On the Equivalence of Money Growth and Interest Rate Policy

Andreas Schabert[1]
University of Cologne

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Abstract

Central bank behavior is often summarized by simple rules for operating targets, i.e., for a short-run nominal interest rate or for a money growth rate. In this paper we examine conditions under which these rules lead to identical fundamental solutions of a conventional business cycle model. When prices are flexible, forward looking interest rate rules can be equivalent to money growth policy. In particular, the consumption Euler equation implies that constant money growth is equivalent to a passive interest rate regime, while an active interest rate rule corresponds to an accommodating money growth policy. When prices are sticky, equivalence further requires either interest rate policy or households' behavior to be history dependent. However, a central bank, which controls the money growth rate, cannot implement a sequence of nominal interest rates satisfying Taylor's (1993) rule on a saddle stable equilibrium path.

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[1] University of Cologne, Department of Economics, D-50923 Koeln, Germany, email: schabert@wiso.uni-koeln.de, fax: +49/221/470-5077, tel: +49/221/470-4532.

1 Introduction

Changes in the monetary policy stance are today often announced in terms of a short-run nominal interest rate; the latter serving as an operating target, when a central bank adjusts the supply of reserves, e.g., via open market operations. Correspondingly, growing interest has emerged in the recent past on the ability of simple interest rate rules to summarize the behavior of central banks, as, for example, demonstrated by Taylor (1993) or Clarida et al. (2000). On the other hand, it has been argued that the central bank actually controls the growth rate of a narrow monetary aggregate (see, e.g., Eichenbaum, 1992). According to this view, monetary policy might as well be summarized by an exogenous money growth rule as recently shown by Christiano et al. (2001) and presumed in various contributions to the monetary business cycle literature. This raises the question of how money growth and interest rates are actually related and if simple rules for both can be used interchangeably. Taylor (1999), for example, suggests interest rate rules as 'a good description of monetary policy in a fixed money growth regime', whereas simulation results point to substantial differences between these monetary policy procedures in a standard business cycle model (see, e.g., Galí, 2002).

In this paper we aim at deriving conditions for interest rate and money growth rules to lead to identical allocations in the consensus monetary business cycle framework, which is known as the New Neoclassical Synthesis or the New Keynesian model.[2] In particular, for monetary policy rules to be equivalent we claim that they have to implement the same fundamental solution, also known as the bubble-free solution, for the perfect foresight equilibrium.[3] Focussing on the structural part of monetary policy, our analysis thus complements the research on monetary policy shock effects, in particular, on liquidity effects of monetary injections (see, e.g., Christiano et al., 1997, or, Andrés et al., 2002). Comparable studies, which relate money growth to interest rate policy, depart from our approach by either employing different theoretical frameworks, such as Monnet and Weber (2001) using a segmented market model with flexible prices or Vegh (2001) applying a continuous time framework with sticky inflation, or by focussing on implied structural relations instead of reduced form solutions (see Minford et al., 2002).[4]

In the consensus model the so-called forward looking IS-curve, which stems from the consumption Euler equation, governs the relation between the money growth rate and the nominal interest rate. It predicts that consumption grows with a rise in the real interest rate. When demand for real balances is positively related to consumption, a higher real interest rate induces, ceteris paribus, real balances to grow. Given that the model exhibits

[2] For a discussion of this framework see Goodfriend and King (1997) and Clarida et al. (1999).

[3] To facilitate the comparison, money growth rates are set contingent on the inflation rate.

[4] A comparative welfare analysis of simple money growth and interest rate rules in a flexible price framework can be found in Carlstrom and Fuerst (1995).

no endogenous dynamics for flexible prices such that the bubble-free equilibrium path is identical with the steady state, equivalent policy regimes can immediately be identified from structural relations. For example, a constant money growth policy is equivalent to a passive interest rate rule, while an active interest rate policy is equivalent to an accommodating money growth policy, i.e., money growth rates rising with inflation.

When prices are sticky, the analysis of the structural relation between the policy instruments is, however, not sufficient for the derivation of equivalence conditions. The fundamental solution is now history dependent for a money growth regime, where the central bank adjusts money supply contingent on a predetermined stock of money. Further, given that prices evolve sluggishly, real money becomes a relevant endogenous state variable in this case. On the contrary, an interest rate policy, implemented in a non-backward looking way, leads to a fundamental solution, which lacks any history dependence. In order to be equivalent to a money growth rule, interest rate policy must therefore feature a backward looking element, which is, for example, also recommended for optimal interest rate policy (see Woodford, 2000). Specifically, equivalence requires interest rate setting to depend on lagged values of real balances or of output, whereas interest rate smoothing is insufficient for this purpose. These results are demonstrated to be robust with regard to changes in the money demand specification as long as households are purely forward looking. When, on the other hand, households' behavior is history dependent in the sense that it relies on beginning-of-period real balances,[5] non-backward looking interest rate policy can in fact be equivalent to money growth policy. Under equivalence, however, interest rate rule coefficients are not in general associated with positive values. Moreover, a non-destabilizing money growth policy is incompatible with the Taylor-principle, as the latter, which ensures bubble-freeness for a forward looking consumption path (see Woodford, 2001), violates the requirements for saddle path stability in a history dependent environment.[6] Accordingly, the Taylor (1993) rule cannot mimic Friedman's k-percent money growth rule.

The remainder is organized as follows. In section 2 we develop a standard cash-in-advance model. Section 3 presents equivalence conditions for flexible and sticky prices. In section 4 we examine alternative money demand specifications including the original Clower (1967) constraint and the money-in-the-utility-function approach. Section 5 concludes.

2 The model

This section presents a cash-in-advance model with staggered price setting, featuring the main ingredients of standard New Keynesian models, i.e., a consumption Euler equation

[5] In this case the main characteristics of the model and the results are comparable to those in Vegh (2001).

[6] The determinacy properties for interest rate policy are comparable to those in Carlstrom and Fuerst (2001), where different specifications for money in the utility function are considered. In our paper changes in the saddle path stability conditions are due to changes in the state space dimension, whereas their results rely on 'different pricing equations for the nominal interest rate' induced by changes in the timing of markets.

(the 'forward looking IS-curve'), a forward looking aggregate supply constraint (the 'New Keynesian Phillips curve'), and a simple monetary policy rule.

Households Throughout the paper nominal (real) variables are denoted by upper-case (lower-case) letters. There is a continuum of identical and infinitely lived households of measure one. The objective of a representative household is given by:

$$\sum_{t=0}^{\infty} \beta^t U\left(c_t, l_t\right), \qquad \text{with } \beta \in (0,1), \tag{1}$$

where c denotes consumption, l working time, and β the subjective discount factor. The instantaneous utility function $U\left(c_t, l_t\right)$ is assumed to be strictly increasing in c, decreasing in l, strictly concave, twice continuously differentiable, and to satisfy the usual Inada conditions. To obtain a consumption Euler equation, which leads to the so-called 'forward looking IS curve', we further assume that $U\left(c_t, l_t\right)$ is separable: $U\left(c_t, l_t\right) = u(c_t) - v(l_t)$.

