^\alpha\rho_u & \eta\rho\xi_b^\alpha & \kappa+\eta\xi_\phi^\alpha \\ \beta\rho_u\pi_u^\alpha+1 & \nu+\beta\rho\pi_b^\alpha & \beta\pi_\phi^\alpha(1-\alpha) \\ \beta\left(\alpha S_{bu}^\alpha+(1-\alpha)\rho\xi_u^\alpha\rho_u\right) & \beta\left(\alpha S_{bb}^\alpha+(1-\alpha)\rho^2\xi_b^\alpha\right) & \beta\left((1-\alpha)\left(\rho\xi_\phi^\alpha-\nu\right)-\alpha\pi_b^\alpha\right) \end{bmatrix}$$

and parameters S_{bu}^α and S_{bb}^α are a part of the solution to the following matrix equation:

$$\begin{bmatrix} S_{uu}^\alpha & S_{ub}^\alpha & S_{u\phi}^\alpha \\ S_{ub}^\alpha & S_{bb}^\alpha & S_{b\phi}^\alpha \\ S_{u\phi}^\alpha & S_{b\phi}^\alpha & S_{\phi\phi}^\alpha \end{bmatrix} = \begin{bmatrix} (\pi_u^\alpha)^2+\lambda(c_u^\alpha)^2 & \pi_b^\alpha\pi_u^\alpha+\lambda c_b^\alpha c_u^\alpha & \pi_u^\alpha\pi_\phi^\alpha+\lambda c_u^\alpha c_\phi^\alpha \\ \pi_b^\alpha\pi_u^\alpha+\lambda c_b^\alpha c_u^\alpha & (\pi_b^\alpha)^2+\lambda(c_b^\alpha)^2 & \pi_b^\alpha\pi_\phi^\alpha+\lambda c_b^\alpha c_\phi^\alpha \\ \pi_u^\alpha\pi_\phi^\alpha+\lambda c_u^\alpha c_\phi^\alpha & \pi_b^\alpha\pi_\phi^\alpha+\lambda c_b^\alpha c_\phi^\alpha & \left(\pi_\phi^\alpha\right)^2+\lambda\left(c_\phi^\alpha\right)^2 \end{bmatrix} \tag{46}$$

$$+\beta(1-\alpha)\begin{bmatrix} \rho_u & -\eta c_u^\alpha & -\pi_u^\alpha \\ 0 & \rho-\eta c_b^\alpha & -\pi_b^\alpha \\ 0 & -\eta c_\phi^\alpha & 1-\pi_\phi^\alpha \end{bmatrix}\begin{bmatrix} \frac{S_{uu}^\alpha}{(1-\alpha)} & \frac{S_{ub}^\alpha}{(1-\alpha)} & S_{u\phi}^\alpha \\ \frac{S_{ub}^\alpha}{(1-\alpha)} & \frac{S_{bb}^\alpha}{(1-\alpha)} & S_{b\phi}^\alpha \\ S_{u\phi}^\alpha & S_{b\phi}^\alpha & S_{\phi\phi}^\alpha \end{bmatrix}\begin{bmatrix} \rho_u & 0 & 0 \\ -\eta c_u^\alpha & \rho-\eta c_b^\alpha & -\eta c_\phi^\alpha \\ -\pi_u^\alpha & -\pi_b^\alpha & 1-\pi_\phi^\alpha \end{bmatrix}$$

Coefficients $\{\pi_u, \pi_b, \pi_\phi, c_u, c_b, c_\phi, \xi_u, \xi_b, \xi_\phi, S_{uu}, S_{ub}, S_{u\phi}, S_{bb}, S_{b\phi}, S_{\phi\phi}\}$ uniquely define the trajectories $\{b_t, \pi_t, c_t\}_{t=0}^\infty$ which solve system (37)-(41) for any given $b_0 = \bar{b}$. Conversely, if the sequence $\{b_t, \pi_t, c_t\}_{t=0}^\infty$ solves system (37)-(41), then there is a unique set of coefficients $\{\pi_u$, π_b, π_ϕ, c_u, c_b, c_ϕ, ξ_u, ξ_b, ξ_ϕ, S_{uu}, S_{ub}, $S_{u\phi}$, S_{bb}, $S_{b\phi}$, $S_{\phi\phi}\}$ that satisfies equations (45)-(46). We call the set of coefficients $\{\pi_u$, π_b, π_ϕ, c_u, c_b, c_ϕ, ξ_u, ξ_b, ξ_ϕ, S_{uu}, S_{ub}, $S_{u\phi}$, S_{bb}, $S_{b\phi}$, $S_{\phi\phi}\}$ a limited commitment equilibrium.

We can plot the solution to system (45)-(46) using the same approach as we used for commitment in Section 3.2. Suppose we guess the response of the private sector to the state variable, π_b^α. Then, we can iterate system (45)-(46), but do not update π_b^α. If the procedure has converged, we iterate it once to obtain the update $\pi_b^{\alpha*}$. Solutions to the system (45)-(46) will be among the points where $\pi_b^{\alpha*} = \pi_b^\alpha$.

For the baseline calibration of $\alpha = 1/2$ (which implies an average regime duration of two quarters) the graph of $\pi_b^{1/2*}\left(\pi_b^{1/2}\right)$ intersects the 45^o degree line in three points labelled A, B and C, see the third panel in Figure 1.[13] Therefore, if we move from the case of pure discretionary policy to the case were the policy maker stays in office on average for two periods then all three equilibria survive.

The survival of all discretionary equilibria under some degree of precommitment is not obvious. Note that if $\alpha = 1$, the policy maker defaults with certainty every period. Then, the Lagrangian

[13] Again, we can verify with standard methods (based on Oudiz and Sachs (1985) and Backus and Driffill (1986), and discussed in Appendix C) that these are indeed solutions to the optimization problem.

