Tight Guarantees for fair division: a general model

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Abstract

A context-free problem of Fair Division is a function $\mathcal{W}$ from n-profiles of "types" x_i to a freely transferable amount of "surplus" $\mathcal{W}(x_1, \cdots, x_n)$ they must share in the common property regime. A pair of *tight guarantees* assigns to each type an upper and a lower bound on its share under *any* profile of types of the other agents, and these bounds cannot be improved. The choice of a particular pair of such guarantees when the types and $\mathcal{W}$ have an economic interpretation vindicates only some familiar "fair" sharing rules, and suggests many new ones. Our examples include the allocation of an indivisible good or bad, the classic model of a "commons" where types enter additively in the function $\mathcal{W}$, and sharing the cost of a capacity or of the transportation costs to a location on a line.

JEL classification codes: D6, D63

1 Introduction

The modern mathematical discussion of fair division starts with the concept of "fair share" in Steinhaus' cake cutting model ([**30**]): irrespective of the n other participants' utilities I know that my share will be worth to me at least $\frac{1}{n}$-th of the whole cake. This first step toward defining the fair and efficient exploitation of resources is critical in a variety of other contexts: the division of family heirlooms and other private assets; sharing a workload or the cost of a public facility; dividing the output of a common production function with variable returns or the discount from a joint purchase etc..

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We bring here new arguments in support of the simplicity and versatility of the fair share concept in an abstract model where agents are described by their type (which may represent preferences, needs, skills, efforts, location etc..) and the resources by a black box transforming the profile of types into an certain amount of utility (or disutility, e. g., cost) that they must share. We assume that the agents have identical rights or liabilities toward these resources and can only be distinguished by their types. Differences in the way those types came about – such as objective needs vs frivolous tastes, proactive effort vs lucky skills – will be ignored: agents are fully responsible for their own types.

In this context-free interpretation of types the first uncontroversial test of fairness is Equal Treatment of Equals (ETE) (aka horizontal equity): two agents with identical types must be treated equally. We impose the stronger property of Anonymity: swapping the (possibly dfferent) types of two agents exchanges their shares and does not affect those of other agents.

The Unanimity test comes next. Fix a profile of types $(x_1, \cdots, x_n)$ and an agent i. At the hypothetical *unanimous* profile where all agents have the same type x_i they all end up by ETE with the same utility level, $\frac{1}{n}$-th of the corresponding efficient total utility: we call it agent i's unanimity utility and write $una(x_i)$. Differences in individual types are a collective externality that they are jointly and equally responsible for. The test rules out a distribution of shares where some agent i's share is larger than her unanimity utility $una(x_i)$ whereas another agent j ends up with a smaller share than $una(x_2)$: they must all end up (weakly) above, or all weakly below, their unanimity utility.[1]

In the cake cutting model where utilities are additive the unanimity utility is precisely $\frac{1}{n}$-th of that for the whole cake. When we divide Arrow Debreu commodities and preferences are convex, it is the utility for $\frac{1}{n}$-th of the bundle of goods we are dividing ([35], see more references in section 2). In the provision of a public good problem it sets an *upper* bound on individual utilities ([20]).[2]

Two simple observations apply in any Fair Division problem, irrespective of the technical details of the model. Suppose that for any profile of types we can and do allocate the resources so that every agent i's utility is at least $una(x_i)$. At the x_i-unanimous profile we cannot give more than $una(x_i)$ to each agent, therefore $una(x_i)$ is type x_i's *worst case* utility over the adversarial choice of types chosen by the other agents. Moreover if we don't pay attention to unanimity utilities and distribute the resources in any other way, the worst case utility of a type x_i agent is *at most* $una(x_i)$ (think again of the x_i-unanimous profile). So $una(x_i)$ is unambiguously the best (largest) *worst case* utility we can offer to the agents.

An endogenous lower bound on my utility that depends only on my type is what we call a *lower guarantee*: it minimises the adverse influence that the other agents can have on me. We just showed

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[1] The unanimity terminology and the corresponding test are introduced in **[19]**. The test is called "diversity of preferences dividend, (or burden)" in [33], p.112-114.

[2] Let $c(z)$ be the cost of producing z units of public good and $u_i(z, y_i)$ be agent i's (standard) utility if her cost share is y_i. The unanimity utility is then $una(u_i) = \max_z u_i(x, \frac{1}{n}c(z))$; clearly the utility profile $(una\,(x_1)\,, \cdots, una\,(x_n))$ is either unfeasible or Pareto optimal, so $una(u_i)$ is a the resulting upper bound on individual utilities.

that if the function $x_i \to una(x_i)$ is a lower guarantee, it is larger than any other lower guarantee.

A similar upper bound on my utility (an upper guarantee) protects the other agents, as a group, by preventing me from grabbing too much surplus. Placing a tight cap on each agent's *best case* welfare, although less common than working to increase the worst case welfare, appears regularly in the form of "no subsidy" constraints in the natural monopoly and cost sharing literatures. Recall the public good provision example above. The unanimity utility plays for such upper bounds the symmetric role of the one explained above for lower bounds. If we can allocate the resources efficiently so that the profile $((una(x_1), una(x_2), \cdots, una(x_n))$ always upper-bounds individual utilities, then $una(x_i)$ is the best case utility for type x_i and the function una is smaller than any other upper guarantee.

Our first goal is to describe the best upper and lower guarantees of a given (black box) function from types to transferable utility (or disutility). That is, the lower guarantees that cannot be increased and the upper ones that cannort be decreased: we call them *tight* guarantees. We find that for the large class of super (resp sub) modular functions the unanimity utility is the unique tight upper (resp lower) guarantee and make good progress toward understanding the typically very large set of tight guarantees on the other side of the unanimity bound.

Next we add a microeconomic interpretation to the types and the black box function and capture many classic fair division problems: the allocation of an indivisible good or bad, the classic model of a commons, cost sharing of a capacity or of the transportation costs to a facility, and more; see the examples in sections 2,5, and 6. In each problem the choice of a pair of tight guarantees, one lower and one upper, severely restricts the set of feasible divisions of the surplus or cost. Each such pair conveys its own normative meaning, its own nuance of fairness. Tight guarantees vindicate some of the familiar division rules (egalitarian, proportional, Shapley value, serial etc..) and dismiss others. They also inspire many new division rules.

1.1 overview of the results

The iconic example in section 2 illustrates the power of our methodology before we unfold it to full effect. A type x_i is a number between 0 and H and we must share $\mathcal{W}(x_1, \cdots, x_n) = \max_{i \in [n]}\{x_i\}$. In the first story this is a desirable surplus: the agents own a single indivisible good in common property and x_i is agent i's willingness to pay for the good. In the second story it is the cost of a facility they share and x_i is the amount of capacity i needs (as in [16]). The unanimity share $una(x_i) = \frac{1}{n}x_i$ is the compelling worst share of surplus or minimal (best case) cost share for agent i. On the other side there is a one dimensional choice of upper guarantees described in Lemma 2.1. At one end the Egalitarian share $\frac{1}{n}H$ is natural in the indivisible good story (everyone is the common owner of the efficient surplus), much less so in the cost sharing context. At the other end the Stand Alone share x_i is natural for the lcapacity story, and is a meaningful alternative to the egalitarian share in the indivisible good story. There is a one dimensional set of upper guarantees compromising between these two (different from their convex combinations) each one with an original normative interpretation.

Section 3 introduces the model and general properties of tight guarantees. Types vary in the real interval $[L, H]$; the function $\mathcal{W}$ inputs a profile of types $x \in [L, H]^n$ and returns $\mathcal{W}(x)$ that must be shared between the agents. It is symmetric in all its variables.

The real valued functions g^- and g^+ with domain $[L, H]$ are respectively a *lower* and an *upper guarantee* if for each n-profile x we have

$$\sum_1^n g^-(x_i) \leq \mathcal{W}(x) \leq \sum_1^n g^+(x_i) \tag{1}$$

The lower guarantee g^- is *tight* if increasing $g^-(x_i)$ at any x_i violates the left hand (LH) inequality (1) at some profile containing x_i. For the tightness of g^+ replace increasing by decreasing and LH by RH. Tight guarantees are the closest separably additive approximations of the function $\mathcal{W}$ from above and below.

Notation: $(\overset{k}{z})$ is the k-vector with identical coordinates z. At the unanimous profile $(\overset{n}{x_i})$ the system (1) implies

$$g^-(x_i) \leq una(x_i) \leq g^+(x_i) \text{ for any type } x_i \tag{2}$$

where $una(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{x_i})$ is the unanimity utility.

Recall that the symmetric function $\mathcal{W}$ is supermodular (resp submodular) if whenever $x_1 < x_1^*$ the difference $\mathcal{W}(x_1^*, x_2, x_{-1,2}) - \mathcal{W}(x_1, x_2, x_{-1,2})$ increases (resp decreases) weakly in x_2. The results in section 4 go a long way toward solving system (1) for such functions.

The unanimity function of a super (resp sub) modular function is an upper (resp lower) guarantee: Proposition 4.1. By (2) this means that una is the unique tight upper (resp lower) guarantee. All results and almost all examples below take advantage of this instant answer to one half of our problem.

As a first step to describe the set of tight guarantees on the other side of the unanimity one, we find, like in Example 2.1, two canonical elements of this large set:

$$g_L(x_i) = \mathcal{W}(x_i, \overset{n-1}{L}) - (n-1)una(L) \; ; \; g_H(x_i) = \mathcal{W}(x_i, \overset{n-1}{H}) - (n-1)una(H)$$

If $\mathcal{W}$ is supermodular g_L is a tight lower guarantee. Because $g_L(L) = una(L)$ and una is the unique tight guarantee, the pair (g_L, una) implies that type L gets the share $una(L)$ *irrespective of other agents' types.* Any other type x_i gets the share $una(L)$ plus the increment $\mathcal{W}(x_i, \overset{n-1}{L}) - \mathcal{W}(\overset{n}{L})$. It is as if agent i is "Standing Alone" while all other agents are of type L.

Theorem 4.1 explains that g_L and g_H have a book-end role in the set of lower guarantees. Still assuming that $\mathcal{W}$ is supermodular, for any other tight lower guarantee g^- we have

$$g_H(L) \le g^-(L) \le g_L(L) = una(L) \; ; \; g_L(H) \le g_H(H) \le g_H(H) = una(H)$$

Moreover, everywhere in $[L, H]$, $g^-(x_i)$ grows in x_i slower than g_H and faster than g_L.