At the beginning of each period households are endowed with money M_{t-1} and risk-free government bonds B_{t-1}. Before households enter the goods market in period t, they are not able to adjust their assets' holdings such that they rely on their predetermined asset holdings M_{t-1} and B_{t-1}. After the goods market is closed, they decide on how to adjust their financial assets M_t and B_t. While government bonds earn a nominal interest, $i_t B_{t-1}$, money serves as a means of payment. In particular, consumption expenditures are restricted by the following cash constraint:

$$P_t c_t \leq M_{t-1} + P_t \tau_t, \tag{2}$$

where P denotes the aggregate price level. According to the cash constraint (2) consumption expenditures are restricted by the after tax money holdings.[7] The government transfer $P_t \tau_t$ will rise with the seignorage such that a monetary expansion eases the cash restriction in equilibrium. Hence, this establishes a positive link between a monetary injection and aggregate demand.[8] The representative household further receives wage payments $P_t w_t l_t$ and firms' profits ω_t such that the budget constraint reads

$$P_t c_t + B_t + M_t \leq (1 + i_t) B_{t-1} + M_{t-1} + P_t w_t l_t + P_t \tau_t + P_t \omega_t. \tag{3}$$

Maximizing the objective (1), subject to the cash-in-advance constraint (2) the budget constraint (3), and a no-Ponzi-game condition, $\lim_{i \to \infty} \left(B_{t+i} + M_{t+i}\right) \Pi_{v=1}^{i} (1 + i_{t+v})^{-1} \geq 0$, for given initial values B_0 and M_0 leads to the following first order conditions:

$$u_c(c_t) = \lambda_t + \psi_t, \qquad v_l(l_t) = w_t \lambda_t, \qquad \psi_{t+1} = \lambda_{t+1} i_{t+1}, \qquad \frac{1}{\beta}\lambda_t = \frac{1 + i_{t+1}}{\pi_{t+1}} \lambda_{t+1}, \tag{4}$$

[7] This specification can, for example, be found in the textbook of Walsh (1998).

[8] Variations of the money demand specification and their effects on the conditions for policy equivalence will be discussed in section 4.

and the cash constraint (2) with $\psi_t\left(m_{t-1}\pi_t^{-1}-c_t+\tau_t\right)=0$ and $\psi_t\geq 0$, where π denotes the inflation rate ($\pi_t\equiv P_t/P_{t-1}$), m real balances ($m_t\equiv M_t/P_t$), λ the shadow price of wealth, and ψ the Lagrange multiplier on the cash constraint. Furthermore, the budget constraint (3) holds with equality and the transversality condition, $\lim_{i\to\infty}\lambda_{t+i}\beta^{t+i}(B_{t+i}+M_{t+i})/P_{t+i}=0$, must be satisfied.

Production sector The final consumption good is an aggregate of differentiated goods produced by monopolistically competitive firms indexed with $i\in(0,1)$. The CES aggregator of differentiated goods is defined as $y_t^{1-1/\epsilon}=\int_0^1 y_{it}^{1-1/\epsilon}di$, with $\epsilon>1$, where y is the number of units of the final good, y_i the amount produced by firm i, and ϵ the elasticity of substitution. Let P_i and P denote the price of good i set by firm i and the price index for the final good. The demand for each differentiated good is derived by minimizing the total costs of obtaining $y: y_{it}=(P_{it}/P_t)^{-\epsilon}y_t$, with $P_t^{1-\epsilon}=\int_0^1 P_{it}^{1-\epsilon}di$. A firm i produces good y_i with the technology: $y_{it}=l_{it}$. Following Calvo (1983), firms may reset their prices with the probability $1-\phi$ independent of the time elapsed since the last price setting. The fraction ϕ of firms are assumed to adjust their previous period's prices according to the following simple rule: $P_{it}=\overline{\pi}P_{it-1}$, where $\overline{\pi}$ denotes the average of the inflation rate. The linear approximation to the aggregate supply constraint then reads (see Yun, 1996)

$$\widehat{\pi}_t=\chi\widehat{mc}_t+\beta\widehat{\pi}_{t+1},\ \text{with}\ \chi=(1-\phi)(1-\beta\phi)\phi^{-1},\tag{5}$$

where $\widehat{x}$ denotes the percent deviation of any variable x_t from its steady state value $\overline{x}$, $\widehat{x}=\log(x_t)-\log(\overline{x})$, and mc the real marginal costs. Demand for aggregate labor input in a symmetric equilibrium relates the real marginal costs to the real wage: $mc_t=w_t$.

Public sector The public sector consists of a monetary and a fiscal authority. The latter is assumed to issue one-period bonds, while the former issues money. The consolidated flow budget constraint of the public sector is given by $B_t+M_t=(1+i_t)B_{t-1}+M_{t-1}+P_t\tau_t$. Public policy is assumed to be Ricardian, i.e., to satisfy $\lim_{i\to\infty}(B_{t+i}+M_{t+i})\Pi_{v=1}^i\left(1+i_{t+v}\right)^{-1}=0$ (see Benhabib et al., 2001). The sequence for government bonds is further restricted, for convenience, in that they are issued at a net supply equal to zero.[9]

The monetary authority either controls the nominal interest rate or the money growth rate according to simple rules, encompassing rules commonly applied in the literature. In the first case, it sets the money growth rate $\mu_t\equiv M_t/M_{t-1}$ according to

$$m_t\pi_t/m_{t-1}=\mu_t=\kappa_\mu\pi_t^{\mu_\pi},\qquad \text{with}\quad \mu_t\geq 1,\tag{6}$$

where the parameter κ_μ is assumed to satisfy $\kappa_\mu=\overline{\pi}^{1-\mu_\pi}$, with $\overline{\pi}>\beta$. Generalizing the

[9] The occurence of bonds in the consolidated cash constraint can alternatively be avoided by assuming that only seignorage, $P_t\tau_t^s=M_t-M_{t-1}$, is transferred in form of cash such that $P_t\tau_t^s$ instead of $P_t\tau_t$ enters the right hand side of (2).

case of a constant money growth policy ($\mu_\pi = 0$), we allow the growth rate μ_t to depend on the current realization of the inflation rate in order to facilitate a direct comparison with common interest rate rules. Apparently, one would expect the parameter μ_π to be non-positive for a central bank which aims at stabilizing inflation (see McCallum, 1999).

In the second case, the central bank is assumed to control the nominal interest rate on government bonds. In particular, we allow the gross nominal rate $R \equiv 1 + i$ to be set contingent on a variety of indicators: $R_t = \rho(\pi_t, \pi_{t+1}, R_{t-1}, m_{t-1})$, with $R_t \geq 1$. This fairly general policy rule is assumed to be consistent with the steady state for a money growth policy, i.e., to satisfy the steady state condition $\overline{\rho} = \overline{\pi}/\beta = \kappa_\mu^{1/(1-\mu_\pi)}/\beta$ for $\overline{\rho} > 1$.