(36) takes the form of (13), which describes the discretionary optimization problem. The first order conditions for the limited commitment optimization problem (37)-(41) are left-discontinuous at point $\alpha = 1$. System (37)-(41) does not collapse to (14)-(18) as taking the limit $\alpha \to 1$ in system (37)-(41) does not eliminate the Lagrange multiplier on the previous-period constraint ϕ_t in equation (38). Because for any $\alpha < 1$ the private sector does not expect the occurrence of default with certainty in the next period, this property holds at the limit and implies discontinuity of the first order conditions. Nevertheless, the number of equilibrium is a locally continuous function of α at $\alpha = 1$, as we prove next.

Proposition 1 *Assume that all roots of the polynomial system (19)-(25) are of multiplicity one so that there are $K \in \{1, 3\}$ distinct solutions under discretionary policy. There exists $\overline{\alpha}$, $0 < \overline{\alpha} < 1$ such that if $\alpha \in (\overline{\alpha}, 1]$ then there are as many quasi-commitment policy equilibria as under discretion.*

Proof. First, we prove that the system of first order conditions to the limited commitment problem, taken at the limit $\alpha \to 1$, has as many solutions as has the system of first order conditions to the discretionary problem.

Indeed, taking the limit $\alpha \to 1$ of system (45)-(46) we obtain system the following system

$$
\begin{aligned}
S_u^1 &= \left(\pi_u^1\right)^2 + \lambda \left(c_u^1\right)^2 + \beta \left(\rho_u^2 S_u^1 + \eta^2 \left(c_u^1\right)^2 S_{bb}^1 - 2\eta\rho_u c_u^1 S_{bu}^1\right), & (47)\\
S_{ub} &= \pi_b^1 \pi_u^1 + \lambda c_b^1 c_u^1 + \beta\rho_u \left(\rho - \eta c_b^1\right) S_{ub}^1 + \beta\eta c_u^1 \left(\eta c_b^1 - \rho\right) S_{bb}^1, & (48)\\
S_{bb} &= \left(\pi_b^1\right)^2 + \lambda \left(c_b^1\right)^2 + \beta \left(-\rho + \eta c_b^1\right)^2 S_{bb}^1, & (49)\\
\eta\beta S_{bu}^1 \rho_u &= \kappa\pi_u^1 + \lambda c_u^1 + \eta^2 \beta S_{bb}^1 c_u^1 - \eta\beta\pi_b^1\pi_u^1, & (50)\\
\eta\rho\beta S_{bb}^1 &= \kappa\pi_b^1 + \lambda c_b^1 + \eta^2\beta S_{bb}^1 c_b^1 - \eta\beta\pi_b^1\pi_b^1, & (51)\\
\beta\rho_u\pi_u^1 + 1 &= \pi_u^1 - \kappa c_u^1 + \beta\eta\pi_b^1 c_u^1, & (52)\\
\nu + \beta\rho\pi_b^1 &= \pi_b^1 - \kappa c_b^1 + \beta\eta\pi_b^1 c_b^1, & (53)
\end{aligned}
$$

$$\xi_u^1 = \beta S_{bu}^1, \quad \xi_b^1 = \beta S_{bb}^1, \quad \xi_\phi^1 = -\beta \pi_b^1, \quad \pi_\phi^1 = \frac{\left(\kappa - \eta\beta\pi_b^1\right)^2}{\left(\left(\kappa - \eta\beta\pi_b^1\right)^2 + \lambda + \eta^2\beta S_{bb}^1\right)}, \tag{54}$$

$$c_\phi^1 = \frac{\kappa - \eta\beta\pi_b^1}{\left(\left(\kappa - \eta\beta\pi_b^1\right)^2 + \lambda + \eta^2\beta S_{bb}^1\right)}, \quad S_{\phi\phi}^1 = \frac{\left(\kappa - \eta\beta\pi_b^1\right)^2}{\left(\left(\kappa - \eta\beta\pi_b^1\right)^2 + \lambda + \eta^2\beta S_{bb}^1\right)}, \tag{55}$$

$$S_{u\phi}^1 = \left(\kappa - \eta\beta\pi_b^1\right) \frac{\pi_u^1\left(\kappa - \eta\beta\pi_b^1\right) + c_u^1\left(\lambda + \beta\eta^2 S_{bb}^1\right) - \beta\eta\rho_u S_{bu}^1}{\left(\left(\kappa - \eta\beta\pi_b^1\right)^2 + \lambda + \eta^2\beta S_{bb}^1\right)}, \tag{56}$$

$$S_{b\phi}^1 = \left(\kappa - \eta\beta\pi_b^1\right) \frac{\pi_b^1\left(\kappa - \eta\beta\pi_b^1\right) + c_b^1\left(\lambda + \beta\eta^2 S\right)_{bb}^1 - \beta\eta\rho S_{bb}^1}{\left(\left(\kappa - \eta\beta\pi_b^1\right)^2 + \lambda + \eta^2\beta S_{bb}^1\right)}, \tag{57}$$

which can be split into two sub-systems, (47)-(53) and (54)-(57). The first sub-system (47)-(53) does not depend on $\{\pi_\phi^1, c_\phi^1, \xi_u^1, \xi_b^1, \xi_\phi^1, S_{u\phi}^1, S_{b\phi}^1, S_{\phi\phi}^1\}$; and it is equivalent to system (19)-(25), which determines solution to the discretionary problem, $\{\pi_u, \pi_b, c_u, c_b, S_{uu}, S_{ub}, S_{bb}\}$. The second sub-system (54)-(57) is linear in variables $\{\pi_\phi^1, c_\phi^1, \xi_u^1, \xi_b^1, \xi_\phi^1, S_{u\phi}^1, S_{b\phi}^1, S_{\phi\phi}^1\}$ and has a unique solution given $\left\{\pi_u^1, \pi_b^1, c_u^1, c_b^1, S_{uu}^1, S_{ub}^1, S_{bb}^1\right\}$. If follows that the solutions to system (19)-(25) and to system (47)-(57) are identical.

Second, we prove that the system of first order conditions to the limited commitment problem has the same number of solutions in some neighborhood of $\alpha = 1$.