We focus in section 5 on the "commons" model: $\mathcal{W}(x) = F(x_N)$ with the notation $x_N = \sum_1^n x_i$; the types are perfect substitutes in the production of surplus or cost. Examples include the production of divisible output from input types, of a cost to share from demand types, as well as queuing and location models. The problem is super (resp sub) modular iff F is convex (resp concave). Although we do not describe the full set of tight guarantees (on the other side of una) we identify two subsets gradually compromising between g_L and g_H. First a discrete sequence of $n-2$ tight guarantees with the similar form: $g(x_i) = \mathcal{W}(x_i, c) - C$, where c is a $(n-1)$-profile mixing types L and H, and C a constant: Proposition 5.1. Second, a continuous line of tight guarantees also linking g_L to g_H: most are simply a tangent to the graph of the unanimity function, except at both ends where they turn into the hybrid of a tangent and a simple guarantee: Proposition 6.1.

In section 6 we introduce a class of modular functions generalising the function $\mathcal{W}(x) = \max_{i \in [n]}\{x_i\}$ for which *every* tight guarantee (on the opposite side to *una*) is *simple* as we just described. Write the order statistics of $(x_i)_1^n$ as $(x^k)_1^n$ (where $x^1 = \max_i\{x_i\}$ and $x^n = \min_i\{x_i\}$) and call $\mathcal{W}$ *rank separable* if $\mathcal{W}(x) = \sum_{k \in [n]} w_k(x^k)$ for some continuous functions w_k. Such a function is modular if and only if $w_{k+1} - w_k$ is weakly increasing for $1 \le k \le n-1$.

If $\mathcal{W}$ is modular and rank separable, Theorem 6.1 shows that its tight guarantees (opposite to the unanimity) are parametrised by *all* the profiles c in $[L, H]^{n-1}$ and take the simple form $g_c^-(x_i) = \mathcal{W}(x_i, c) - C$, where C is determined by the choice of c. We give several applications of this result to capacity, facility location and queuing problems.

Our last main result, Theorem 7.1, is a full characterisation of all tight guarantees for two person commons with one-dimensional types, when $\mathcal{W}(x_1, x_2)$ is *strictly* modular. They are parametrised by the choice of a decreasing, continuous and symmetric function from the set of types into itself. This is evidence that the full set of tight guarantees is much larger, in fact of infinite dimension, when the modularity property is strict. Note that a rank separable function is emphatically *not* strictly modular.

After concuding comments in section 8, the Appendix section 9 contains many long or minor proofs.

1.2 tight guarantees and sharing rules

Tight guarantees contribute to solve a fair division problem in two ways. First by promoting participation in an unscripted negotiation by minimising its stakes: a type x_i agent can reject any agreement where her share falls outside the interval $[g^-(x_i), g^+(x_i)]$, confident that if no agreement is reached the manager will pick an (arbitrary) share within these bounds for each i.

Alternatively, the shares of $\mathcal{W}(x)$ may be computed by a deterministic division rule: a mapping φ from profiles x the individual shares $(\varphi_i(x))_1^n$ such that $\mathcal{W}(x) = \sum_{i-1}^n \varphi_i(x)$ for all x. Each rule φ implements its own lower and upper guarantees for each type x_i: $g^-(x_i) = \min_{x_{-i}}\{\varphi_i(x_i, x_{-i})\}$ and $g^+(x_i) = \max_{x_{-i}}\{\varphi_i(x_i, x_{-i})\}$. Our test dismisses the rules implementing non tight guarantees and partitions the others according to the tight pair they generate.

For instance if $\mathcal{W}(x) = F(x_N)$ as in section 5, the venerable Average Return (AR) rule $\varphi_i(x) = \frac{x_i}{x_N}F(x_N)$ fails spectacularly the unanimity test ([22]).[3] If F is concave it gives to each input type x_i below the average $\frac{1}{n}x_N$ a smaller share than the unanimity lower guaranteed share $\frac{1}{n}F(nx_i)$: the latter is a justified guaranteed share because agent i is not responsible for the low returns generated by larger inputs; she is entitled to her fair share of the best (i. e, first) marginal returns of the technology F, captured by her unanimity share. If F is convex AR allows a below average input x_i to free ride on high returns to which they did not contribute, contradicting this time the unanimity upper guarantee.

By contrast the increasing and decreasing Serial rules (Definition 4.4) implement respectively the tight pairs (una, g_L) and (una, g_H), and this for any modular function $\mathcal{W}$: Proposition 4.2.

It may not be easy, given a particular sharing rule, to decide if the guarantees it implements are tight or not. But the answer to the converse question is easy: given a pair of tight guarantees any sharing rule delivering shares within the interval it defines implements exactly these guarantees (Lemma 3.7). We can construct such rules by simple extrapolations of the guarantees, or by "trimming" an arbitrary sharing rule when its shares violate the guarantees.

1.3 related literature

After its introduction in the mathematical discussion of cake cutting ([30], [14] [11]), the concept of *endogenous fair share* (in our terminology a guaranteed worst case utility against adversarial types of other participants) was picked up by economists in the early 70-s. If we divide a bundle ω of private Arrow Debreu (AD) commodities in common property and preferences are convex, the allocation$\frac{1}{n}\omega$ delivers precisely the unanimity utility ([35], [32]); the latter is also recognised in the allocation of indivisible items with cash transfers ([31], [2]).

In the following two decades, endogenous lower and upper bounds on individual welfare play an important role in the axiomatic discussion of cooperative production (the commons problem). The stand alone utility (from using a private copy of the production function) joins the unanimity utility and sits on the opposite side of the Pareto frontier when the returns to scale are monotonic ([28], [19], [22], [37]). The same is true in the public good provision model irrespective of returns ([20]); see also a proposal of weaker bounds in ([12]). Endogenous guarantees appear also in the axiomatic bargaining model ([34]) with a focus on variations in the set of agents rather than the agents' preferences.

[3] The same is true for the Shapley value rule derived from the Stand Alone TU game (ibid.).

In this century computer scientists and others are still searching for a compelling concept of fair share for the allocation of indivisible items (good or bad) even when utilities are additive. Only a little more than a $\frac{3}{4}$ fraction of the plausible *MaxMin Share* (MMS) is feasible at all profiles ([6], [26], [1]) and even if the exact fraction is ever identified, this lower guarantee may not be tight. Other concepts of endogenous fair share appear in ([10]), ([18]), ([15]) and ([3]).

An even more widely open question is the search for tight guarantees in cake division with non additive utilities, or for dividing AD commodities with non convex preferences ([4]).

Our interpretation of self-ownership as a pair of lower and upper bounds on welfare introduces the new viewpoint that minimising the best case utility (against adversarial others) is as important as maximising the worst case utility. Our assumption that the agents are equally responsible for their own type (which could mean fully or not at all) avoids a familiar controversy around the "neo-Lockean" self-ownership maxim, defended by Nozick ([25]) but criticised for its potentially libertarian implications by Roemer ([27]) and Cohen ([9]).

We also diverge from the context dependent interpretation of common ownership derived from the Resource Monotonicity property: when the shared resources improve, all the participants should weakly benefit. This fairness test is in fact incompatible with the unanimity fair share if we divide AD commodities ([24]), and with the unanimity utility in the cooperative production commons ([23]).

Our abstract model of the commons delivers general results (in Sections 3, 4) that we apply to a great variety of microeconomic examples. But the assumption that utility is transferable by some numeraire is typically absent in the above literature.

2 Example 2.1: *one $\mathcal{W}$, two stories*

The type x_i varies in the interval $[0, H]$ and the function $\mathcal{W}$ is

$$\mathcal{W}(x) = \max_{1 \leq i \leq n} \{x_i\}$$

Story 1: *The agents inherit an indivisible good and share it efficiently. Type x_i is agent i's willingness to pay for the good. One of the efficient agents will get it and compensate the others in cash. What compensation is fair?*

Story 2: *They share the cost (linear 1 to 1) of a public capacity: the width of a channel, the length of a runway ([16]) or the amount of broadband they jointly buy. Type x_i is the amount of capacity agent i needs and the largest need must be met. Who should pay how much?*

We generate some lower and upper guarantees, feasible but not necessarily tight, from any deterministic sharing rule. In story 1 a natural rule splits equally of the efficient surplus $\mathcal{W}(x)$ between the heirs, treating that surplus as a common property. This rule generates the following lower and upper guarantees[4]

$$g^-(x_i) = \min_{x_{-i}\in[0,H]^{n-1}} \frac{1}{n}\max\{x_i, \max_{j\in N\setminus i} x_j\} = \frac{1}{n}x_i = una(x_i)$$

$$g_H^+(x_i) = \max_{x_{-i}\in[0,H]^{n-1}} \{\frac{1}{n}\max\{x_i, \max_{j\in N\setminus i} x_j\}\} = \frac{1}{n}H$$

The unanimity guarantee is the uncontroversial Proportional Share.

To check that g_H^+ is a tight upper bound fix x_i and consider the profile $(x_i, \overset{n-1}{H})$: all agents other than i cannot get more than $\frac{1}{n}H$ and we distribute the surplus H so all the agents i must get $\frac{1}{n}H$.

In story 2 the Proportional Share $\frac{1}{n}x_i$ is still reasonable as a minimal payment by i, but the upper guarantee $\frac{1}{n}H$ (now i's worst case outcome) is less so: it takes the extreme view of ignoring all differences in individual demands, so an agent with very small needs may end up paying as much as one imposing the largest capacity H. Without any reference to the context, Lemma 1.1 below confirms the extreme nature of this normative position.

The simplest way to account for differences in needs is to divide total cost in proportion to individual needs, which produces the guarantees

$$g^-(x_i) = \min_{x_{-i}\in[0,H]^{n-1}} \{\frac{x_i}{x_i + x_{N\setminus i}} \max_{j\in[n]}\{x_j\}\} = \frac{1}{n}x_i = una(x_i)$$

$$g_0^+(x_i) = \max_{x_{-i}\in[0,H]^{n-1}} \frac{x_i}{x_i + x_{N\setminus i}} \max_{j\in[n]}\{x_j\} = x_i$$

The upper guarantee g_0^+ is the familiar Stand Alone cost (discussed in subsection above). It is tight: at the profile $(x_1, \overset{n-1}{0})$ no agent other than 1 pays anything (because $una(0) = g_0^+(0) = 0$), therefore i covers the full cost.

In story 1 the guarantee $g_0^+(x_i) = x_i$ does make sense if the good (perhaps a heirloom of sentimental value) is worthless to anyone but these n agents: each one is at most entitled to the amount of surplus that she is creating.

Clearly a convex combination of g_0^+ and g_H^+ is a feasible upper guarantee but not a tight one; we check this right after the proof of Lemma 2.1. Our first result describes the simple line of tight upper guarantees connecting g_0^+ and g_H^+. We use the notation $z_+ = \max\{z, 0\}$.