3 Relating interest rate to money growth policy

In this section, we aim at revealing conditions for money growth and interest rate policy to lead to the same allocation. In particular, we define policy regimes to be equivalent when they are associated with the identical fundamental solution for the model's perfect foresight equilibrium. We choose this particular definition of equivalence provided that the fundamental solution, which is also known as the bubble-free solution, satisfies commonly used equilibrium selection devices. For example, the fundamental solution is identical with the unique solution satisfying the criterium for saddle path stability, which is also applied in this paper,[10] and it satisfies McCallum's (1983, 1999) minimum state variable criterion.[11] Hence, we abstain from considering non-fundamental solutions, i.e., solutions with extraneous states, which would enlarge the set of policy rules with identical solutions.[12]

We restrict our attention to cases where the nominal interest rate is strictly larger than zero, $i_t > 0$, such that the cash-in-advance constraint always binds. Before we turn to the – probably more realistic – case where prices are set in a staggered way, we start with the case where prices are flexible ($\phi = 0$). We will demonstrate for both cases how money growth and interest rate policy are related to each other by deriving the requirements for simple interest rate rules to implement the fundamental solution of the model's perfect foresight equilibrium for the money growth rule (6).

Flexible prices For the case where prices are flexible we assume, for convenience, that the utility function exhibits constant and strictly positive elasticities of intertemporal substitution: $U^{flex}(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \theta \frac{l_t^{1+\vartheta}}{1+\vartheta}$, where $\theta, \vartheta, \sigma > 0$. Collecting the equilibrium conditions for flexible prices, (2), (4), (6), $w = \frac{\epsilon-1}{\epsilon}$, $(\mu_t - 1)m_t\pi_t^{-1} = \tau_t$ and $c_t = l_t$, and reducing the model, a perfect foresight equilibrium can be defined as follows:[13]

[10] A model exhibits saddle path stability if the number of stable roots is equal to the number of predetermined variables (see Blanchard and Kahn, 1980).

[11] Gauthier (2003) further shows that the fundmental solution is also locally stable under learning.

[12] Non-fundamental solutions are known to allow for sunspot equilibria (see, e.g., Clarida et al., 2000).

[13] Explosive equilibrium paths are, apparently, not ruled out by the transversality condition, which can here be written as $\lim_{i\to\infty} \theta\epsilon(\epsilon-1)^{-1}\beta^{t+i}m_{t+i}^{1+\vartheta} = 0$.

Definition 1 (Flexible price equilibrium) *A perfect foresight equilibrium of the flexible price model ($\phi = 0$) with $R_t > 1$ and U^{flex} is a set of sequences $\{\pi_t, m_t, c_t, R_t\}_{t=0}^{\infty}$ satisfying*

$$(c_{t+1}/c_t)^{\sigma} = \beta R_t/\pi_{t+1}, \tag{7}$$

$$c_t^{\sigma+\vartheta} R_t = (\epsilon - 1)/(\epsilon\theta), \tag{8}$$

$$m_t = c_t, \tag{9}$$

the money growth rule (6), and the transversality condition, given an initial value M_0.

Combining the equilibrium conditions (7)-(9) with the policy rule (6), the model can be reduced to a forward looking difference equation in real balances, $m_t = m_t(m_{t+1})$, such that the single predetermined variable, i.e., nominal balances M_{t-1}, is irrelevant for equilibrium determination. Hence, the fundamental solution exhibits no endogenous state variable implying that a stable and bubble-free equilibrium path must be identical with the steady state itself, satisfying $\overline{\mu} = \overline{\pi} = \beta R$ and the long-run versions of (8) and (9). Examining the difference equation $m_t(m_{t+1})$ reveals that if $\frac{\sigma-1}{\sigma} < \mu_\pi < 1$ or if $1 + \frac{2}{\vartheta-\sigma} < \mu_\pi$ and $\sigma < \vartheta$, the model is saddle path stable such that the fundamental solution is the unique solution (see appendix 7.1). The consumption Euler equation (7) and the money demand condition (9) deliver the following structural relation between money growth, inflation and the nominal interest rate

$$\beta R_t \pi_{t+1}^{\sigma-1} = \mu_{t+1}^{\sigma}. \tag{10}$$

Given that the model exclusively consists of jump variables, equivalent policy regimes can immediately be identified by (10). The following proposition presents the equivalence condition for a forward looking interest rate rule $R_t = \kappa_\rho \cdot \pi_{t+1}^{\rho_\pi}$, where the parameter κ_ρ is assumed to satisfy $\kappa_\rho = \beta^{-1}\overline{\pi}^{1-\rho_\pi}$ to ensure consistency of the steady state.

Proposition 1 (Equivalence, $\phi = 0$) *Suppose that the central bank sets the nominal interest rate according to $R_t = \kappa_\rho \cdot \pi_{t+1}^{\rho_\pi}$. Then interest rate policy is equivalent to the money growth policy* (6) *if and only if*

$$\rho_\pi = 1 + \sigma(\mu_\pi - 1). \tag{11}$$

Condition (11) reveals that a constant money growth policy is equivalent to a passive interest rate policy ($\rho_\pi = 1 - \sigma < 1$), which even features a non-positive inflation elasticity for common degrees of risk aversion, $\sigma \geq 1$.[14] On the other hand, an active interest rate policy ($\rho_\pi > 1$), which is commonly recommended for macroeconomic stability (see Clarida et al., 1999) and often examined in theoretical contributions (see, e.g., Benhabib et al., 2001, or Carlstrom and Fuerst, 2001), mimics an accommodating money growth policy ($\mu_\pi > 1$), which allows for non-fundamental solutions and, thus, for sunspot equilibria. It should

[14] For an interest rate policy satisfying $R_t = \kappa_\rho \cdot \pi_{t+1}^{\rho_\pi}$ the model can be reduced to a difference equation in $\widetilde{\pi}_t = \log \pi_t$ with the eigenvalue $1 + (1 - \rho_\pi)(\sigma + \vartheta)/(\sigma\rho_\pi)$. Hence, a policy rule satisfying $0 < \rho_\pi < 1$ is associated with equilibrium determinacy, which corresponds to Carlstrom and Fuerst's (2001) result for a money-in-the-utility-function model with a cash-in-advance timing.

further be noted that the condition (11) also ensures equivalence for an interest rate rule featuring current inflation, $R_t = \kappa_\rho \cdot \pi_t^{\rho_\pi}$, given that a stable and bubble-free equilibrium path is equal to the steady state itself.