Indeed, system (45)-(46) is a polynomial system in $\{\pi, c, S\}$, which coefficients are polynomial functions of α. Therefore, all solutions to (45)-(46) at $\alpha = 1$ are continuous functions in α. For any solution j, $j \in \{1, 3\}$, to system (45)-(46) for $\alpha = 1$ there exists a $\alpha^j < 1$ such that solution j is a continuous function of α for $\alpha \in (\alpha^j, 1]$. A solution which exists for $\alpha = 1$ also exists for $\alpha \in (\bar{\alpha}, 1]$ where $\bar{\alpha} = \max_j\{\alpha^j\}$. If there are K solutions to (45)-(46) for $\alpha \in (\bar{\alpha}, 1]$, then there are K paths which solve (37)-(41) for $\alpha \in (\bar{\alpha}, 1]$.

Therefore, the system of first order conditions to the limited commitment problem has as many solutions as has the system of first order conditions to the discretionary problem if $\alpha \in (\overline{\alpha}, 1]$. ■

We plot the case $\alpha \to 1$ in the fourth panel in Figure 1. The $\pi_b^{1*}\left(\pi_b^1\right)$ line intersects the 45^o degree line in three points, which are the same points as under pure discretion.[14]

In Figure 2 we show the responses of all variables to a positive 1% cost push shock under a quasi-commitment policy. We set $\alpha = 1/2$, which implies average regime duration of two quarters. We also demonstrate impulse responses under commitment and discretion (equilibria A and C).

Panel I of Figure 2 shows the impulse response functions of Type (i).[15] These impulse re-

[14] The shape of $\pi_b^{1*}\left(\pi_b^1\right)$ is different than in Panel I because we take into account the Lagrange multipliers when computing $\pi_b^{1*}\left(\pi_b^1\right)$. But in equilibrium $\pi_b^1 = \pi_b$.

[15] The categorization of the impulse response functions follows Schaumburg and Tambalotti (2007).

sponses demonstrate the evolution of the economy if no reoptimization happens over the horizon of interest, while the private sector expects them to happen every period with probability 1/2. In this scenario a central banker stays in office unexpectedly long, which becomes more and more unrealistic over time. To generate these impulse responses we use the transition matrix given by the conditions (37)-(41). Similar to discretion we plot the two quasi-commitment equilibria A and C. We use solid and dash-dotted lines correspondingly. Compared to the full commitment policy, quasi-commitment policy in the active monetary policy equilibrium A delivers a stronger and longer lasting decrease in consumption. As reoptimizations are expected to happen the price setters expect future policy makers to increase consumption and therefore expect a high inflation in the future. Therefore, if the policy maker wants to exploit private sector expectations he has to pay a higher cost in from of a stronger recession. In the absence of reoptimizations this results in stronger future deflation and higher debt, compared to commitment.

Type (i) impulse responses under quasi-commitment policy in equilibrium C are explosive. In this case the 'passive' monetary policy is not able to stabilize inflation, while trying to keep debt under control. After the shock occurred the policy maker cannot move consumption by much, since he has to avoid excessive debt accumulation. This behavior is similar to the one in discretionary equilibrium C. Because the private sector expects defaults in the future and hence high future inflation, inflation can only be controlled with low demand. However, lower consumption would result in excessive debt accumulation. Therefore the reduction in consumption counteracts the effort of the central bank to ensure fiscal solvency and therefore the economy exhibits explosive behavior. As the fourth chart in the first panel shows, the Lagrange multiplier ϕ_t which measures the shadow price of controlling the private sector inflation expectations is much higher in equilibrium C and explodes with time.[16] The result is not surprising, given that the monetary policy maker has to control debt in the passive equilibrium. This task becomes incompatible with inflation stabilization if expected defaults do not happen.[17]

Impulse responses of Type (ii) in Panel II of Figure 2 characterize a more typical behavior of the economy under quasi-commitment. Suppose reoptimizations happen in periods 2, 3, 6 and 8 after the initial shock. In each of these periods the reoptimizing policy maker reneges on the plan of its predecessor. When the policy maker defaults on the promises of his predecessor, he resets the predetermined Lagrange multiplier to zero. The policy maker takes this opportunity to end the promised recession of his predecessor and raises consumption back to its initial level.

[16] This Lagrange multiplier is set on the Phillips curve in the optimization problem of the policy maker.

[17] Using an analogy with a roulette game, system (37)-(41) describes the history when 'red' never realizes while it is expected – and it is bet on – with probability 1/2.

The increase in consumption also leads to a faster reduction of government debt.

Type (iii) impulse responses (Panel III in Figure 2) are the ex ante averages of all the possible conditional IRFs integrated over the distribution of the corresponding reoptimization draws. Therefore they demonstrate the expected evolution of the system following the initial shock. Naturally, they are in between the IRF of the respective discretionary equilibria and the IRF under full commitment.

4.2 Equilibrium Selection

Proposition 2 *There exists* $\underline{\alpha}$, $0 < \underline{\alpha} \leq 1$ *such that if (i)* $\alpha \in [0, \underline{\alpha})$ *and (ii) a quasi-commitment equilibrium exists, than the equilibrium is unique.*

Proof. Under commitment the policy equilibrium in LQ RE models, if it exists, is always unique, see e.g. Backus and Driffill (1986).[18] Equations (45)-(46) determine the parameters of the solution to the limited commitment problem. Equation (46) collapses to a symmetric discrete algebraic Riccati equation for the value function if $\alpha \to 0$; this equation is known to have a unique symmetric positive semi-definite solution, see e.g. Lancaster and Rodman (1995). Equation (45) collapses to a Riccati equation (34) if $\alpha \to 0$; if a solution to this equation exists, it is unique. System (45)-(46) is a polynomial system in $\{\pi_k, c_k, \xi_k\}_{k \in \{u,b,\phi\}}$, and in components of S, which coefficients are polynomial functions of α. Therefore, all solutions to (45)-(46) at $\alpha = 0$ are continuous functions in α. If a solution to system (45)-(46) exists for $\alpha = 0$ there exists a $\underline{\alpha} > 0$ such that the solution is a continuous functions of α for $\alpha \in [0, \underline{\alpha})$. These three solutions determine three paths which solve (26)-(30) for $\alpha \in [0, \underline{\alpha})$. ■

By continuity the selected equilibrium is always Pareto-optimal because the commitment equilibrium, to which the selected equilibrium converges at the limit, delivers the lowest loss. The value of $\underline{\alpha}$ which selects the unique equilibrium can be smaller than the value of $\overline{\alpha}$ which ensures the same number of equilibria as under discretion. How big are these values for our model? In particular, what is the sufficient degree of precommitment $\underline{\alpha}$ such that only one equilibrium survives?