Lemma 2.1: *The tight upper guarantees g_p^+ of $\mathcal{W}(x) = \max_{1\leq i\leq n}\{x_i\}$ are parametrised by a type $p \in [0, H]$ as follows:*

$$g_p^+(x_i) = \frac{1}{n}p + (x_i - p)_+ \text{ for } x_i \in [0, H] \tag{3}$$

So g^+_0 and g^+_H are the two end-points of the interval of tight upper guarantees.

The Lemma follows from the much more general Theorem 5.1. We give here a simple proof to develop an intuition for our problem.

Proof: Checking that g^+_p meets the RH inequalities in (1) is routine and omitted for brevity.

Pick now an arbitrary tight upper guarantee g^+ and set $p = ng^+(0)$. At the unanimous profile $(\overset{n}{0})$ inequality (1) implies $p \geq 0$. Tightness implies that g^+ increases weakly (this is easy to check or see Lemma 3.1) so $g^+(x_i) \geq \frac{1}{n}p$ for all x_i. The constant function $\frac{1}{n}H$ is an upper guarantee therefore if $p > H$ the guarantee g^+ is not tight. So $p \in [0, H]$.

Applying now (1) to $(x_i, \overset{n-1}{0})$ gives $g^+(x_i) \geq x_i - \frac{n-1}{n}p$. Combining this with $g^+(x_i) \geq \frac{1}{n}p$ gives $g^+ \geq g^+_p$. Because g^+ is tight and g^+_p is an upper guarantee this must be an equality, which also implies that g^+_p is tight. ■

For any $\lambda \in [0, 1]$ choose $p = \lambda H$: it is easy to check that the upper guarantee $g^+ = (1 - \lambda)g^+_0 + \lambda g^+_H$ coincides with g^+_p at types 0 and H, and is strictly larger everywhere else.

If we adopt the tight pair (una, g^+_p) the benchmark type p gets the share $\frac{1}{n}p$ irrespective of other agents' types. In story 1 p could be an estimate of the market value of the good: if the good is worth less than p for an agent i, she will receive at most a fair share of p; an agent who values it more than p *could* receive, in addition to $\frac{1}{n}p$ the full surplus $x_i - p$. This will happen for sure if x_i is the only type above p: then i gets the good and pays $\frac{1}{n}p$ to everyone else.

Similarly in story 2 p could be the "normal" or status quo capacity; only agents with needs larger than p can be charged more than $\frac{1}{n}p$, and the surcharge can reach the full incremental cost $x_i - p$.

Overall an agent with small needs prefer a low benchmark capacity p and one with large needs a high p. And vice versa in story 1 where the agents of large type likes the parameter p to be low.

3 Guarantees: definition and general properties

The set of agents is $\{1, \cdots, n\}$ is written N or $[n]$ depending if the individual labels matter or not, and $\mathcal{X} = [L, H] \subset \mathbb{R}$ is the common set of types. The notation x_i may represent the type of a specific agent identified by the context, or simply a generic single type, to distinguish it from a profile of types $x \in X^N$.

At the profile $x = (x_i)_{i \in [n]} \in \mathcal{X}^{[n]}$ we must divide the benefit or cost $\mathcal{W}(x)$. The function $\mathcal{W}$ is symmetric in the n variables x_i and continuous on $\mathcal{X}^{[n]}$.

Definition 3.1 *The functions g^- and g^+ from $\mathcal{X}$ into $\mathbb{R}$ are respectively a lower and an upper guarantee of $\mathcal{W}$ if they satisfy the inequalities:*

$$\sum_{i \in [n]} g^-(x_i) \leq \mathcal{W}(x) \leq \sum_{i \in [n]} g^+(x_i) \text{ for all } x \in \mathcal{X}^{[n]} \tag{4}$$

We write $\boldsymbol{G}^{-}, \boldsymbol{G}^{+}$ the (clearly non empty) sets of such guarantees.

Definition 3.2 *Given two lower guarantees $g_1^-, g_2^- \in G^-$ we say that g_1^- dominates g_2^- if $g_1^-(x_i) \geq g_2^-(x_i)$ for $x_i \in \mathcal{X}$ and $g_1^- \neq g_2^-$. The guarantee $g^- \in G^-$ is tight if it is not dominated in G^-; equivalently increasing g^- at a single $x_1 \in \mathcal{X}$ creates a violation of the LH inequality in (4) for some $x_{-1} \in \mathcal{X}^{[n-1]}$.*

The isomorphic statement for upper guarantees in G^+ flips the domination inequality around; for tightness it replaces increasing by decreasing and LH by RH.

We write $\mathcal{G}^-$ and $\mathcal{G}^+$ for the subsets of tight guarantees in $\boldsymbol{G}^-$ and $\boldsymbol{G}^+$.

Lemma 3.1

i) For $\varepsilon = +, -$ every guarantee $g \in \boldsymbol{G}^\varepsilon$ is either tight or dominated by a tight one. So $\mathcal{G}^\varepsilon$ is not empty.

ii) A tight guarantee is weakly monotonic if $\mathcal{W}$ is.

iii) A tight guarantee is continuous because $\mathcal{W}$ is.

Statement *i)* is a simple application of Zorn's Lemma. For statement *ii)* fix $g \in \mathcal{G}^-$. If $x_i > x_i^*$ and $g(x_i) < g(x_i^*)$ define $\widetilde{g}(x_i) = g(x_i^*)$ and $\widetilde{g} = g$ otherwise, then check that $\widetilde{g}$ is still in $\boldsymbol{G}^-$. This contradicts that g is tight. The longer proof of statement *iii)* is in section 9.1.

Lemma 3.2

A guarantee g in $\boldsymbol{G}^\varepsilon$ is tight if and only if for all $x_i \in \mathcal{X}$ there exists $x_{-i} \in \mathcal{X}^{[n-1]}$ s. t.

$$g(x_i) + \sum_{j \in [n] \setminus i} g(x_j) = \mathcal{W}(x_i, x_{-i}) \tag{5}$$

Then we call (x_i, x_{-i}) a contact profile of g at x_i ; the set of such profiles is the contact set $\mathcal{C}(g)$ of g.

Proof in the section 9.1.of the Appendix.

Tight guarantees with a simple contact set are easy to describe and use. Such a class of guarantees is defined two subsections below: it plays a critical role in the results of sections 4,5 and 6.

Lemma 3.3 *Fix a tight guarantee $g \in \mathcal{G}^+$. For any x_i, x_i^* and contact profile $x = (x_i, x_{-i})$ of g at x_i we have*

$$g(x_i^*) - g(x_i) \geq \mathcal{W}(x_i^*, x_{-i}) - \mathcal{W}(x_i, x_{-i}) \tag{6}$$

and the opposite inequality if $g \in \mathcal{G}^-$.

Proof In the inequality

$$g(x_i^*) + \sum_{j \neq i} g(x_j) \geq \mathcal{W}(x_i^*, x_{-i})$$

we replace each term $g(x_j)$ by $\mathcal{W}(x) - g(x_i) - \sum_{k \neq i,j} g(x_k)$ and rearrange it as follows

$$(n-1)(\mathcal{W}(x) - g(x_i)) - (n-2)\sum_{j \neq i} g(x_j) \geq \mathcal{W}(x_i^*, x_{-i}) - g(x_i^*)$$

$$\Longleftrightarrow \mathcal{W}(x) - g(x_i) + (n-2)(\mathcal{W}(x) - \sum_{[n]} g(x_j)) \geq \mathcal{W}(x_i^*, x_{-i}) - g(x_i^*)$$

The term in parenthesis is zero by our choice of x_{-i} so we are done. ■

Corollary *Suppose K is a positive constant, $\mathcal{X} \subset \mathbb{R}^A$ and the function $\mathcal{W}$ is K-Lipschitz in each x_i, uniformly in $x_{-i} \in \mathcal{X}^{[n-1]}$. Then so is each tight guarantee $g \in \mathcal{G}^{\varepsilon}$* for $\varepsilon = +, -$.

We state without proof two useful invariance properties of guarantees.

Lemma 3.4 *Fix $\mathcal{X} = [L, H]$, $\mathcal{W}$ and $\varepsilon = +, -$.*
i) If $\mathcal{W}_0$ is additively separable, $\mathcal{W}_0(x) = \sum_{[n]} w_0(x_i)$, then

$$\mathcal{G}^{\varepsilon}(\mathcal{W} + \mathcal{W}_0) = \mathcal{G}^{\varepsilon}(\mathcal{W}) + \{w_0\}$$

ii) Fix θ is a bicontinuous increasing bijection $x_i = \theta(z_i)$ from the interval $\mathcal{Z} = [\theta^{-1}(L), \theta^{-1}(H)]$ into $\mathcal{X}$, and change variables to a new problem $\widetilde{\mathcal{W}}(z) = \mathcal{W}(\theta(z))$ where $\theta(z)_i = \theta(z_i)$. Then if *$g \in \mathcal{G}^{\varepsilon}(\mathcal{W})$ in the original problem, $g \circ \theta \in \mathcal{G}^{\varepsilon}(\widetilde{\mathcal{W}})$* in the new problem. *If θ is decreasing, ceteris paribus, then $g \circ \theta \in \mathcal{G}^{-\varepsilon}(\widetilde{\mathcal{W}})$.*

For instance the problem $\mathcal{W}(x) = F(\max_{i \in [n]}\{x_i\})$ reduces to $\widetilde{\mathcal{W}}(z) = \max_{i \in [n]}\{z_i\}$ by the change $x_i = F^{-1}(z_i)$; and $\mathcal{W}(x) = \min_{i \in [n]}\{x_i\}$ reduces to $\widetilde{\mathcal{W}}(z) = \max_{i \in [n]}\{z_i\}$ by the change of variable $x_i = -z_i$.

3.1 the unanimity shares and guarantees

The restriction of $\mathcal{W}$ to the diagonal of $\mathcal{X}^{[n]}$ defines the *unanimity* share of agent i:

$$una(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{x_i}) \tag{7}$$

(recall $(\overset{k}{z})$ is the k-vector with z in each coordinate). We repeat and amplify two observations made in the Introduction.