Sticky prices Turning to the sticky price case ($\phi > 0$), the model is log-linearized at the steady state with a target inflation rate $\overline{\pi} : \overline{\pi} \geq \beta$. Interest rate rules, which will only be presented in a log-linear form, are assumed to be associated with the same steady state. The perfect foresight equilibrium of the log-linear model is defined as follows:

Definition 2 (Sticky price equilibrium) *A perfect foresight equilibrium of the log-linear approximation to the model at the steady state with sticky prices ($\phi > 0$) and $R_t > 1$ is a set of sequences $\{\widehat{\pi}_t, \widehat{m}_t, \widehat{c}_t, \widehat{R}_t\}_{t=0}^{\infty}$ satisfying*

$$\sigma\widehat{c}_t = \sigma\widehat{c}_{t+1} - \widehat{R}_t + \widehat{\pi}_{t+1}, \qquad \text{with } \sigma \equiv -u_{cc}(\overline{c})\overline{c}/u_c(\overline{c}) > 0, \tag{12}$$

$$\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \omega\widehat{c}_t + \chi\widehat{R}_t, \qquad \text{with } \omega \equiv \chi(\vartheta + \sigma) \text{ and } \vartheta \equiv v_{ll}(\overline{l})\overline{l}/v_l(\overline{l}) > 0, \tag{13}$$

$$\widehat{m}_t = \widehat{c}_t, \tag{14}$$

$$\widehat{m}_t = \widehat{m}_{t-1} - \widehat{\pi}_t + \mu_\pi\widehat{\pi}_t, \tag{15}$$

and the transversality condition, given an initial value for real balances $m_0 = M_0/P_0$.

When prices are rigid beginning-of-period real balances $\widehat{m}_{t-1}$, which is predetermined, serves as an endogenous state. Hence, the equilibrium sequences of all endogenous variables rely on the history of real balances. It should be noted that the equilibrium conditions (12) and (13) differ from the prototype New Keynesian model in Clarida et al. (1999) just with regard to the gross nominal interest rate entering the aggregate supply constraint (13). However, all results summarized in the propositions of this paper are unchanged for the conventional specification $\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \omega\widehat{c}_t$.[15]

Given that the model exhibits one predetermined, $\widehat{m}_{t-1}$, and one jump variable, $\widehat{\pi}_t$, saddle path stability requires exactly one stable eigenvalue. Examining the characteristic equation reveals that the model is saddle path stable if and only if $\underline{\mu_\pi} < \mu_\pi < 1$, where $\underline{\mu_\pi} = -\infty$ for $\vartheta \geq \sigma$ and $\underline{\mu_\pi} = 1 - 2\frac{1+\beta+\chi}{\chi(\sigma-\vartheta)}$ for $\vartheta < \sigma$ (see appendix 7.2). Hence, equilibrium determinacy is usually ensured by a constant money growth policy.[16] If, on the other hand, the central bank raises the money growth rate by more than one for one with a rise in inflation ($\mu_\pi > 1$), sunspot equilibria or explosiveness cannot be ruled out. Provided that the unique solution under $\underline{\mu_\pi} < \mu_\pi < 1$ is identical to the fundamental solution, the characteristics of the latter are presented in the following lemma.

[15] The latter can, for example, be obtained when the cash constraint is modified to $P_tc_t \leq M_{t-1} + P_tw_t(l_t - L_t) + P_t\tau_t$, where L_t denotes aggregate labor input. This specification, which can be found in Jeanne (1998), avoids the cash-credit distortion between consumption an leisure.

[16] For $\vartheta < \sigma$ the lower bound $\underline{\mu_\pi}$ takes extremely small (negative) values for reasonable parameter values. It can easily be shown that $\underline{\mu_\pi}$ always equals $-\infty$ for the conventional aggregate supply constraint.

Lemma 1 (Fundamental solution) *The fundamental solution of the sticky price model given in definition 2 satisfies*

$$\widehat{m}_t = \widehat{c}_t = \delta_m \widehat{m}_{t-1}, \quad \widehat{\pi}_t = \delta_{\pi m} \widehat{m}_{t-1}, \quad \widehat{R}_t = \delta_{\pi m} \widehat{m}_{t-1}, \tag{16}$$
$$\text{with} \quad \delta_{\pi m} \equiv (1-\delta_m)/(1-\mu_\pi) > 0, \quad \text{and} \quad \delta_{rm} \equiv (1+\sigma(\mu_\pi - 1))\,\delta_m \delta_{\pi m},$$

The eigenvalue δ_m *satisfies* $0 < \delta_m < 1$ *if the model is saddle path stable,* $\underline{\mu_\pi} < \mu_\pi < 1$.

Proof. See appendix 7.3.

As stated in the last part of lemma 1, saddle path stability implies that the eigenvalue δ_m lies between zero and one. According to the solution for the stable eigenvalue (see appendix 7.3), endogenous persistence rises with the inflation elasticity μ_π. For example, applying standard parameter values ($\beta = 0.99$, $\sigma = \vartheta = 2$, $\phi = 0.8$), we obtain $\delta_m = 0.60$ for $\mu_\pi = -0.5$, $\delta_m = 0.65$ for $\mu_\pi = 0$, and $\delta_m = 0.73$ for $\mu_\pi = 0.5$. Using the fundamental solution of the model, one can easily derive conditions for policy equivalence. For this the structural relation between money growth, inflation, and the interest rate, $\sigma\widehat{\mu}_{t+1} = \widehat{R}_t + (\sigma-1)\widehat{\pi}_{t+1}$, is again employed, which can either be obtained from combining (12) and (14) or by log-linearizing (10) at the steady state. In contrast to the flexible price case, the requirements for equivalence cannot solely be inferred from this equation. As can be seen from definition 2, the equilibrium path is history dependent as monetary policy relies on beginning-of-period real balances, $\widehat{m}_{t-1}$. While this always applies for money growth policy, interest rate policy induces the same history dependence if interest rate policy relies on lagged values of real balances, $\widehat{m}_{t-1}$. The following proposition summarizes this result.