Panel I in Figure 3 plots the expected welfare loss for each equilibrium as a function of the average duration of the period of precommitment $1/\alpha$ for a given fiscal feedback parameter $\mu = 0.05$. In the case of pure discretion ($\alpha = 1$) we have three equilibria denoted by triangular markers. With higher degrees of precommitment all three equilibria survive. The losses in the corresponding equilibria are marked with crosses. Panel I suggests that for the benchmark value

[18] A commitment policy which stabilises the economy may not exist, see Appendix A.

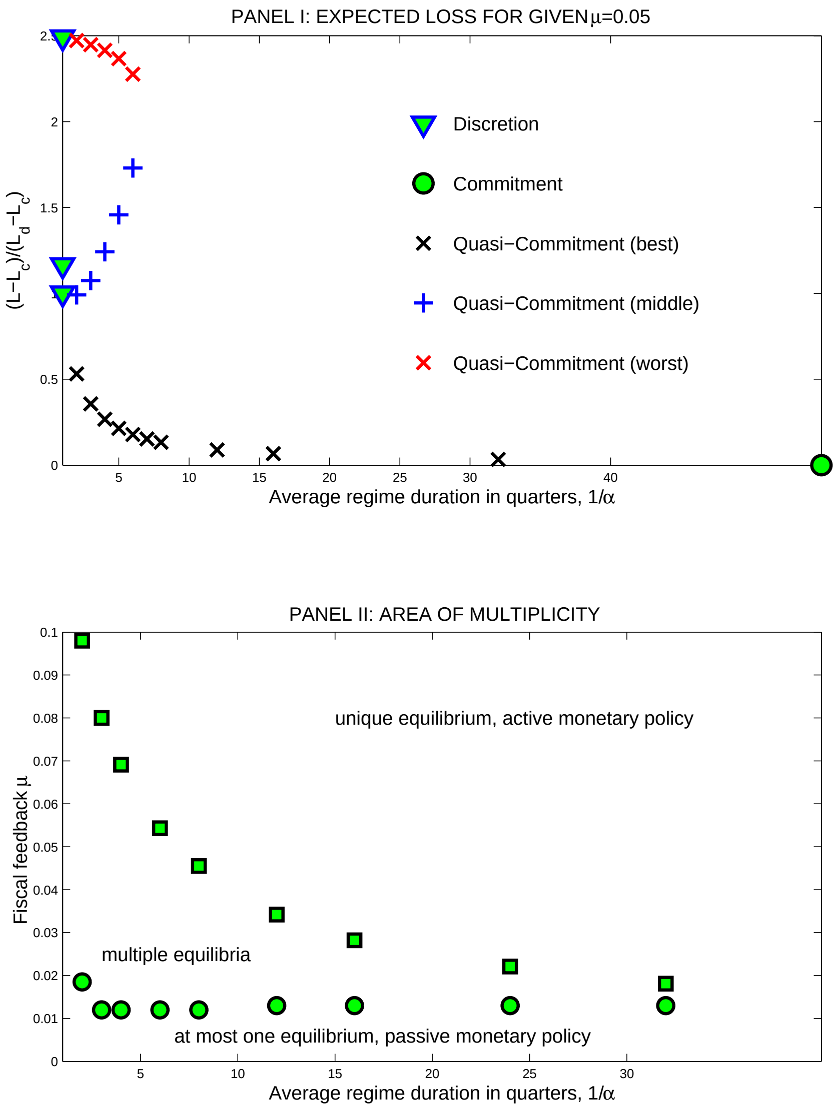

Figure 3: Equilibrium Selection

of the fiscal feedback parameter the worst and the middle equilibria are eliminated if $\overline{\alpha} = \underline{\alpha} = 1/8$, as we also report in row (7) of Table 1. If the policy maker stays in the office only for two years on average this will guarantee the unique equilibrium under quasi-commitment policy.[19]

To summarize, only a relatively small degree of commitment is required to select the best equilibrium. If a limited commitment technology is available, then it is a more powerful selection mechanism than a formation of a coalition of consequent discretionary policy makers, see Dennis and Kirsanova (2009). If consequent policy makers form coalitions and reoptimize under discretion only in the first period of each coalition tenure, sticking to the same time-consistent policy between reoptimizations, it requires a tenure period of three years to select the best equilibrium in this model.[20] An access to the limited commitment technology reduces the necessary tenure period which is required to avoid falling into an expectation trap. Panel I in Figure 3 shows that for $\mu = 0.05$ multiplicity is eliminated, if a policy maker can commit on average for 2 years.[21] Moreover, and more generally, Proposition 2 claims that there is some sufficient degree of commitment which will certainly select the Pareto-preferred equilibrium (if the corresponding commitment equilibrium exists), while no coalition of discretionary policy makers might exist to select it (Dennis and Kirsanova (2009)).

Panel II in Figure 3 investigates the robustness of the above result for different values of the fiscal feedback parameter μ, which is crucial for multiplicity. We concentrate on the range of the fiscal feedback μ which generates multiplicity of quasi-commitment equilibria for a given average regime duration, $1/\alpha$. For every (discrete) regime duration the square marker denotes the minimum level of μ above which there is a unique equilibrium characterized by an 'active' monetary policy. The area below the round markers displays unique equilibria characterized by a 'passive' monetary policy. In the area between the two markers we observe multiplicity of policy equilibria. Panel II demonstrates that with longer periods of precommitment the area of multiplicity shrinks very quickly: if the average period of precommitment is more than five years then expectation traps only exist for very small and empirically irrelevant values of the fiscal feedback μ.