Lemma 3.5
i) For any $(g^-, g^+) \in \boldsymbol{G}^- \times \boldsymbol{G}^+$ and for $x_i \in \mathcal{X}$

$$g^-(x_i) \leq una(x_i) \leq g^+(x_i) \tag{8}$$

ii) If una is a lower (resp upper) guarantee it dominates each lower (resp upper) guarantee. And if $\mathcal{G}^{\varepsilon}$ *is a singleton then it must be the unanimity guarantee. For* $\varepsilon = +, -$:

$$una \in \boldsymbol{G}^{\varepsilon} \Longrightarrow \mathcal{G}^{\varepsilon} = \{una\} \text{ and } |\mathcal{G}^{\varepsilon}| = 1 \Longrightarrow \mathcal{G}^{\varepsilon} = \{una\}$$

iii) For $\varepsilon = +, -$ *and for any* $x_i \in \mathcal{X}$ *there is a tight guarantee* $g \in \mathcal{G}^{\varepsilon}$ *s.t.* $g(x_i) = una(x_i)$.

Statement $i)$ and the first part of $ii)$, already discussed in section 1, follow by applying inequalities (4) at a unanimous profile. The rest of the proof is in section 9.2.

We see that $\mathcal{W}$ is additively separable, $\mathcal{W}(x) = \sum_1^n w(x_i)$, if and only if both sets $\mathcal{G}^{\varepsilon}$ are singletons and the unanimity shares are justified by the tight guarantee requirement. In any other case we will find that on at least one side of (4) there are infinitely many tight guarantees.

3.2 a special class of guarantees

In Example 2.1 for any x_i, a contact profile of g_p at x_i is $(x_i, \overset{n-1}{p})$.[5] The guarantees for which a fixed $(n-1)$-profile serves as contact set for any type are easy to describe and play a key role in sections 4 and 6.

Definition 3.3

i) An upper or lower guarantee is called simple if all types have a common contact profile $c \in \mathcal{X}^{n-1}$

$$g_c(x_i) + \sum_{\ell=1}^{n-1} g_c(c_\ell) = \mathcal{W}(x_i, c) \text{ for all } x_i \in \mathcal{X} \tag{9}$$

This implies that g_c *is given by*

$$g_c(x_i) = \mathcal{W}(x_i, c) - \frac{1}{n}(\sum_{\ell=1}^{n-1} \mathcal{W}(c_\ell, c)) \text{ for all } x_i \in \mathcal{X} \tag{10}$$

ii) A Stand Alone guarantee g_{c_0} *of* $\mathcal{W}$ *is a simple guarantee such that* $c = (\overset{n-1}{c_0})$:

$$g_{c_0}(x_i) = \mathcal{W}(x_i, \overset{n-1}{c_0}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{c_0})$$

If property (9) is true, g_c can be written $g_c(x_i) = \mathcal{W}(x_i, c) - C$ for some constant C. Replace accordingly each term $g_c(c_\ell)$ by $\mathcal{W}(c_\ell, c) - C$ and rearrange to find $g_c(x_i)$ as announced in (10). The converse transformation of (10) to (9) is just as easy.

Lemma 3.6 *For any* $c \in \mathcal{X}^{n-1}$ *if* g_c *is a guarantee, it is is tight.*

[5] There are more: e. g., if $x_i \leq p$ we can choose any x_{-i} where just one type is larger than p.

This follows at once from (9) and Lemma 3.2.

If the parameter $c = \binom{n-1}{c_0}$ is unanimous the Stand Alone guarantee g_{c_0} "touches" the unanimity guarantee at c_0: $g_c(c_0) = una(c_0)$. In Lemma 2.1 g_p is the Stand Alone guarantee with parameter $c_0 = p$.

We do not expect a simple but not Stand Alone guarantee to touch the unanimity guarantee: see statement ii) in Proposition 5.1.

3.3 implementing a guarantee by a sharing rule

A sharing rule φ for the function $\mathcal{W}$ maps each profile $x \in \mathcal{X}^{[n]}$ to a division $y = \varphi(x)$ of $\mathcal{W}(x)$: $\sum_{[n]} y_i = \mathcal{W}(x)$.

Given a pair (g^-, g^+) of tight guarantees it is easy to find a rule φ of which the guarantees are precisely this pair.

Lemma 3 7. *Fix the function $\mathcal{W}$ and a tight pair $(g^-, g^+) \in \mathcal{G}^- \times \mathcal{G}^+$. If the sharing rule φ satisfies $g^-(x_i) \leq \varphi_i(x) \leq g^+(x_i)$ for all i and x then it implements (g^-, g^+):* for all i and x

$$\min_{x_{-i}}\{\varphi_i(x_i, x_{-i})\} = g^-(x_i) \ ; \ \max_{x_{-i}}\{\varphi_i(x_i, x_{-i})\} = g^+(x_i)$$

To check the left equality note that $\min_{x_{-i}}\{\varphi_i(x_i, x_{-i})\}$ is a lower guarantee, at the same time bounded below by $g^{-\cdot}$ then invoke the tightness of g^-. A similar argument works for the right equality.

The moving average of g^- and g^+ is the simplest sharing rule implementing this pair in $\mathcal{G}^- \times \mathcal{G}^+$:

$$\varphi_i(x) = \lambda g^-(x_i) + (1 - \lambda) g^+(x_i)$$

where for all $x \in \mathcal{X}^{[n]}$ we choose λ s. t. $\sum_{[n]} \varphi_i(x) = \mathcal{W}(x)$.

Alternatively we can pick *any* sharing rule φ that does not implement (g^-, g^+) and adjust it *only* at those profiles where if fails at least one of these bounds; in this way the adjusted rule $\widetilde{\varphi}$ does implement the desired pair of guarantees and preserves the choices of φ as much as possible.

4 Modular functions $\mathcal{W}$

In this class of benefit and cost functions that includes most of our examples, the analysis of tight guarantees simplifies on both sides of the system of inequalities (4). The following definitions take into account of the fact that $\mathcal{W}$ is symmetric in its variables.

Definition 4.1 *We call $\mathcal{W}$ super (resp sub) modular if for all $x \in \mathcal{X}^{[n]}$ and all x_1^* such that $x_1^* > x_1$ the function $\mathcal{W}(x_1^*, x_{-1}) - \mathcal{W}(x_1, x_{-1})$ is weakly increasing (resp decreasing) in x_{-1}. And*

$\mathcal{W}$ is strictly super (sub) modular if this monotonicity is strict.

We say that $\mathcal{W}$ is modular if it is either supermodular or submodular.

The following consequence of Definition 4.1 can be taken as an alternative definition of supermodularity (or submodularity by reversing the inequality). For every x_1, x_1^*, x_2, x_2^* in $\mathcal{X}$ and x_{-12} in $\mathcal{X}^{n-2}$ such that $x_i \leq x_i^*$ for $i = 1, 2$, we have

$$\mathcal{W}(x_1, x_2^*, x_{-12}) + \mathcal{W}(x_1^*, x_2, x_{-12}) \leq \mathcal{W}(x_1, x_2, x_{-12}) + \mathcal{W}(x_1^*, x_2^*, x_{-12}) \tag{11}$$

If $\mathcal{W}$ is twice differentiable it is modular if and only if the sign of $\partial_{ij}\mathcal{W}(x)$ is constant in $\mathcal{X}^{[n]}$.

Simple examples of (non differentiable) modular functions include $\max_i\{x_i\}$ submodular (Example 2.1), $\min_i\{x_i\}$ supermodular. But the median of $\{x_i\}_{i\in[n]}$ (say n is odd) is not modular. We describe its guarantees in Example 6.3.

An important remark: with the exception of section 5, none of our general results requires $\mathcal{W}$ to be monotonic in the x_i-s, or convex or concave even in a single variable.

4.1 the unanimity guarantee

Proposition 4.1 *If $\mathcal{W}$ is super (resp sub) modular the unanimity function (7) is the unique tight upper (resp lower) guarantee: $\mathcal{G}^+ = \{una\}$.*

Notation: as $\mathcal{W}$ is symmetric in the x_i-s, we write $(z; \overset{k}{y})$ for any $(k+1)$-vector where one coordinate is z and k coordinates are y.

Proof The proofs for the submodular and supermodular cases simply exchange the sign of all inequalities. We assume that $\mathcal{W}$ is supermodular without loss of generality.

Suppose $n = 2$. By Lemma 3.5 we only need to prove that una is an upper guarantee, $una \in \boldsymbol{G}^+$:

$$\mathcal{W}(x_1, x_2) \leq \frac{1}{2}(\mathcal{W}(x_1, x_1) + \mathcal{W}(x_2, x_2)) \text{ for any } x_1, x_2$$

which follows inequality (11) and $\mathcal{W}(x_1, x_2) = \mathcal{W}(x_2, x_1)$.

Reasoning by induction we assume the statement is true up to $(n-1)$ agents. We fix a n-person supermodular function $\mathcal{W}$ and a profile $x \in \mathcal{X}^{[n]}$. For all i the unanimity function of the supermodular $(n-1)$-function $\mathcal{W}(\cdot; x_i)$ is an upper guarantee, therefore

$$\{\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{j\in[n]\setminus\{i\}} \mathcal{W}(x_i; \overset{n-1}{x_j}) \text{ for all } i\} \Longrightarrow n\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{(i,j)\in P} \mathcal{W}(x_i; \overset{n-1}{x_j}) \tag{12}$$

where P is the set of ordered pairs (i, j) in $[n]$. Fix now a pair i, j and use again the inductive assumption for $\mathcal{W}(\cdot; x_j)$ at the $(n-1)$-profile $(x_i, \overset{n-2}{x_j})$:

$$\mathcal{W}(x_i; \overset{n-1}{x_j}) \leq \frac{1}{n-1}((n-2)\mathcal{W}(\overset{n}{x_j}) + \mathcal{W}(x_j; \overset{n-1}{x_i}))$$

Summing up both sides over $(i,j) \in P$ and writing S for the summation in the RH inequality of (12) gives

$$S \leq (n-2)\sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j}) + \frac{1}{n-1}S \Longrightarrow S \leq (n-1)\sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j})$$

Combining the RH of (12) with the latter inequality concludes the proof. ■

We saw in Lemma 3.5 (statement *iii*) that for a fully general function $\mathcal{W}$, modular or not, and for every type x_i there is a tight lower guarantee g^- touching the unanimity function at x_i: $g^-(x_i) = una(x_i)$ (and the same is true for at least one tight upper guarantee).

Our next result says that if $\mathcal{W}$ is strictly supermodular a tight lower guarantee can only meet the unanimity function at a single type (or not at all). Combining these two facts we see that if $\mathcal{W}$ is strictly modular there is for each type x_i a different tight guarantee g on the other side of unanimity touching una at x_i: in particular there is a continuum of such guarantees.

In Example 2.1 the tight upper guarantees are parametrised by the type where they touch the unanimity function. But for a general strictly modular function $\mathcal{W}$ we expect for each type a large continuum of tight guarantees touching una at that type, and many more never touching una: this will be clear from Proposition 5.2 and 7.1 below.