Proposition 2 (Equivalence, $\phi > 0$) *Suppose that the central bank sets the nominal interest rate according to* $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_m \widehat{m}_{t-1}$. *Then interest rate policy is equivalent to the money growth policy* (15) *if and only if*

$$\rho_m = [(1+\sigma(\mu_\pi - 1))\,\delta_m - \rho_\pi]\,(1-\delta_m)/(1-\mu_\pi), \quad \text{and} \quad \rho_m \neq 0. \tag{17}$$

Proof. Applying the state space solution form $\widehat{m}_t = \delta_m \widehat{m}_{t-1}$ and $\widehat{\pi}_t = \delta_{\pi m}\widehat{m}_{t-1}$ for the model (12)-(14) with an interest rate policy satisfying $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_m \widehat{m}_{t-1}$, leads to the conditions $(1-\beta\delta_m - \chi\rho_\pi)\,\delta_{\pi m} = \omega\delta_m + \chi\rho_m$ and $(1-\delta_m)\,\sigma\delta_m + \rho_m = (\delta_m - \rho_\pi)\,\delta_{\pi m}$ for the undetermined coefficients δ_m and $\delta_{\pi m}$. A comparison with the solution given in (6) then reveals that the policy parameter ρ_π and ρ_m has to satisfy (17) in order to implement the same fundamental solution as the money growth policy. As the fundamental solution for interest rate policy exhibits coefficients on the endogenous state equal to zero ($\delta_m = \delta_{\pi m} = \delta_{rm} = 0$) for $\rho_m = 0$, equivalence further requires $\rho_m \neq 0$. ■

Given that the solution (16) implies $\widehat{\pi}_{t+1} = \delta_m \widehat{\pi}_t$, an analogous equivalence condition can immediately be derived for an interest rate rule featuring future inflation rates.[17] According

[17] For a policy rule satisfying $\widehat{R}_t = \rho_\pi \widehat{\pi}_{t+1} + \rho_m \widehat{m}_{t-1}$ the equivalence condition (17) changes slightly to: $\rho_m = [1+\sigma(\mu_\pi - 1) - \rho_\pi]\delta_m(1-\delta_m)/(1-\mu_\pi)$ and $\rho_m \neq 0$.

to proposition 2 equivalence requires the coefficient on real balances ρ_m to be non-positive for reasonable parameter values and to decline with the inflation elasticity ρ_π. For example, using the parametrization from above equivalence demands: $\rho_m = -0.35\rho_\pi - 0.23$ for $\mu_\pi = 0$, $\rho_m = -0.27\rho_\pi - 0.32$ for $\mu_\pi = -0.5$, and $\rho_m = -0.54\rho_\pi$ for $\mu_\pi = 0.5$. Condition (17), together with the condition for saddle path stability, further has an important implication for the set of interest rate sequences, which can be implemented by a money growth policy. For example, a money growth rule aimed to mimic a simple active interest rate rule ($\rho_m = 0$, $\rho_\pi > 1$), has to satisfy, by (17), $\mu_\pi = 1 + \left(\rho_\pi \delta_m^{-1} - 1\right)/\sigma > 1$. Such a value for μ_π, however, violates the condition for saddle path stability. It implies that a higher inflation rate leads to a rise in the growth rate of real balances, which further stimulates aggregate demand feeding higher prices such that inflation expectations can be self-fulfilling. When prices are extremely rigid, this mechanism can even lead to explosive paths.[18] Hence, a sequence of interest rates satisfying $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$, with $\rho_\pi > 1$, which is often found to be associated with equilibrium uniqueness for interest rate policy (see, e.g., Woodford, 2001),[19] allows for unstable or sunspot equilibria when the central bank sets the money growth rate.

According to the equivalence condition (17), an interest rate policy characterized by $\rho_m = 0 \Rightarrow \rho_\pi = (1 + \sigma(\mu_\pi - 1))\,\delta_m$ is not equivalent to a money growth policy, even though, it implies a sequence of constant money growth rates. In this case, the equilibrium sequences are independent of the history of real balances implying that the fundamental solution exhibits no relevant endogenous state ($\delta_m = \delta_{\pi m} = \delta_{rm} = 0$). Thus, history dependence of interest rate rules, which is also recommended for optimal interest rate policy (see Woodford, 2000), is necessary for equivalence, whereas the inflation elasticity can be equal to zero. Consider, for example, the lagged value of output, which enters the optimal interest rate rule under commitment in the standard New Keynesian model (see Clarida et al., 1999), as an alternative interest rate rule argument. Replacing real balances by output in the policy rule, $\widehat{R}_t = \rho_y \widehat{y}_{t-1} + \rho_\pi \widehat{\pi}_t$, again leads to equivalence for (17) with $\rho_y = \rho_m$, given that $\widehat{y}_t = \widehat{m}_t$ holds. However, at least one coefficient, ρ_y or ρ_π, of an interest rate rule, which is equivalent to a constant money growth policy, is non-positive for reasonable degrees of risk aversion, $\sigma \geq 1$. We can further conclude, using $\widehat{y}_t = \delta_m \widehat{y}_{t-1}$, that a money growth policy cannot implement an interest rate sequence satisfying Taylor's (1993) rule on a saddle stable equilibrium path, while Friedman's k-percent rule is for $\sigma \geq 1$ even incompatible with the so-called Taylor-principle (see Woodford, 2001).[20]

[18] It can be shown that an inflation elasticity satisfying $\mu_\pi > 1$ leads to explosiveness if $\chi < 1 - \beta$.

[19] For our model the determinacy condition for $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$ reads: $1 < \rho_\pi < 1 + (\vartheta/\sigma) + (1-\beta)/\chi$, which is related to Carltrom and Fuerst's (2001) result for sticky prices. The derivation of this condition can be found in Brueckner and Schabert (2003) for a sticky price model with working capital exhibiting a reduced and log-linearized form, which is isomorphic to the model, (12) and (13), given in definition 2.

[20] As defined by Woodford (2001), the Taylor-principle demands $\rho_\pi + \kappa\rho_{y0} > 1$ for a policy rule $\widehat{R}_t = \rho_{y0}\widehat{y}_t + \rho_\pi \widehat{\pi}_t$, where the parameter κ is strictly positive and defined by $\kappa \equiv (1-\beta)/\omega > 0$.

It should further be noted that interest rate smoothing, where lagged values of the interest rate are included in the policy rule, $\widehat{R}_t = \rho \widehat{R}_{t-1} + \rho_\pi \widehat{\pi}_t$, does not facilitate equivalence. With a non-zero coefficient on past interest rates, interest rate policy can certainly induce any degree of history dependence, including an eigenvalue $\delta_r \equiv \widehat{R}_t / \widehat{R}_{t-1}$ equal to δ_m. However, the policy rule parameters ρ and ρ_π cannot be set in a way that reproduces all structural relations between inflation, real balances and the interest rate implied by the solution (16) for a money growth regime (see appendix 7.4).[21]

4 Alternative money demand specifications

In this section we consider two variations of the cash-constraint, before we briefly examine the case where money demand is induced by assuming that money enters the utility function.

Including net bonds trading Though, the notion of a 'cash-in-advance' constraint seems to be unambiguous, the literature has employed a variety of specifications, which depart from the original Clower (1967) constraint, including the version presented in (2). An alternative specification, which is for example applied in Lucas and Stokey (1987), is $P_t c_t \leq M_{t-1} + R_t B_{t-1} + B_t + P_t \tau_t$. Allowing net bonds trading to ease the cash constraint can be rationalized by assuming that the bonds market opens before the goods market is closed (see Carlstrom and Fuerst, 2001). While the consolidated cash constraint still reads $c_t = m_t$ in an equilibrium with $i_t > 0$, the model changes with regard to the consumption Euler equation (7), which now features the future interest rate $R_{t+1} : (c_{t+1}/c_t)^\sigma = \beta R_{t+1}/\pi_{t+1}$. Hence, the structural relation between the money growth rate and the nominal interest rate becomes: $\beta R_{t+1} \pi_{t+1}^{\sigma-1} = \mu_{t+1}^\sigma$. Regarding the issue of equivalence this modification has only minor consequences. While the equivalence condition for flexible prices in proposition 1 remains unchanged, it changes slightly when prices are sticky.[22] Hence, including net bonds trading in the cash constraint does not alter the main conclusions.