Parameter μ is crucial for multiplicity and, as we argued in Section 3.1, the range of fiscal

[19] Panel I in Figure 3 also demonstrates that the welfare loss is quickly reduced for a higher degree of precommitment. The initial gap between the loss in the best discretionary equilibrium A and commitment is nearly halved after one year of precommitment. A further reduction in α demonstrates that the gains from even minimal levels of credibility are substantial.

[20] For the base line calibration of the model with $\mu = 0.05$.

[21] Our numerical experiments with different (and more complex) models show consistently that the best equilibrium is selected only after a few periods of precommitment.

feedbacks in Panel II in Figure 3 is empirically relevant. The baseline value of $\mu = 0.05$ creates multiplicity under pure discretionary policy. However, it is enough for a policy maker to stay in the office for two years in order to select the best equilibrium.

Our results are robust to different calibrations of other parameters of the model. Two parameters were found to affect the quantitative results most. More myopic agents (i.e. lower β) put higher relative weight on stabilization of the economy in the first periods after the shock. This eliminates the good equilibrium for low values of the fiscal feedback parameter μ, because in this equilibrium the adjustment is relatively slow. Differently, a higher degree of price stickiness (bigger Calvo parameter γ) slows the adjustment process down and eliminates the bad equilibrium for high values of μ. However, the shape of the curve in Panel I of Figure 3 stays in both cases the same and quantitative differences are not very large.

5 Conclusion

In this paper we study monetary policy in a limited commitment framework using a simple New Keynesian model with government debt. We show by example the existence of multiple equilibria under quasi-commitment policy using a model with government debt accumulation. We demonstrate the existence of expectation traps similar to those under pure discretionary policy. Because the private sector expects eventual re-optimizations to happen the current policy maker formulates its policy based on the forecast of the private sector about future policy makers' behavior. We find that there can be at least as many limited commitment policy equilibria as in the corresponding discretionary policy problem.

Although the previously developed equilibrium selection mechanism may suggest that economic agents are likely to coordinate on the best equilibrium, our example demonstrates that a limited commitment technology helps the policy maker to avoid falling into an expectation trap even if the degree of precommitment is very small.

In this paper we also provide an algorithm for computing quasi-commitment equilibria in the general class of LQ RE models with endogenous state variables and with exogenous probability of default. We leave the numerical investigation of properties of quasi-commitment policy in a wider class of non-linear dynamic models for future research. This research might investigate how much commitment is required to select the best equilibrium in a King and Wolman (2004) type of model with multiple discretionary equilibria. Once a robust algorithm to solve non-linear models is developed, future research will be able to endogenize the probability of default along the lines suggested in Debertoli and Nunes (2010).

A Commitment FOCs in the Form of a Riccati Equation

When optimizing, the policy maker internalizes the effect of its choice on private sector's expectations and solves the following Lagrangian

$$\mathcal{L}^c = \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} \left(\pi_t^2 + \lambda c_t^2 \right) + \xi_{t+1} \left(\rho b_t - \eta c_t - b_{t+1} \right) + \phi_{t+1} \left(\pi_t - \kappa c_t - \nu b_t - u_t - \beta \pi_{t+1} \right) \right).$$

The corresponding first order conditions are:

$$0 = -\xi_t + \rho\beta\xi_{t+1} - \nu\beta\phi_{t+1}, \tag{58}$$

$$0 = \pi_t + \phi_{t+1} - \phi_t, \tag{59}$$

$$0 = \lambda c_t - \eta\xi_{t+1} - \kappa\phi_{t+1}, \tag{60}$$

$$0 = \rho b_t - \eta c_t - b_{t+1}, \tag{61}$$

$$0 = \beta\pi_{t+1} + \kappa c_t + \nu b_t + u_t - \pi_t, \tag{62}$$

for $t \geq 0$; with initial conditions $b_0 = \bar{b}$ and $\phi_0 = 0$, and the transversality condition $\lim_{t\to\infty} b_t < \infty$.

Assume that $\rho \neq 0$ and $\eta \neq 0$ in system (6)-(7). Then the system (6)-(7) is controllable, and there always exists a unique path $\{c_t, \pi_t, b_t\}_{t\geq 0}$ which (i) satisfies system (58)-(62) and the initial conditions and (ii) all eigenvalues of the resulting transition matrix are less than $1/\sqrt{\beta}$ in modulus (see, e.g. Kwakernaak and Sivan (1972), Backus and Driffill (1986)). For the rest of this paper we use the following definition: The economy is *stabilized* by a policy if all eigenvalues of the transition matrix are inside the unit circle. If the economy is stabilized by a policy we call such a policy *stabilizing*. In general, because $\beta < 1$ a stabilizing commitment policy may not exist for all problems in the LQ RE class.

One way to solve the system (58)-(62) is to use the Schur decomposition, see e.g. Söderlind (1999). Alternatively, and more convenient for our purpose, we can also solve the system using an iterative scheme.