Lemma 4.1 *If* $\mathcal{W}$ is *strictly modular then a tight guarantee g on the other side of the unanimity one touches its graph in at most one type: the equation $g(x_i) = una(x_i)$ has at most one solution.*

Proof in section 9.3.

4.2 two canonical Stand Alone guarantees

On the other side of the unanimity we discover two Stand Alone guarantees (Definition 3.3) already mentioned in section 1.1 and the literature review.

Definition 4.2 *The two canonical guees g_L, g_H are*

$$\begin{aligned} g_L(x_i) &= \mathcal{W}(x_i; \overset{n-1}{L}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{L}) \\ g_H(x_i) &= \mathcal{W}(x_i; \overset{n-1}{H}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{H}) \end{aligned} \qquad (13)$$

In Example 2.1 g_L^+ and g_H^+ they are the end points of $\mathcal{G}^+$. They keep this role in all modular problems.

Theorem 4.1 *Fix a supermodular function* $\mathcal{W}$.
i) Then g_L *and* g_H *are tight lower guarantees of* $\mathcal{W}$.
ii) The property $g_L(L) = una(L)$ *characterises* g_L *in* $\mathcal{G}^-$*; so does* $g_H(H) = una(H)$ *for* g_H.
iii) Any tight lower guarantee g^- *grows slower than* g_H *and faster than* g_L: for any $x < y$

$$g_L(y) - g_L(x) \leq g^-(y) - g^-(x) \leq g_H(y) - g_H(x)$$

iv) Any tight lower guee g^- *starts (at* L*) above* g_H *and below* g_L*, and ends (at* H*) above* g_L *and below* g_H

$$g_H(L) \leq g^-(L) \leq g_L(L) = una(L) \tag{14}$$

$$g_L(H) \leq g^-(H) \leq g_H(H) = una(H)$$

Proof in section 9.4.

Clearly if $\mathcal{W}$ is a supermodular *surplus*, g_L favors the types x_i close to L who get a share close to their best case $una(x_i)$, and g_H favors those close to H. These comments are inverted if $\mathcal{W}(x)$ is a cost or if $\mathcal{W}$ is a submodular surplus.

Whether $\mathcal{W}$ is super or submodular, statement *iii*) implies that the spread $\Delta(g) = g(H) - g(L)$ is smallest at one of g_L, g_H and largest at the other, over all tight guees on the other side of unanimity. For instance in Example 2.1 $\Delta(g_p^+) = H - p$ varies from H for g_0^+ to 0 at g_H^+. It is easy to check that the spread of our two canonical guarantees is another way to characterise them.[6]

4.3 implementing the Stand Alone guarantees: the serial rules

We adapt to our model these well known sharing rules, originally introduced for the commons problem with substitutable inputs ([22], [29]) discussed in the next section.

Definition 4.4 *The increasing Serial sharing rule (Ser↑)* $\varphi^{ser\uparrow}$ *is defined by the combination of two properties a) it is symmetric in its variables and b) the share of agent* i *with type* x_i *is independent of other agents' larger shares.*[7]

When the agents are labelled by increasing types as $x_1 \leq x_2 \leq \cdots \leq x_n$ *agent* i*'s share is:*

$$\varphi_i^{ser\uparrow}(x) = \frac{\mathcal{W}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})}{n-i+1} - \sum_{j=1}^{i-1} \frac{\mathcal{W}(x_1, \cdots, x_{j-1}, \overset{n-j+1}{x_j})}{(n-j+1)(n-j)} \tag{15}$$

We omit this computation for brevity: see the details in ([21]) where this is equation (6).

[6] For instance if $\mathcal{W}$ is supermodular and $\Delta(g) = \Delta(g_L)$ for some $g \in \mathcal{G}^-$, we sum up $g(x) - g^-(L) \geq g_L(x) - g_L(L)$ with $g(H) - g(x) \geq g_L(H) - g_L(x)$ to get an equality and conclude $g = g_L$.

[7] The share $\varphi_i(x)$ does not change if agent j's type changes from x_j to x'_j both weakly larger than x_i.

The decreasing Serial rule Ser↓ is defined symmetrically by property a) and b)*: agent i' share is independent of other agents' smaller shares. It is given by the same expression (15) if we label the agents by decreasing types.

In Example 2.1 the Ser↑ rule is proposed first in [LO] and interpreted as the Shapley value. The SER↓ rule reduces to the Equal Split rule.

Proposition 4.2 *Fix a supermodular function $\mathcal{W}$ in $[L,H]^{[n]}$.*
The SER↑ rule implements the pair of guarantees (g_L, una). The Ser↓ rule implements the pair (g_H, una).

The isomorphic statement for submodular functions exchanges the g_L and g_H guarantees.

Proof in section 9.5.

Dismissing the Proportional, Equal Split and Shapley rules These two familiar rules, $\varphi_i^{pro}(x) = \frac{x_i}{x_N}\mathcal{W}(x)$ and $\varphi_i^{es}(x) = \frac{1}{n}\mathcal{W}(x)$, make sense in particular when $\mathcal{X} \subset \mathbb{R}_+$ and $\mathcal{W}$ increases in x. They cannot play a general role in our approach to fair division because they do not point to the unanimity shares when the function $\mathcal{W}$ is separably additive. Moreover the guarantees they generate are often not tight: Example 2.1 is a spectacular exception where they actually implement respectively the pairs (una, g_L) and (una, g_H).

In the commons problem $\mathcal{W}(x) = F(x_N)$ to which we now turn the Proportional rule is popular. But we noted in subsection 1.2 that it violates the unanimity guarantee, whether the problem is super or submodular. Lemma 9.1 in section 9.6 explains that on the other side it implements g_L but only if $L = 0$. It also shows the same failings for the Shapley value division rule. That the Equal Split rule fails both lower and upper tightness tests is clear.

5 Substitutable inputs

Definition 5.1 *A commons problem is defined by the domain of types $[L,H] \subset \mathbb{R}$ and $\mathcal{W}(x) = F(x_N)$. It is super (resp sub) modular if and only if F is convex (resp concave).*

We do not require F to be monotonic or $[L,H]$ to be non negative. This extends the usual context of a production function in common property where types are input contributions or output demands (see references in subsection 1.3). For instance the function F can be single-peaked if $F(x_N)$ is the profit of n suppliers acting as a monopolist. We also take advantage of the invariance properties in Lemma 3.4 to solve problems with complementary inputs (Example 5.1) and to propose ways to divide the cost or benefit of the variance of types located along a line (Example 5.3).

The next two subsections introduce two new families of tight guarantees for modular commons problems.

5.1 the n simple guarantees

They are a sequence of simple guarantees (Definition 3.3) progressively compromising from one Stand Alone guarantee to the other.

Proposition 5.1 *Fix a n-person commons problem $([L,H],F)$ and consider the functions $g_{\ell,h}$ where ℓ,h are integers s. t. $0 \leq \ell,h \leq n-1$ and $\ell+h=n-1$:*

$$g_{\ell,h}(x_i) = F(x_i + (\ell L + hH)) - \frac{1}{n}\{\ell F((\ell+1)L + hH) + hF(\ell L + (h+1)H)\} \tag{16}$$

so that $g_{n-1,0} = g_L$ and $g_{0,n-1} = g_H$.

i) If F is convex (resp concave) $g_{\ell,h}$ is a simple tight lower (resp upper) guarantee.

ii) If F is strictly convex (resp concave) the $n-2$ guarantees other than g_L and g_H do not touch the unanimity one (their graphs do not intersect). The gap $una(x_i) - g_{\ell,h}(x_i)$ (resp $g_{\ell,h}(x_i) - una(x_i)$) is minimal at the type $x_i = \frac{\ell}{n-1}L + \frac{h}{n-1}H$

Proof in section 9.7.

A rephrasing of statement *ii)* is that the guarantees $g_{\ell,h}$ are not Stand Alone ones in the sense of Definition 3.3.

There is perhaps a sequence of natural division rule between Ser↑ and Ser↓ to implement the $n-2$ lower guarantees between g_L and g_H. Short of discovering one we must use the ready made interpolation rules explained after Lemma 3.7.

Example 5.1 *Complementary inputs*

A project wil return one unit of surplus if and only if all agents complete their own part in full. Agent i's input $x_i \in [L,H]$ is the probability that i is successful, and $0 < L < H \leq 1$. The agents share the expected return

$$\mathcal{W}(x) = x_1 x_2 \cdots x_n \text{ for } x \in [L,H]^{[n]}$$

The function $\mathcal{W}$ is supermodular so $una(x_i) = \frac{1}{n}x_i^n$ is the unique tight upper bound on type x_i's share.

The change of variable $x_i = e^{z_i}$ (Lemma 3.4) transforms $\mathcal{W}$ into $\widetilde{\mathcal{W}}(z) = e^{z_N}$ to which we apply Proposition 5.1 then write the guarantees $\widetilde{g}_{\ell,h}$ in terms of the original problem:

$$g_{\ell,h}(x_i) = L^\ell H^h (x_i - \frac{1}{n}(\ell L + hH))$$

So our n simple guarantees are linear in type.

Here $g_L(x_i) = \frac{1}{n}L^n + L^{n-1}(x_i - L)$ helps an agent supplying minimal effort as much as possible (as $g_L(L) = una(L)$) and gives a little bit more for any "voluntary" larger effort. At the other end $g_H(x_i) = \frac{1}{n}H^n - H^{n-1}(H - x_i)$ expects maximal effort and rewards it as if everyone else does the same. The minimal share of a "slacker" falls rapidly with his effort, and become negative if $x_i < \frac{n-1}{n}H$: this tax is needed to subsidize the hard working agent if she is the only one! The $n-2$

other guarantees $g_{\ell,h}$ allow the manager to adjust, along a grid increasingly fine as n grows, the critical effort level $\frac{1}{n}(\ell L + hH)$ guaranteeing a positive share of output (slightly smaller than the level where the gap between the upper and lower bounds is minimal).

5.2 the tangent and hybrid guarantees

If the general function $\mathcal{W}$ is globally convex and differentiable in $[L,H]^{[n]}$ the tangent θ_a at any point $(a, una(a))$ of its unanimity graph

$$\theta_a(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{a}) + \partial_1\mathcal{W}(\overset{n}{a})(x_i - a)$$

defines a feasible but not necessarily tight lower guarantee, $\theta_a \in \mathbf{G}^-$. Indeed for any $x_i \in [L,H]$ the LH of inequalities (4) is

$$\sum_{[n]}\theta_a(x_i) = \mathcal{W}(\overset{n}{a}) + \partial_1\mathcal{W}(\overset{n}{a})(x_N - na)) \leq \mathcal{W}(x)$$

precisely the tangent hyperplane inequality of $\mathcal{W}$ at $(\overset{n}{a})$ because $\mathcal{W}$ is symmetric.