History dependent households In contrast, the results in the previous section are considerably altered when transfers are omitted from the cash constraint providing the original Clower (1967) constraint: $P_t c_t \leq M_{t-1}$.[23] It implies that current consumption can only respond to shocks via changes in the price level, and leads to the following equilibrium condition: $c_t = m_{t-1} \pi_t^{-1}$. While the change in the cash restriction leaves the equivalence condition for the flexible price model unaffected, it has substantial consequences when prices are sticky. Replacing the cash restriction (14) in definition 2 by

$$\widehat{c}_t = \widehat{m}_{t-1} - \widehat{\pi}_t, \tag{18}$$

[21] The reason is that the fundamental solution now features the lagged nominal interest rate as the single endogenous state variable, which implies that the remaining coefficients of the fundamental solution are only indirectly affected by ρ and ρ_π via their impact on the eigenvalue $\delta_r(\rho, \rho_\pi)$.

[22] The equivalence condition is now given by $\rho_m = [(1 + \sigma(\mu_\pi - 1)) - \rho_\pi](1 - \delta_m)/(1 - \mu_\pi)$ and $\rho_m \neq 0$.

[23] This specification can, for example, also be found in the textbook of Obstfeld and Rogoff (1996).

causes the bubble-free equilibrium path to be history dependent regardless of the prevailing policy regime.[24] On the one hand, the perfect foresight equilibrium of the model for money growth policy is again characterized by (16).[25] On the other hand, the fundamental solution for an interest rate policy now exhibits an endogenous state variable, i.e., real balances $\widehat{m}_{t-1}$, even for a non-backward looking policy rule. For example, it is shown in appendix 7.5 that an interest rate rule solely featuring the current inflation rate $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$ is associated with saddle path stability if interest rate policy is passive $\rho_\pi < 1$. In this case, the fundamental solution is given by: $\widehat{m}_t = \delta_m \widehat{m}_{t-1}$, $\widehat{c}_t = (1-\delta_{\pi m})\widehat{m}_{t-1}$, $\widehat{\pi}_t = \delta_{\pi m}\widehat{m}_{t-1}$, with $0 < \delta_m < 1$ and $\delta_{\pi m} = \omega(1-\beta\delta_m + \omega - \chi\rho_\pi)^{-1}$. Equivalence of money growth and interest rate policy can then be established in the following way.

Proposition 3 *Suppose that consumption expenditures are restricted by* $P_t c_t \leq M_{t-1}$. *An interest rate rule* $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$ *is then equivalent to a money growth policy* (15) *if and only if*

$$\rho_\pi = (1-\sigma)\,\delta_m + \sigma\mu_\pi. \tag{19}$$

Proof. Replacing consumption in (12) with the cash constraint (18) gives: $\widehat{R}_t = \sigma\widehat{\mu}_t + (1-\sigma)\,\widehat{\pi}_{t+1}$. For money growth policy, the fundamental solution for the nominal interest rate, thus, reads: $\widehat{R}_t = [\sigma\mu_\pi + (1-\sigma)\,\delta_m]\,\delta_{\pi m}\widehat{m}_{t-1}$. Reintroducing the current inflation rate by its solution $\widehat{\pi}_t = \delta_{\pi m}\widehat{m}_{t-1}$ then leads to the equivalence condition (19). ■

Hence, history dependence of households' behavior allows simple interest rate rules to be equivalent to a money growth policy. For example, condition (19) implies that the central bank can for $\sigma = 1$ mimic a constant money growth policy by pegging the interest rate.

Money-in-the-utility-function At last we consider the widely used money-in-the-utility-function approach to demonstrate the robustness of the main results. Specifically, suppose that the utility function takes the form $U^m(c_t, l_t, a_t) = U(c_t, l_t) + \frac{\gamma}{1-\sigma_m}(A_t/P_t)^{1-\sigma_m}$, with $\gamma, \sigma_m > 0$.[26] The relevant stock of nominal balances A_t either equals *B*eginning-of-period money holdings $A_t = M_{t-1}$ (version *B*) or *E*nd-of-period money holdings $A_t = M_t$ (version *E*).[27] The corresponding first order condition, which is given by

$$\gamma m_t^{-\sigma_m} = \lambda_{t+1} i_{t+1} \text{ for version } B, \quad \text{and} \quad \gamma m_t^{-\sigma_m} = \lambda_{t+1} i_{t+1}\beta/\pi_{t+1} \text{ for version } E, \tag{20}$$

reveals that households' behavior does not depend on past realizations of real balances, regardless whether beginning-of-period or end-of-period money holdings enter the utility

[24] This case, thus, accords to Vegh's (2001) continuous time framework, where real balances are assumed to be predetermined even for a forward looking interest rate policy.

[25] For sufficiently rigid prices, $\chi < \beta/(\sigma-1)$, the condition for saddle path stability is then given by $-(\chi\vartheta)^{-1} < \mu_\pi < 1$, which ensures that the stable root lies between zero and one (see appendix 7.5).

[26] Empirical evidence, for example provided by Ireland (2003), indicates separability of the utility function with regard to consumption and money to be a reasonable assumption.

[27] The latter is also called cash-when-I'm-done (see Carlstrom and Fuerst, 2001).

function. As in our benchmark case, the model exhibits an endogenous state variable only if monetary policy is backward looking. The log-linearized version of (20) and the consumption Euler equation,[28] leads to the following structural relation between the nominal interest rate, the money growth rate, and inflation

$$(R-1)^{-1}\left(R\widehat{R}_t-\widehat{R}_{t+1}\right)-\sigma_m\widehat{\mu}_t=\begin{cases}\widehat{\pi}_{t+1}-\sigma_m\widehat{\pi}_t & \text{version } B\\ 2\widehat{\pi}_{t+1}-(\sigma_m+1)\widehat{\pi}_t & \text{version } E\end{cases}. \tag{21}$$

The fact that money demand also depends on the nominal interest rate apparently makes the problem more awkward.[29] Nevertheless, equation (21) implies that the nominal interest rate is in the long-run, i.e., after gradual adjustments are completed, related to the inflation rate by $1+\sigma_m\left(\mu_\pi-1\right)$, which corresponds to the relation in the benchmark model (see 11). Further, equivalence in a sticky price environment still requires interest rate policy to depend on lagged values of real balances. On the contrary, a B-version of a utility function U^{mn}, which is non-separable in consumption and real balances, leads to a first order condition for consumption featuring beginning-of-period money holdings, $\lambda_t=U_c^{mn}(c_t,m_{t-1}/\pi_t)$. History dependence is then induced by the households' behavior – as in the case of the Clower (1967) constraint (see 18) – such that equivalence then becomes possible even for non-backward looking interest rate rules.