System (58)-(62) can be written as

$$\begin{bmatrix} 0 & 0 & \eta \\ 0 & \beta & 0 \\ 0 & 0 & \rho\beta \end{bmatrix} \begin{bmatrix} c_{t+1} \\ \pi_{t+1} \\ \xi_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\kappa \\ -1 & -\nu & 0 \\ 0 & 0 & \nu\beta \end{bmatrix} \begin{bmatrix} u_t \\ b_t \\ \phi_t \end{bmatrix} + \begin{bmatrix} \lambda & \kappa & 0 \\ -\kappa & 1 & 0 \\ 0 & -\nu\beta & 1 \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \\ \xi_t \end{bmatrix} \tag{63}$$

$$\begin{bmatrix} u_{t+1} \\ b_{t+1} \\ \phi_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_u & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ b_t \\ \phi_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\eta & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \\ \xi_t \end{bmatrix} \tag{64}$$

Substitute (31)-(33) into both sides of (63) and use (64) to substitute out $u_{t+1}, b_{t+1}, \phi_{t+1}$. We obtain

$$\begin{bmatrix} \lambda+\eta^2\xi_b & \kappa+\eta\xi_\phi & 0 \\ \beta\eta\pi_b-\kappa & \beta\pi_\phi+1 & 0 \\ \beta\eta\rho\xi_b & -\beta\left(\nu-\rho\xi_\phi\right) & 1 \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} \eta\xi_u\rho_u & \eta\rho\xi_b & \kappa+\eta\xi_\phi \\ \beta\rho_u\pi_u+1 & \nu+\beta\rho\pi_b & \beta\pi_\phi \\ \beta\rho\xi_u\rho_u & \beta\rho^2\xi_b & -\beta\left(\nu-\rho\xi_\phi\right) \end{bmatrix} \begin{bmatrix} u_t \\ b_t \\ \phi_t \end{bmatrix}$$

Substitution of (31)-(33) yields the matrix algebraic Riccati equation (34) in the main text. We can guess all feedback coefficients in (31)-(33) and thus in the right hand side of the equation above. Then, the Riccati equation (34) gives an update of these coefficients: in the next step we update the right hand side of it and iterate until convergence. The algorithm will converge (Lancaster and Rodman (1995)).

Although the baseline calibration delivers a stabilizing solution, note that if the fiscal feedback is weak, $0<\mu<\mu^*$, where $\mu^*=\left(1-\tilde{\Upsilon}\right)(1-\beta)\kappa/\left(\tilde{\Upsilon}\left(\left(1-\tilde{\Upsilon}\right)\kappa-\zeta\theta\tilde{\Upsilon}\right)\right)$, the economy is not stabilized by policy. The optimal monetary policy still delivers a finite value of the loss function (8), but all variables exhibit slow explosion with a rate of explosion less than $1/\sqrt{\beta}$. However, this solution should be disregarded as it violates the assumption of a finite working week.[22]

Finally, note that equation (59) implies price stability: if $\phi_t=0$ and $\lim_{t\to\infty}\phi_t=0$ it follows that $\sum_{t=0}^{\infty}\pi_t=0$.

B Limited Commitment and Matrix Equations

System (37)-(41) can be written as

$$\begin{bmatrix} 0 & 0 & \eta \\ 0 & \beta(1-\alpha) & 0 \\ 0 & 0 & \rho\beta(1-\alpha) \end{bmatrix} \begin{bmatrix} c_{t+1} \\ \pi_{t+1} \\ \xi_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda & \kappa & 0 \\ \beta\alpha\eta\pi_b^\alpha-\kappa & 1 & 0 \\ 0 & -\nu\beta(1-\alpha) & 1 \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \\ \xi_t \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 & -\kappa \\ -\beta\alpha\pi_u^\alpha\rho_u-1 & -\beta\alpha\pi_b^\alpha\rho-\nu & 0 \\ -\beta\alpha S_{ub}^\alpha & -\beta\alpha S_{bb}^\alpha & \beta\left(\nu(1-\alpha)+\alpha\pi_b^\alpha\right) \end{bmatrix} \begin{bmatrix} u_t \\ b_t \\ \phi_t \end{bmatrix} \tag{65}$$
$$\begin{bmatrix} u_{t+1} \\ b_{t+1} \\ \phi_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\eta & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} \rho_u & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ b_t \\ \phi_t \end{bmatrix} \tag{66}$$

Substitute (31)-(33) into both sides of (65) and use (66) to substitute out $u_{t+1}, b_{t+1}, \phi_{t+1}$. We obtain the matrix equation (45) in the main text.

[22] This result was shown in a similar model in Schmitt-Grohe and Uribe (2004) and in Kirsanova and Wren-Lewis (2011).

Parameters S_{bb}^{α} and S_{ub}^{α} are the components of the value function; to close the system we write (36) in the form of Bellman equation and substitute (31)-(33). We obtain the matrix equation (46). Equations (45) and (46) solve the limited commitment problem.

C Limited Commitment Policy in General LQ RE Framework

We assume a non-singular linear deterministic rational expectations model, augmented by a vector of control instruments. Specifically, the evolution of the economy is explained by the linear system

$$\begin{bmatrix} y_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_t + C \begin{bmatrix} \xi_{t+1} \\ 0 \end{bmatrix}, \tag{67}$$

where y_t is an n_1-vector of predetermined variables with initial conditions y_0 given, x_t is n_2-vector of non-predetermined (or jump) variables with $\lim_{t\to\infty} x_t = 0$, u_t is a $k-$vector of policy instruments of the policy maker, and ξ_t is a vector of i.i.d. shocks with covariance matrix Σ. For notational convenience we define the n-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$. We assume A_{22} is non-singular.

The inter-temporal policy maker's welfare criterion is defined by the quadratic loss function

$$L_0 = \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t g_t' \mathcal{Q} g_t = \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{s-t} \left(z_t' Q z_t + 2 z_t' P u_t + u_t' R u_t \right). \tag{68}$$

The elements of vector g_s are the goal variables of the policy maker, $g_t = \mathcal{C}(z_t', u_t')'$. Matrix $\mathcal{Q}$ is assumed to be symmetric and positive semi-definite.[23]

Schaumburg and Tambalotti (2007) and then Debertoli and Nunes (2010) demonstrate that the optimization problem can be written as

$$\min \mathbb{E}_0 \sum_{t=0}^{\infty} \left(\beta \left(1-\alpha\right)\right)^t \left(z_t' Q z_t + 2 z_t' P u_t + u_t' R u_t + \beta\alpha y_{t+1}' S y_{t+1} \right) \tag{69}$$

subject to

$$\begin{aligned} y_{t+1} &= A_{11} y_t + A_{12} x_t + B_1 u_t + C\xi_{t+1} \\ \left(1-\alpha\right)\mathbb{E}_t x_{t+1} + \alpha H y_{t+1} &= A_{21} y_t + A_{22} x_t + B_2 u_t \end{aligned}$$

[23] It is standard to assume that R is symmetric positive definite (see Anderson et al. (1996), for example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $\mathcal{Q}$ being positive definite can be weakened to $\mathcal{Q}$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer (2003)). The analysis in this paper is valid for $R \equiv 0$.

where H and S are components of the solution to the corresponding discretionary problem, $x_t = Hy_t$ and the loss is $L_t(y_t) = \frac{1}{2}y_t' S y_t$.