For $\mathcal{W}(x) = F(x_N)$ with F convex we find that most tangents to the unanimity graph are *tight* lower guarantees: those touching that graph inside the subinterval of $[L,H]$ left after deleting $\frac{1}{n}$-th of the interval at each end. And on the deleted left and right interval, concatenating a tangent with the translated of one of our two Stand Alone guarantees completes the continuous line of tight guarantees connecting g_L to g_H

In the commons model F is either convex or concave so has well defined left and right derivatives. At a point z where they differ, we write $\frac{dF}{dx}(z)$ the corresponding closed interval and $\Theta_a(x_i) = \frac{1}{n}F(na) + \frac{dF}{dx}(na)(x_i - a)$ for the set of tangent(s) to the unanimity graph at na, with generic element θ_a: $\theta_a(x_i) = \frac{1}{n}F(na) + \partial\theta_a \times (x_i - a)$ where $\partial\theta_a \in \frac{dF}{dx}(na)$.

Proposition 5.2: *If F is convex in $[nL, nH]$ the supermodular commons $\mathcal{W}(x) = F(x_N)$ admits the following tight lower guarantees g_a, where $a \in [L,H]$:*

Case 1: $\frac{n-1}{n}L + \frac{1}{n}H \leq a \leq \frac{1}{n}L + \frac{n-1}{n}H$ *then* $g_a = \theta_a \in \Theta_a$ *in* $[L,H]$;

Case 2: $L \leq a \leq \frac{n-1}{n}L + \frac{1}{n}H$ *then*

$$g_a = \theta_a \in \Theta_a \text{ in } [L, na-(n-1)L] \text{ ; } g_a = g_L + (n-1)(una(L) - \theta_a(L)) \text{ in } [na-(n-1)L, H]]$$

Case 3: $\frac{1}{n}L + \frac{n-1}{n}H \leq a \leq H$ then

$$g_a = g_H + (n-1)(una(H) - \theta_a(H)) \text{ in } [L, na-(n-1)H] \text{ ; } g_a = \theta_a \in \Theta_a \text{ in } [na-(n-1)H, H]$$

If F is concave in $[nL, nH]$ the same family of $g_a, a \in [L,H]$, are tight upper guarantees of $\mathcal{W}$.

Proof in section 9.8.

An important remark: at a given contact point on the graph of the unanimity function we have many other (a large infinity in fact) of tight guarantees. This is clear from the discussion of Theorem 7.1 two sections below.

5.3 more examples

Example 5.2 *Sharing the cost of a public bad*

Each agent can engage in a potentially polluting activity at a level x_i in $\mathcal{X} = [0, 2d]$. The cleaning cost is $F(x_N) = (x_N - nd)_+$: total activity x_N below nd is costless, but requires cleaning at price 1 beyond this threshold. This cost function is convex.

The unanimity cost share $una(x_i) = (x_i - d)_+$ is the single tight upper guarantee (worst cost share) for the different types. A "clean" type $x_i \leq d$ will never pay (but could be paid): the cost can only be shared by the dirty agents. The latter agent may pay up to the cost of his pollution in excess of the threshold d.

The Stand Alone guarantee g_L is $g_L(x_i) \equiv 0$. A clean agent never pays anything (because $g_L = una$ in $[0, d]$) but cannot be compensated either.

The other Stand Alone $g_H(x_i) = x_i - d$ takes a radically different viewpoint: a dirty agent i pays his worst unanimity cost for sure ($g_H = una$ in $[d, 2d]$) even if the other agents are so clean that total pollution incurs no cost; in that case the tax $x_i - d$ will go the the clean agents; a zero polluter can receive as much as d in cash. Under the guarantee g_H a clean agent j is effectively selling her unused pollution credit $d - x_j$ to "absorb" the pollution of the dirty ones.

There is only one other simple guarantee and only if $n = 2m + 1$ is odd[8], striking a reasonable compromise between the extremists g_L and g_H:

$$g_{m.m}(x_i) = (x_i - d)_+ - \frac{m}{n} d \simeq (x_i - d)_+ - \frac{1}{2} d$$

Here an agent will pay something for sure only if his pollution is 50% larger than the threshold d, and even a very dirty type can expect a rebate of $\frac{1}{2}d$ if the others are sufficiently clean. At the other end the superclean agent, $x_i = 0$, cannot be rewarded more than $\frac{1}{2}d$.

By contrast the infinitely many tangents at the kink $(d, 0)$ of the unanimity graph are tight lower guarantees, and in this case do not need to be adjusted near the endpoints of $\mathcal{X}$:[9]

$$g_\lambda(x_i) = \lambda(x_i - d) \text{ for } 0 \leq \lambda \leq$$

Now dirtiness is (at least) taxed at the fixed rate λ and the same rate applies to the subsidy of clean types.

[8] This is because the cost function is two-piece linear.

[9] Case 2 in Proposition 5.2 gives g_L for a in the corresponding neighborhood of 0, and case 3 gives similarly g_H.

Keep in mind a consequence of Theorem 7.1 two sections below: the lower tight guarantees in Propositions 5.1, 5.2 far from exhaust the whole menu.

Example 5.3 *Sharing the cost of the variance*

Agents choose a type x_i in $[0,1]$ and must share (n times) the variance of their distribution:

$$\mathcal{W}(x) = \sum_{[n]} x_i^2 - \frac{1}{n}(\sum_{[n]} x_i)^2 \tag{17}$$

In a cost sharing interpretation agent i's deviation from the mean $\frac{1}{n}x_N$ has a quadratic cost $(x_i - \frac{1}{n}x_N)^2$ and $\mathcal{W}(x)$ is the sum of these costs. For instance given the profile of locations x_i the sum of quadratic travel costs to a facility located at the mean (to minimise this sum) is precisely (17)

Cash compensations for agents incurring a large travel cost are feasible. The cost function $\mathcal{W}$ is submodular and $una(x_i) \equiv 0$. The unanimity test limits these transfers to each type x_i: they should never exceeed i's travel cost to the facility: no one should end up with a net subsidy from the others.

The natural cost sharing rule $\varphi_i(x) = (x_i - \frac{1}{n}x_N)^2$ rules out cash compensations. It meets the unanimity lower bound but the upper bound (worst case) it assigns to type x_i is too high:. Its easy to check that it implements the upper guarantee:

$$\max_{x_{-i}} \varphi_i(x_i, x_{-i}) = \frac{n-1}{n}(x_i - 1)^2 \text{ if } x_i \leq \frac{1}{2} \ ; \ = \frac{n-1}{n}x_i^2 \text{ if } x_i \geq \frac{1}{2}$$

This is the maximum of the upper guarantees $g_L(x_i) = \frac{n-1}{n}x_i^2$ and $g_H(x_i) = \frac{n-1}{n}(x_i - 1)^2$, and is strictly larger everywhere than the tight upper guarantee $g_{\frac{1}{2}}(x_i) = (x_i - \frac{1}{2})^2$ discussed below.

To compute the tight guarantees identified in Propositions 5.1 and 5.2 we apply the invariance Lemma 3.4. By statement *i*) there and a change of sign, every tight upper guarantee g^+ of $\mathcal{W}$ obtains from a tight lower guarantee g^* of $\mathcal{W}^*(x) = (x_N)^2$ as $g^+(x_i) = x_i^2 - \frac{1}{n}g^*(x_i)$.

The n simple upper guarantees in Proposition 5.1 are indexed as follows by the integer h from 0 to $n-1$:

$$g_h^+(x_i) = \frac{n-1}{n}(x_i - \frac{h}{n-1})^2 + \delta_h$$

where $\delta_h = \frac{h(n-1-h)}{n^2(n-1)}$.

So g_h^+ focuses on the location $\frac{h}{n-1}$ where the the worst cost share δ_h is small because $\delta_h \leq \frac{1}{4n}$ for all h. Contrast this with the two Stand Alone guarantees g_L and g_H that do not charge anything to their respective benchmark types 0 and 1.

The tangent lower guarantees of $\mathcal{W}^*$ (case 1 in Proposition 5.2) are $g_a^*(x_i) = na(2x_i - a)$: they correspond to the tight upper guarantees $g_a^+(x_i) = (x_i - a)^2$ of $\mathcal{W}$ for $a \in [\frac{1}{n}, \frac{n-1}{n}]$: the location

a is free for a type a, and the worst cost share at other locations is exactly the travel cost to the benchmark. But if a is L or H the guarantees $g(x_i) = x_i^2$ and $(1-x_i)^2$ are dominated by g_L and g_H.

Note that if $a \simeq \frac{h}{n-1}$ the guarantees g_a^+ and g_h^+ are similar: g_h^+ is $\frac{n-1}{n}$flater than g_a^+ and smaller at 0 and 1, but unlike g_a^+, it never vanishes.

6 Rank separable functions

In this large class of functions $\mathcal{W}$ the tight guarantees are all simple and parametrised by a $(n-1)$-profile c in $[L,H]^{[n-1]}$.

The *decreasing order statistics* of the profile $x \in [L,H]^{[n]}$ is written $(x^k)_{k=1}^n$ with $x^1 = \max_i\{x_i\}$ and $x^n = \min_i\{x_i\}$. The statement "x_i is of rank k in profile x" is unambiguous if x_i is different from every other coordinate; otherwise we mean that x_i appears at rank k for some weakly increasing ordering of the coordinates of x.

Definition 6.1 *The function $\mathcal{W}$ on $[L,H]^{[n]}$ is rank-separable if there exist n continuous real valued functions w_k on $[L,H]$ s. t.*

$$\mathcal{W}(x) = \sum_{k=1}^{n} w_k(x^k) \text{ for all } x \text{ in } [L,H]^{[n]} \tag{18}$$

An equivalent definition of rank separability phrased as a functional equation in the unknown $\mathcal{W}$ makes clear where the functions w_k come from:

$$\mathcal{W}(x) = \sum_{k=1}^{n} \mathcal{W}(\overset{k-1}{H}, x^k \overset{n-k}{L}) \text{ for all } x \text{ in } [L,H]^{[n]}$$

The routine proof of the equivalence is omitted.

Lemma 6.1 *The rank separable function $\mathcal{W}$ is supermodular (resp submodular) if and and only if w_k grows weakly slower (resp faster) than w_{k+1} in $[L,H]$: $w_k(y) - w_k(z) \leq w_{k+1}(y) - w_{k+1}(z)$ for all $z \leq y$ and $k \in [n-1]$.*

Proof in section 9.9.