5 Conclusion

In this paper we have examined the relation between money growth and interest rate policy in a standard business cycle model. When money demand increases with consumption, the consumption Euler equation predicts that real balances rise with the real interest rate. As a consequence, an active interest rate policy is associated with accommodating money growth rates, while a constant money growth rule mimics a passive interest rate rule. When prices are sticky, interest rate policy must further exhibit a backward looking component to be able to implement the same fundamental solution as a money growth regime. While the inclusion of lagged values of real balances or of output in an interest rate rule facilitates equivalence, interest rate smoothing is not sufficient for this purpose. However, the arguments of an interest rate rule, that is equivalent to Friedman's k-percent rule, are in general not associated with positive coefficients, which clearly runs counter to conventional expectations about a monetary policy regime, which aims at targeting inflation. The results further suggest that Taylor's (1993) rule cannot be interpreted as a description of an interest rate sequence implemented by a non-destabilizing money growth policy, i.e., a money growth regime satisfying the requirements for saddle path stability.

[28] Note that the consumption Euler equation, now reads $\sigma(\widehat{c}_{t+1}-\widehat{c}_t)=\widehat{R}_{t+1}-\widehat{\pi}_{t+1}$.

[29] In particular, the characteristic polynomial becomes cubic, as the interest rate now enters the structural relation (21) with two different time indices.

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7 Appendix

7.1 Appendix to the flexible price model

Taking logs, the equilibrium conditions given in definition 1 can be written as $\sigma(\widetilde{c}_{t+1}-\widetilde{c}_t) = \widetilde{\beta}+\widetilde{R}_t-\widetilde{\pi}_{t+1}$, $\widetilde{c}_t = \widetilde{m}_t$, and $\widetilde{c}_t = -\gamma_1\widetilde{R}_t+\gamma_2$, with $\gamma_1 \equiv (\sigma+\vartheta)^{-1}$, $\gamma_2 \equiv \widetilde{\frac{\epsilon-1}{\epsilon\theta}}(\sigma+\vartheta)^{-1}$, where $\widetilde{x}_t$ denotes the log of x_t ($\widetilde{x}_t = \log x_t$). Replacing the nominal interest rate and consumption yields $\sigma\widetilde{m}_{t+1} = -\vartheta\widetilde{m}_t - \widetilde{\pi}_{t+1} + \widetilde{\beta} + \gamma_2\gamma_1^{-1}$. Combining the latter with the logged money growth rule for period $t+1$, $\widetilde{m}_{t+1} = \widetilde{m}_t + (\mu_\pi - 1)\,\widetilde{\pi}_{t+1} + \widetilde{\kappa_\mu}$, then leads to the following difference equation in real balances

$$\widetilde{m}_{t+1} = (1+\gamma_3)\,\widetilde{m}_t + \xi, \qquad \text{with } \gamma_3 \equiv (1-\mu_\pi)\,(\sigma+\vartheta)\,/[1+\sigma\,(\mu_\pi-1)], \tag{22}$$

and where ξ denotes a constant. Given that the endogenous variable can jump, uniqueness requires the eigenvalue of (22) to be unstable, i.e., to lie outside the unit circle. Using that γ_3 is strictly positive if and only if $\frac{\sigma-1}{\sigma} < \mu_\pi < 1$, and that $\gamma_3 < -2$ holds if and only if $\sigma < \vartheta$ and $1 + \frac{2}{\vartheta-\sigma} < \mu_\pi$, the model with flexible prices is uniquely determined if $\frac{\sigma-1}{\sigma} < \mu_\pi < 1$ or if $1 + \frac{2}{\vartheta-\sigma} < \mu_\pi$ for $\sigma < \vartheta$.

7.2 Appendix to the sticky price model

To derive conditions for saddle path stability, the model in definition 2 is written as

$$M_0\begin{pmatrix}\widehat{m}_t\\ \widehat{\pi}_{t+1}\end{pmatrix} = M_1\begin{pmatrix}\widehat{m}_{t-1}\\ \widehat{\pi}_t\end{pmatrix}, \qquad \text{with } M_0 = \begin{pmatrix}\omega & \eta\\ 1 & 0\end{pmatrix} \text{ and } M_1 = \begin{pmatrix}0 & 1\\ 1 & \mu_\pi-1\end{pmatrix}, \tag{23}$$

where $\eta \equiv \beta + \chi + \chi\sigma\mu_\pi - \chi\sigma$. The characteristic polynomial of $M_0^{-1}M_1$ is given by

$$F(X) = X^2 + (-1+\omega\mu_\pi - \omega - \chi - \chi\sigma\mu_\pi + \chi\sigma - \beta)\,\eta^{-1}X + \eta^{-1}. \tag{24}$$

The model exhibits exactly one predetermined variable such that stability and uniqueness requires one stable and one unstable eigenvalue. To identify the conditions for saddle path stability, we distinguish two cases. Consider first the case where $\eta > 0 \Leftrightarrow 1 - \frac{\chi+\beta}{\chi\sigma} < \mu_\pi$. Given that $F(X)$ exhibits a minimum and that $F(0) = \eta^{-1} > 0$, $F'(0) = -[1 + \omega\,(1-\mu_\pi)]\eta^{-1} - 1 < 0$ and $F(1) = \omega\eta^{-1}\,(\mu_\pi - 1)$, there is exactly one stable root, which lies between zero and one if $\mu_\pi < 1$. For the case $1 - \frac{\chi+\beta}{\chi\sigma} > \mu_\pi$, which implies that $\mu_\pi < 1$, $F(0) < 0$, $F(1) > 0$ and $F'(0) > 0$ holds, there is again exactly one root between zero and one. In order to rule out equilibrium multiplicity, we further have to ensure that $F(-1) = -2+[2+\omega(1-\mu_\pi)]\eta^{-1}$ is not positive. While $F(-1)$ cannot be positive for $\vartheta \geq \sigma$ given that $1 - \frac{\chi+\beta}{\chi\sigma} > \mu_\pi$, uniqueness additionally requires that $1 - \frac{2\beta+2+2\chi}{\chi(\sigma-\vartheta)} < \mu_\pi < 1$ for $\vartheta < \sigma$. Hence, the model is saddle path stable if $\underline{\mu_\pi} < \mu_\pi < 1$, where $\underline{\mu_\pi}$ is equal to $-\infty$ for $\vartheta \geq \sigma$ or equal to $1 - 2\frac{1+\beta+\chi}{\chi(\sigma-\vartheta)}$ for $\vartheta < \sigma$.