The first order conditions to the appropriate Lagrangian

$$\begin{aligned}
\mathcal{L}^{qc} &= \sum_{t=0}^{\infty} (\beta(1-\alpha))^t \left(z_t' Q z_t + 2z_t' P u_t + u_t' R u_t + \beta\alpha y_{t+1}' S y_{t+1}\right. \\
&\quad +2\varphi_{t+1}' (A_{21}y_t + A_{22}x_t + B_2 u_t - (1-\alpha)x_{t+1} - \alpha H y_{t+1}) \\
&\quad \left.+2\psi_{t+1}' (A_{11}y_s + A_{12}x_s + B_1 u_s + \xi_{t+1} - y_{s+1})\right)
\end{aligned}$$

can be written as

$$\begin{aligned}
&\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \beta A_{22}' & 0 & 0 & \beta A_{12}' \\ 0 & B_2' & 0 & 0 & B_1' \\ \alpha H & 0 & 0 & (1-\alpha)I & 0 \\ 0 & \beta(1-\alpha)A_{21}' & 0 & 0 & \beta(1-\alpha)A_{11}' \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \varphi_{t+1} \\ u_{t+1} \\ x_{t+1} \\ \psi_{t+1} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} & 0 & B_1 & A_{12} & 0 \\ -\beta Q_{12}' & I & -\beta P_2 & -\beta Q_{22} & 0 \\ -P_1' & 0 & -R & -P_2' & 0 \\ A_{21} & 0 & B_2 & A_{22} & 0 \\ -\beta((1-\alpha)Q_{11} + \alpha S) & \alpha H' & -\beta(1-\alpha)P_1 & -\beta(1-\alpha)Q_{12} & I \end{bmatrix} \begin{bmatrix} y_t \\ \varphi_t \\ u_t \\ x_t \\ \psi_t \end{bmatrix}
\end{aligned} \tag{70}$$

Solution to this system (using Schur decomposition, for example, or iteration Riccati equation as we do in the text) can be written in the form

$$\begin{aligned}
\begin{bmatrix} u_t \\ x_t \\ \psi_t \end{bmatrix} &= \begin{bmatrix} X_{uy} & X_{u\varphi} \\ X_{xy} & X_{x\varphi} \\ X_{\psi y} & X_{\psi\varphi} \end{bmatrix} \begin{bmatrix} y_t \\ \varphi_t \end{bmatrix}, \quad \begin{bmatrix} y_{t+1} \\ \varphi_{t+1} \end{bmatrix} = \begin{bmatrix} M_{yy} & M_{y\varphi} \\ M_{\varphi y} & M_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} y_t \\ \varphi_t \end{bmatrix}, \\
W_t(y_t, \varphi_t) &= \frac{1}{2}\left(\begin{bmatrix} y_t \\ \varphi_t \end{bmatrix}' \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} y_t \\ \varphi_t \end{bmatrix} \right).
\end{aligned} \tag{71}$$

Equation (69) yields

$$\begin{aligned}
\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} &= \begin{bmatrix} I & 0 \\ X_{xy} & X_{x\varphi} \\ X_{uy} & X_{u\varphi} \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} & P_1 \\ Q_{12}' & Q_{22} & P_2 \\ P_1' & P_2' & R \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{xy} & X_{x\varphi} \\ X_{uy} & X_{u\varphi} \end{bmatrix} \\
&\quad + M' \begin{bmatrix} \beta(1-\alpha)U_{11} + \beta\alpha S & \beta(1-\alpha)U_{12} \\ \beta(1-\alpha)U_{21} & \beta(1-\alpha)U_{22} \end{bmatrix} M
\end{aligned} \tag{72}$$

A possible iterative scheme is (different order of updates is possible):

1. Guess M, X, U, as part of them we have $H = X_{xy}$, $S = U_{11}$
2. Compute an update of U using (72)

3. Solve (70) using Schur decomposition (with stability threshold as $1/\sqrt{\beta(1-\alpha)}$) to find an update for X and M.

Once the procedure has converged, we can find the loss using the standard approach. Assume that the social welfare loss is given by

$$\begin{aligned} L^S &= \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t g_t' \mathcal{Q}^S g_t = \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{s-t} \begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix}' \begin{bmatrix} Q_{11}^S & Q_{12}^S & P_1^S \\ Q_{12}^{S\prime} & Q_{22}^S & P_2^S \\ P_1^{S\prime} & P_2^{S\prime} & R^S \end{bmatrix} \begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix} \\ &= \frac{1}{2} trace\left(\hat{\mathcal{Q}}^S \hat{\mathcal{P}}\right) \end{aligned}$$

where

$$\begin{aligned} \hat{\mathcal{Q}}^S &= \begin{bmatrix} I & 0 \\ X_{xy} & X_{x\varphi} \\ X_{uy} & X_{u\varphi} \end{bmatrix}' \begin{bmatrix} Q_{11}^S & Q_{12}^S & P_1^S \\ Q_{12}^{S\prime} & Q_{22}^S & P_2^S \\ P_1^{S\prime} & P_2^{S\prime} & R^S \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{xy} & X_{x\varphi} \\ X_{uy} & X_{u\varphi} \end{bmatrix} \\ \hat{\mathcal{P}} &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{s-t} \begin{bmatrix} y_t \\ \varphi_t \end{bmatrix}' \begin{bmatrix} y_t \\ \varphi_t \end{bmatrix} \end{aligned}$$