The simple rank separable functions $\mathcal{W}(x) = x^1$ and x^n are respectively sub and supermodular. But the other rank function $\mathcal{W}(x) = x^k$, $2 \leq k \leq n-1$, are neither sub- nor supermodular. In Example 6.3 below we describe their tight guarantees, infinitely many of them on both sides of inequality *(4)* because the unanimity function is not any kind of guarantee.

More examples with linear functions w_k follow the characterisation theorem.

Recall the Definition 3.3 of the simple guarantee g_c with universal contact profile $c \in [L,H]^{n-1}$:

$$g_c(x_i) = \mathcal{W}(x_i, c) - \frac{1}{n}(\sum_{\ell=1}^{n-1} \mathcal{W}(c_\ell, c))$$

Theorem 6.1 *Fix a (super or sub) modular function $\mathcal{W}$. The two following statements are equivalent:*
i) The function $\mathcal{W}$ is rank separable.
ii) For all $c \in [L,H]^{n-1}$ the function g_c is a tight simple guarantee of $\mathcal{W}$ on the other side of the unanimity one.

The function $\mathcal{W}$ has no other tight guarantee.

The long proof is in section 9.10.

Recall that in the rank separable Example 2.1 the tight upper guarantees form a uni-dimensional interval. For a general modular and rank separable function the set $\mathcal{G}^\varepsilon$ of tight guarantees on the other side of *una* is of dimension at most $n-1$. This is in sharp contrast with what Theorem 7.1 reveals about *strictly* modular and differentiable functions $\mathcal{W}$ for which the dimension of that set is infinite. The rank separable functions are additive in the open subsets of $[L,H]^{[n]}$ where the ordering of the coordinates is strict and constant, therefore not at all strictly modular.

In the next examples $\mathcal{W}$ is also additive w r t the ordered types, so every tight guarantee is piecewise linear and concave (resp convex) if $\mathcal{W}$ is super (resp sub) modular.

Example 6.1 *Team work and Queuing*

As in Example 2.1 the same supermodular function $\mathcal{W}$ has two very different interpretations:

$$\mathcal{W}(x) = x^1 + 2x^2 + \cdots + nx^n \text{ for } x \in [0,H]^n \tag{19}$$

Story 1: *Team work with increasing returns*

When k agents work as a team their productivity is $\frac{1}{2}k(k+1)$ per hour. Agent i can work for x_i hours, so the maximal output (19) is produced by making everyone work from time 0 to x^n, then all but agent n from time x^n to x^{n-1}, etc.. . How should we divide the output?

Story 2: *Queuing with cash compensation* (**[17], [7], [8]**)

A single server processes n jobs, each job takes one day, and agent i's waiting cost is x_i per day, varying in $[0,H]$. High cost agents are served first to minimise total waiting costs and the optimal cost is also (19). What cash compensations from the impatient agents to the patient ones are fair?

The unanimity upper guarantee is $una(x_i) = \frac{1}{2}(n+1)x_i$ and the first Stand Alone lower guarantee is $g_L(x_i) = x_i$. The tight pair (g_L, una) makes sure that working for a very short time never incurs a penalty in the team story or make a profit in the queuing one. By statement *ii*) in Theorem 4.1 *every* other tight lower guarantee g is s. t. $g(0) < 0$ thus taxing (at some contact profiles) an absentee agent in story 1, and allowing the very patient one in story 2 to receive more than her actual waiting cost.

The substantial literature on the queuing model rules out the option of rewarding a patient agent that much. This makes sense if agents are not responsible for their types: I am not blamed for my impatience but only asked to compensate those more patient than me because I will impose a wait on them. For instance the convincing solution proposed in (**Manq**) make me pay to each agent I displace one half of their cost for waiting one more day.

Our tests of tightness are compatible with this viewpoint, but also with many others treating the impatient agents much more agresively in the queuing story. In the team story they allow to tax the absentees for the benefit of those who work long hours, much like in Example 5.1. The stronger such effect is with the other canonical guarantee $g_H(x_i) = n(x_i - \frac{n-1}{2n}H)$ where the absentee may end up (at some profiles) paying as much as as much as $\frac{1}{2}(n-1)H$.

Included in Theorem 6.1 is for each $c_0 \in [L, H]$ the two-piece linear Stand Alone guarantee with all $n-1$ fixed types at c_0:

$$g_{c_0}(x_i) = \begin{array}{c} n(x_i - \frac{n-1}{2n}c_0) \\ x_i + \frac{n-1}{2}c_0 \end{array} \text{ if } \begin{array}{c} x_i \leq c_0 \\ x_i \geq c_0 \end{array}$$

where it takes a type above approximately $\frac{1}{2}c_0$ to avoid any penalty (story 1) or rule out a profit (story 2).

To make the derivative of a simple guarantee g_c decreases more progressively with x_i from n to 1, we can instead choose the fixed types at $\frac{1}{n}H, \frac{1}{n}H, \cdots, \frac{n-1}{n}H$.

Example 6.2 *Sharing the cost of the spread*

The types x_i vary in the interval $[L, H]$ *of length* Δ, they must share the submodular cost of the spread $\mathcal{W}(x) = x^1 - x^n$. For instance types represent a location and $\mathcal{W}(x)$ is the cost of a road or link connecting the agents. Or type x_i is when agent i will show up for service and they share the cost of staffing the desk so everyone will be served upon arrival. Etc.. This is similar in spirit to Example 5.3 where the cost is the variance of the types, but here we discover *all* the tight guarantees whereas there we found infinitely many but certainly not all of them.

The tight lower guarantee is $una(x_i) \equiv 0$: everyone's best case is to pay nothing, but nobody can make a profit. But should an agent be penalised (pay more than the average) for being at the periphery of the distribution of agents, and if so, by how much?

Like in Example 2.1 the simple Equal Split rule $\varphi_i(x) = \frac{1}{n}(x^1 - x^n)$ delivers a plausible upper guarantee $g^{es}(x_i) = \frac{1}{n}\Delta$, tight if $n \geq 3$.

Fixing any $c \in [L, H]^{n-1}$ it is easy to check that for $n \geq 3$ the simple upper guarantee g_c depends only upon the largest and the smallest parameters c_1 and c_{n-1}. In the safe interval $[c_{n-1}, c_1]$ you pay at most $\frac{1}{n}(c_1 - c_{n-1})$.[10] Write $\delta = c_1 - c_{n-1}$ and $\mu = \frac{1}{2}(c_1 + c_{n-1})$ then

$$g_c(x_i) = \max\{\frac{1}{n}\delta, |x_i - \mu| - \frac{n-2}{2n}\delta\} \text{ for all } x_i \in [L, H]$$

If $\delta = H - L$ this is the Equal Split guarantee and for $\delta = 0$ the Stand Alone one $g_c(x_i) = |x_i - c_1|$, in particular the two canonical guarantees for $c_1 = L$ or H.

The upper bound guarantee rises at speed 1 going down from c_{n-1} or up from c_1: those types could pay, in addition to the base cost share $\frac{1}{n}\delta$, the full connecting cost to the safe interval.

Remark We interpreted Example 5.3 as the division of quadratic travel costs to an optimally located facility. With linear travel cost this become the cost of traveling to the median x^{m+1} of the types (say $n = 2m + 1$) and is computed as $\mathcal{W}(x) = \sum_{k=1}^{m} x^k - \sum_{\ell=m+2}^{n} x^\ell$, another rank separable function.

In our last example $\mathcal{W}$ is not modular but simple enough that we can still describe the two sets of tight guarantees, now infinite on both sides.

Example 6.3 *Production with quota*

Fix n and a quota $q, 2 \leq q \leq n-1$. Agent i inputs the effort x_i. To achieve the output $y = F(z)$ we need at

least q agents contributing an effort at least z:

$$\mathcal{W}_q(x) = F(x^q) \text{ for } x \in [L, H]^{n]} \tag{20}$$

If $q = 1$ this is the submodular Example 2.1, up to a change of variable, and if $q = n$ its supermodular mirror image. For other values of q, $\mathcal{W}_q$ is not modular.

Here $una(x_i) = \frac{1}{n}F(x_i)$ is neither a lower guarantee nor an upper guarantee. There is a one dimensional choice of tight guarantees on both sides of (4). The set $\mathcal{G}_q^+$ is parametrised by $p \in [L, H]$:

$$g_p^+(x_i) = \frac{1}{n}F(p) + \frac{1}{q}(F(x_i) - F(p))_+$$

and $\mathcal{G}_q^-$ is similarly parametrised by $p^* \in [L, H]$:

$$g_{p^*}(x_i) = \frac{1}{n}F(p^*) + \frac{1}{n-q+1}(F(x_i) - F(p^*))_-$$

The proof, omitted for brevity, mimicks that of Proposition 2.1.

If $p = p^*$ this "standard" level of effort guarantees the share $\frac{1}{n}F(p)$. If the actual input x^q is below p the "slackers" inputting a sub-standard effort get on average less than $\frac{1}{n}F(p)$ if there are

[10] For $n = 2$ there is a single parameter c_0 and the corresponding tight guarantee is $g_{c_0}(x_i) = |x_i - c_0|$. The Equal Split guarantee is not tight in this case only.

some hard working agents who get at least $\frac{1}{n}F(x^q)$. Symetrically if x^q is above p^* the "slackers" cannot get more than the standard share $\frac{1}{n}F(p)$, and may get less if more than q agents input x_i larger than z^*.

7 Two person strictly modular problems

A commons problem (section 5) is strictly modular if F is strictly convex or strictly concave. But no rank separable function is (as explained just after Theorem 6.1).

The key to our characterisation result is a precise description, in the next two Lemmas, of the contact set of tight guarantees on the other side of *una*. Fixing a modular function $\mathcal{W}(x_1, x_2)$ and a tight guarantee g on either side of (4), we define its contact correspondence γ with range and domain $[L, H]$:

$$\gamma(x_1) = \{x_2 \in [L,H] | g(x_1) + g(x_2) = \mathcal{W}(x_1, x_2)\} \text{ for all } x_1 \in [L,H] \tag{21}$$

It is non empty by Lemma 3.2 and we write its graph as $\Gamma(\gamma)$.

Lemma 7.1 *If $\mathcal{W}$ is supermodular, $g \in \mathcal{G}^-$* and *$\Gamma(\gamma)$ contains (x_1, x_2) and (x_1', x_2') s.t. $(x_1, x_2) \ll (x_1', x_2')$, then $\Gamma(\gamma)$ contains $(x_1, x_2'), (x_1', x_2) \in$ as well, and $\mathcal{W}$ is not strictly supermodular.*

If $\mathcal{W}$ is submodular replace $\mathcal{G}^-$ by $\mathcal{G}^+$.