7.3 Proof of lemma 1

In order to establish the claims made in the lemma, the model given in definition 2 is solved by applying the method of undetermined coefficients (see, e.g., McCallum, 1999) for the solution form: $\widehat{m}_t = \widehat{c}_t = \delta_m \widehat{m}_{t-1}$, $\widehat{\pi}_t = \delta_{\pi m}\widehat{m}_{t-1}$, $\widehat{R}_t = \delta_{rm}\widehat{m}_{t-1}$. Eliminating consumption in (12) and (13) by (14) and replacing the endogenous variables in the equilibrium conditions with the general solution form gives the following conditions for δ_m and $\delta_{\pi m}$:

$$\delta_{\pi m} = (1-\delta_m)/(1-\mu_\pi), \qquad \omega\delta_m = \delta_{\pi m} - (\beta + \chi\sigma(\mu_\pi - 1) + \chi)\,\delta_{\pi m}\delta_m. \tag{25}$$

Eliminating the coefficient $\delta_{\pi m}$ in (25) leads to a quadratic equation in δ_m, which equals the characteristic polynomial (24) in appendix 7.2. There it is shown that if the model is saddle path stable, then the coefficient δ_m, which equals the smaller root of (24), lies between zero and one. Further applying the structural relation $\sigma\widehat{\mu}_{t+1} = \widehat{R}_t + (\sigma-1)\widehat{\pi}_{t+1}$ and the money growth rule (14) gives $\widehat{R}_t = [\sigma\mu_\pi - (\sigma-1)]\widehat{\pi}_{t+1}$ and, thus, $\delta_{rm} = [1+\sigma(\mu_\pi - 1)]\,\delta_{\pi m}\delta_m$. ■

7.4 Appendix to interest rate smoothing

To demonstrate that interest rate smoothing is insufficient to facilitate policy equivalence, we apply the method of undetermined coefficients to derive the fundamental solution of the sticky price model with a backward looking interest rate rule satisfying $\widehat{R}_t = \rho\widehat{R}_{t-1} + \rho_\pi\widehat{\pi}_t$. Replacing the endogenous variables therein and in the equilibrium conditions (12)-(14) with the solution form $\widehat{m}_t = \delta_{mr}\widehat{R}_{t-1}$, $\widehat{\pi}_t = \delta_{\pi r}\widehat{R}_{t-1}$, and $\widehat{R}_t = \delta_r\widehat{R}_{t-1}$ leads to the following conditions for δ_{mr}, $\delta_{\pi r}$, and δ_r :

$$\rho_\pi\delta_{\pi r} = \delta_r - \rho, \qquad \delta_{mr} = \frac{(\delta_{\pi r}-1)\,\delta_r}{\sigma(1-\delta_r)}, \qquad \delta_{\pi r} = \frac{\omega\delta_{mr} + \chi\delta_r}{(1-\beta\delta_r)}. \tag{26}$$

For (26) to be identical to the fundamental solution for a money growth policy (16), the following equalities must hold: $\delta_r = \delta_m$, $\delta_{mr} = \delta_m^2/\delta_{rm}$, and $\delta_{\pi r} = \delta_{\pi m}\delta_m/\delta_{rm}$. Inserting the latter equations into (26) leads to the following three conditions in ρ and ρ_π : $\rho_\pi\delta_{\pi m}\delta_m\delta_{rm} = \delta_m - \rho$, $\delta_m^2/\delta_{rm} = (\delta_{\pi m}\delta_m/\delta_{rm} - 1)\,\delta_m/[\sigma(1-\delta_m)]$, and $\delta_{\pi m}\delta_m/\delta_{rm} = \left(\omega\delta_m^2/\delta_{rm} + \chi\delta_m\right)/(1-\beta\delta_m)$. Hence, it follows immediately from the last two conditions, which are not equivalent and independent of ρ and ρ_π, that the set of values for ρ and ρ_π ensuring the equivalence to a money growth policy (15) is empty.

7.5 Appendix to history dependent households

In order to examine the conditions for saddle path stability when the original Clower (1967) constraint is imposed, the model (12), (13), (15), and (18) is rewritten as (23) with

$$M_0 = \begin{pmatrix} 0 & \beta + (1-\sigma)\,\chi \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} -\omega & 1+\omega-\chi\sigma\mu_\pi \\ 1 & \mu_\pi - 1 \end{pmatrix}.$$

The characteristic polynomial of $M_0^{-1}M_1$ is now given by $F(X) = X^2 - (\omega - \chi\sigma - \chi\sigma\mu_\pi + \beta + 1 + \chi)(\beta + \chi - \chi\sigma)^{-1} X - (\chi\sigma\mu_\pi - 1 - \omega\mu_\pi)(\beta + \chi - \chi\sigma)^{-1} = 0$. Suppose that prices are reasonable rigid such that $\chi < \frac{\beta}{\sigma-1}$. Then $F(0) = \frac{1-\chi\sigma\mu_\pi+\omega\mu_\pi}{\beta-\chi\sigma+\chi} > 0$ for $\mu_\pi > -1/\chi\vartheta$, and $F(1) = \omega\frac{\mu_\pi-1}{\beta-\chi\sigma+\chi} < 0$ for $\mu_\pi < 1$, indicating that exactly one root lies between zero and one if $-1/\chi\vartheta < \mu_\pi < 1$ given that prices are sufficiently rigid, $\chi < \frac{\beta}{\sigma-1}$.

Replacing the money growth rule (15) by the interest rate rule $\widehat{R}_t = \rho_\pi\widehat{\pi}_t$ the matrices in (23) are now given by

$$M_0 = \begin{pmatrix} 0 & \beta \\ \sigma & 1-\sigma \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} -\omega & 1+\omega-\chi\rho_\pi \\ \sigma & \rho_\pi - \sigma \end{pmatrix}.$$

The characteristic polynomial of $M_0^{-1}M_1$, which reads $F(X) = X^2 - (\omega + \sigma - \sigma\chi\rho_\pi + \sigma\beta)(\sigma\beta)^{-1}X + (\omega\rho_\pi + \sigma - \sigma\chi\rho_\pi)(\sigma\beta)^{-1}$, is characterized by $F(0) = (\chi\vartheta\rho_\pi + \sigma)/\beta\sigma$ and by $F(1) = \frac{\omega}{\beta\sigma}(\rho_\pi - 1)$. Hence, the model is saddle path stable if and only if $-\sigma/\chi\vartheta < \rho_\pi < 1$. In this case, $F(0) > 0$ and $F(1) < 0$ such that the stable eigenvalue lies between zero and one. Regarding the fundamental solution, we thus know that $0 < \delta_m < 1$. Further, the solution for the inflation rate, $\delta_{\pi m} = \omega/[(1+\omega-\chi\rho_\pi) - \beta\delta_m]$, can immediately be obtained from the aggregate supply constraint, which now reads $(1+\omega-\chi\rho_\pi)\widehat{\pi}_t - \beta\widehat{\pi}_{t+1} = \omega\widehat{m}_{t-1}$.