and $\mathcal{Q}^S$ is not necessarily the same as $\mathcal{Q}$ in (69) because the policy maker's objectives are not necessarily social.[24] Matrix $\hat{\mathcal{P}}$ can be found from

$$vec(\hat{\mathcal{P}}) = \left(I - \beta\left(\hat{M} \otimes \hat{M}\right)\right)^{-1} vec\left(\frac{\beta}{1-\beta} V + Z_0\right).$$

where $\hat{M} = (1-\alpha) M + \alpha \begin{bmatrix} M_{yy} & 0 \\ 0 & 0 \end{bmatrix}$, $V = \mathbb{E}_0\left(\begin{bmatrix} \xi_{t+1} \\ 0 \end{bmatrix} \begin{bmatrix} \xi_{t+1} \\ 0 \end{bmatrix}'\right)$ and $Z_0 = \begin{bmatrix} y_0 \\ \varphi_0 \end{bmatrix} \begin{bmatrix} y_0 \\ \varphi_0 \end{bmatrix}'$.

References

Albanesi, S., V. V. Chari, and L. Christiano (2003). Expectation Traps and Monetary Policy. *Review of Economic Studies 70*, 715–741.

Anderson, E. W., L. P. Hansen, E. R. McGrattan, and T. J. Sargent (1996). Mechanics of Forming and Estimating Dynamic Linear Economies. In H.M.Amman, D.A.Kendrick, and J.Rust (Eds.), *Handbook of Computational Economics*, pp. 171–252. Elsevier.

Backus, D. and J. Driffill (1986). The Consistency of Optimal Policy in Stochastic Rational Expectations Models. CEPR Discussion Paper 124, London.

[24] This algorithm is more naturally suited to deal with the case of $\mathcal{Q}^S \neq \mathcal{Q}$ than the algorithm given in Schaumburg and Tambalotti (2007).

Benigno, P. and M. Woodford (2003). Optimal Monetary and Fiscal Policy: A Linear-Quadratic Approach. In *NBER Macroeconomics Annual 2003*, pp. 271–333.

Blake, A. P. and T. Kirsanova (2012). Discretionary Policy and Multiple Equilibria in LQ RE Models. *Review of Economic Studies*. Forthcoming.

Blanchard, O. and C. Kahn (1980). The Solution of Linear Difference Models Under Rational Expectations. *Econometrica 48*, 1305–1311.

Calvo, G. (1983). Staggered Prices in a Utility-Maximising Framework. *Journal of Monetary Economics 12*, 383–398.

Clarida, R. H., J. Galí, and M. Gertler (1999). The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature 37*, 1661–1707.

Clements, D. J. and H. Wimmer (2003). Existence and uniqueness of unmixed solutions of the discrete-time algebraic Riccati equation. *Systems Control Letters 50*, 343–346.

Coenen, G. and R. Straub (2005). Does government spending crowd in private consumption? Theory and empirical evidence for the Euro Area. *International Finance 8*, 435–470.

Currie, D. and P. Levine (1993). *Rules, Reputation and Macroeconomic Policy Coordination.* Cambridge: Cambridge University Press.

Davig, T. and E. Leeper (2006). Fluctuating Macro Policies and the Fiscal Theory. In *NBER Macroeconomics Annual*, pp. 247–298. MIT Press.

Debertoli, D. and R. Nunes (2010). Fiscal Policy under Loose Commitment. *Journal of Economic Theory 145*(3), 1005–1032.

Dennis, R. and T. Kirsanova (2009). Expectations Traps and Coordination Failures: Selecting Among Multiple Discretionary Equilibria. Federal Reserve Bank of San Francisco Working paper 10-2.

Dixit, A. and J. Stiglitz (1977). Monopolistic Competition and Optimum product Diversity. *American Economic Review 67*, 297–308.

Forni, L., L. Monteforte, and L. Sessa (2009). The general equilibrium effects of fiscal policy: Estimates for the Euro area. *Journal of Public Economics 93*, 559–585.

King, R. and A. Wolman (2004). Monetary Discretion, Pricing Complementarity, and Dynamic Multiple Equilibria. *Quarterly Journal of Economics 199*(4), 1513–1553.

Kirsanova, T. and S. Wren-Lewis (2011). Optimal Fiscal Feedback on Debt in an Economy with Nominal Rigidities. *Economic Journal*. Forthcoming.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. Wiley-Interscience.

Kydland, F. E. and E. C. Prescott (1977). Rules Rather than Discretion: the Inconsistency of Optimal Plans. *Journal of Political Economy 85*, 473–91.

Lancaster, P. and L. Rodman (1995). *Algebraic Riccati Equations*. Oxford: Oxford University Press.

Leeper, E., M. Plante, and N. Traum (2010). Dynamics of Fiscal Financing in the United States. *Journal of Econometrics*, Forthcoming.

Leeper, E. M. (1991). Equilibria Under 'Active' and 'Passive' Monetary and Fiscal Policies. *Journal of Monetary Economics 27*, 129–147.

Lockwood, B. and A. Philippopoulos (1994). Insider Power, Unemployment Dynamics and Multiple Inflation Equilibria. *Economica 61*(241), 59–77.

Lohmann, S. (1992). Optimal Commitment in Monetary Policy: Credibility versus Flexibility. *The American Economic Review 82*(1), 273–286.

Oudiz, G. and J. Sachs (1985). International policy coordination in dynamic macroeconomic models. In W. H. Buiter and R. C. Marston (Eds.), *International Economic Policy Coordination*. Cambridge: Cambridge University Press.

Roberds, W. (1987). Models of policy under stochastic replanning. *International Economic Review 28*, 731–755.

Schaumburg, E. and A. Tambalotti (2007). An Investigation of the Gains from Commitment in Monetary Policy . *Journal of Monetary Economics 54*(2), 302–324.

Schmitt-Grohe, S. and M. Uribe (2004). Optimal fiscal and monetary policy under sticky prices. *Journal of Economic Theory 114*, 198–230.

Söderlind, P. (1999). Solution and Estimation of RE Macromodels with Optimal Policy. *European Economic Review 43*, 813–823.