Proof We sum up the two equalities in (21) for (x_1, x_2) and $(x_1\prime, x_2\prime)$:

$$\mathcal{W}(x_1, x_2) + \mathcal{W}(x_1', x_2') = \{g(x_1) + g(x_2')\} + \{g(x_1') + g(x_2)\} \leq \mathcal{W}(x_1, x_2') + \mathcal{W}(x_1', x_2)$$

Combined with the supermodular inequality (11) this gives an equality and the desired contradiction. In fact $\mathcal{W}$ is then additive inside the rectangle $[x_1, x_1'] \times [x_2, x_2']$. ■

Lemma 7.2 *Fix a strictly supermodular function $\mathcal{W}$ and a tight guarantee $g \in \mathcal{G}^-$ – or a submodular $\mathcal{W}$ and $g \in \mathcal{G}^+$ – with contact correspondence γ.*
i) $\Gamma(\gamma)$ is symmetric: $x_2 \in \gamma(x_1) \Longleftrightarrow x_1 \in \gamma(x_2)$ for all x_1, x_2.
ii) γ is convex valued: $\gamma(x_1) = [\gamma^-(x_1), \gamma^+(x_1)]$, single-valued a.e., and upper-hemi-continuous (its graph is closed).
iii) γ^- and γ^+ are weakly decreasing and $x_1 \leq x_1' \Longrightarrow \gamma^-(x_1) \geq \gamma^+(x_1')$; γ is the u.h.c. closure of both γ^- and *γ^+.*
iv) $\gamma(L)$ contains H and $\gamma(H)$ contains L.
v) γ has a unique fixed point a: $a \in \gamma(a)$, and a is an end-point of $\gamma(a)$.

Proof in Appendix 9.11.

After picking a correspondence γ as just described, the tight guarantee of which γ describe the contact set obtains by integrating the differential equation $\frac{dg}{dx_i}(x_i) = \frac{\partial \mathcal{W}}{\partial x_i}(x_i, \gamma(x_i))$.

Theorem 7.1 *Fix a strictly super (resp. sub) modular function $\mathcal{W}$, continuously differentiable in $[L,H]^2$.*
i) For any correspondence γ as in Lemma 7.2, the following equation

$$g(x_1) = \int_a^{x_1} \partial_1 \mathcal{W}(t, \gamma(t))dt + una(a) \tag{22}$$

defines a tight lower guarantee $g \in \mathcal{G}^-$ (resp. $\mathcal{G}^+$).
ii) Conversely if g is a guarantee in $\mathcal{G}^-$ (resp. $\mathcal{G}^+$) with contact correspondence γ (as in Lemma 7.2) then g takes the form (22).

Corollary: *If $n = 2$ all guarantees on other side of una touch it (are tangent to it if smooth) at a unique point*

So the sets $\mathcal{G}^\pm$ on the other side of unanimity are parametrised by a large set of functions γ.

The proof starts with a differentiability result (Lemma 9.2) in Appendix 9.12 and concludes in Appendix 9.13.

The contact correspondences of the two Stand Alone guarantees g_L, g_H follow respectively the lower and upper edges of the square $[L,H]^2$.[11] A natural compromise between these two follows the anti-diagonal $\gamma(x_1) = L + H - x_1$ of $[L,H]^2$. With the notation $d = \frac{1}{2}(L+H)$ this is

$$g_d(x_i) = una(d) + \int_d^{x_i} \partial_1 \mathcal{W}(t, 2d - t)dt$$

In general, after selecting the type a at which γ crosses the diagonal of $[L,H]^2$, we can pick any decreasing single-valued function $\overline{\gamma}$ from $[L,a]$ into $[a,H]$ mapping L to H and a to itself, then fill the (countably many) jumps down to create the correspondence γ of which the graph connects (L,H) to (a,a), and finally extend γ to $[a,H]$ by symmetry of its graph around the diagonal of $[L,H]^2$.

We illustrate this embarrassement of riches in the commons of section 5.

Example 7.1 *Commons with substitutable inputs*

We have $\mathcal{W}(x) = F(x_1 + x_2)$ and F is strictly concave on $[0,1]$.

Proposition 5.1 has no bite for $n = 2$.

In Proposition 5.2 statement *i*) delivers a unique full tangent guarantee g_d described above.

The contact functions of the guarantees in statements *ii*) and *iii*) are two-piece linear. For instance if $\alpha \in [0, \frac{1}{2}[$: $\gamma_\alpha(0) = [2\alpha, 1]$; $\gamma_\alpha(x_i) = 2\alpha - x_i$ on $]0, 2\alpha]$ and $\gamma_\alpha(x_i) = 0$ on $[2\alpha, 1]$.

To construct new tight upper guarantees connecting g_L and g_H we can fix $\sigma \in [0,1]$ and use the following piecewise constant contact function:

$$\gamma_\sigma \equiv 1 \text{ on } [0,\sigma[\ ;\ \gamma_\sigma(\sigma) = [\sigma, 1]\ ;\ \gamma_\sigma(x_i) = \sigma \text{ on }]\sigma, 1[\ ;\ \gamma_\sigma(1) = [0, \sigma]$$

[11] For instance $\gamma_L(0) = [0,1]$, $\gamma_L(]0,1]) = 0$.

The equation (22) gives

$$g_\sigma(x_i) = \begin{matrix} F(x_i+1) - F(\sigma+1) + \frac{1}{2}F(2\sigma) \\ F(x_i+\sigma) - \frac{1}{2}F(2\sigma) \end{matrix} \text{ if } \begin{matrix} x_i \leq \sigma \\ x_i \geq \sigma \end{matrix}$$

concatenating two different stand alone-like pieces, connected at $x_i = \sigma$ where they touch the unanimity graph. Unlike the g_α-s in Proposition 5.2 (see γ_α above), the connection is not smooth.

Taking the symmetric of γ_σ around the anti-diagonal we find, after similar computations, a second family of non smooth concatenations of stand alone-like pieces parametrised by $\tau \in [0,1]$:

$$g_\tau(x_i) = \begin{matrix} F(x_i+\tau) - \frac{1}{2}F(2\tau) \\ F(x_i) - F(\tau) + \frac{1}{2}F(2\tau) \end{matrix} \text{ if } \begin{matrix} x_i \leq \tau \\ x_i \geq \tau \end{matrix}$$

8 Concluding comments

We start with three open questions.

Q1 The two generalised serial sharing rules (subsection 4.2) implement the unanimity guarantee and the two canonical Stand Alone ones. Can we find similarly natural sharing rules to implement the $n-2$ simple guarantees in Proposition 5.1? Or the tangent guarantees in Case 1 of Proposition 5.2? Or the simple guarantees of rank separable problems?

Q2 *Generalising Theorem 7.1 for $n \geq 3$*

The key for this Theorem is the full description of the contact correspondence of any tight guarantee (Lemmas 7.1, 7.2). We could not gain a similar understanding of this correspondence with three or more agents.

In a *two agent* problem the contact set of *every* tight guarantee g in $\mathcal{G}^\varepsilon$ intersects the diagonal: g touches *una* (Lemma 7.2). This gives the starting point of the integral equation (22). But if $n \geq 3$ in Proposition 5.1 and Theorem 6.1 we found many tight guarantees of which the contact set does not intersect the diagonal.

Q3 *Multi-dimensional types*

All definitions and results of section 3 are preserved if the type space $\mathcal{X}$ is a compact subset of a metric space. An obstacle to further develop the multidimensional analysis is the following very challenging *decentralisation* question.

The following claim is obvious from the definitions and Lemma 3.5. Suppose each type has two components $x_i = (x_i^a, x_i^b) \in \mathcal{X}_a \times \mathcal{X}_b = \mathcal{X}$ and pick two functions $\mathcal{W}_a$ on $\mathcal{X}_a^{[n]}$ and $\mathcal{W}_b$ on $\mathcal{X}_b^{[n]}$. If g_a^+ and g_b^+ are two tight guarantees of, respectively, $\mathcal{W}_a$ and $\mathcal{W}_b$, then $g_a^+ + g_b^+$ is clearly a tight guarantee of their "sum" $\mathcal{W}(x) = \mathcal{W}_a(x^a) + \mathcal{W}_b(x^b)$ on the domain $\mathcal{X}$.

We do not know for which domain of functions $\mathcal{W}$ the converse decentralisation property holds: *every tight guarantee g of $\mathcal{W}_a + \mathcal{W}_b$ (two functions in the domain) is the sum of two tight guarantees in the component problems.*

The answer eludes us even for the simple problem of assigning more than one indivisible object and cash transfers when utilities over objects are additive : the corresponding function $\mathcal{W}$ is the sum of problems $\mathcal{W}_a(x^a) = \max_{i\in[n]}\{x_i^a\}$ over several objects a. With much sweat we showed that the decentralisation property holds for two agents and two objects![12]

relation to optimal transport The tight guarantees g^- and g^+ to a given symmetric function $\mathcal{W}$ are its best approximations by symmetric additively separable functions from above and below. There is a formal connection[13] to the celebrated Optimal Transport problem ([36], [13]), specifically to its dual formulation as the Kantorovitch- Rubinstein Lemma:

$$\max_{\Pi:\Pi_i=\lambda_i}\{\int \mathcal{W}(x)d\Pi(x)\} = \min_{g_i:\sum_i g_i(x_i)\geq\mathcal{W}(x)}\{\sum_i \int g_i(x_i)d\lambda_i\}$$

where $\mathcal{W}(x)$ is the abstract transport cost, and Π the transportation protocol with fixed marginals λ_i over the n coordinates of x.

The symmetry assumption is central to our approach: it restricts the marginals λ_i to be identical and the function Π symmetric, which is not the case in a standard Monge transportation problem or the matching models discussed in ([13]). The insights of that literature may still be useful for symmetric fair division problems such as those discussed here.

9 Appendix: missing proofs

9.1 Lemma 3.1, statement *iii*) and Lemma 3.2

Step 1: a tight guarantee is upper-hemi-continuous. We fix $g \in \mathcal{G}^-$ and check that it is u.h.c.. If it is not, there is in $\mathcal{X}$ some x_1, a sequence $\{x_1^t\}$ converging to x_1, and some $\delta > 0$ such that $g(x_1^t) \geq g(x_1) + \delta$ for all t. Then we have, for any $x_{-1} \in \mathcal{X}^{[n-1]}$

$$\mathcal{W}(x_1^t, x_{-1}) \geq g(x_1^t) + \sum_{i=2}^n g(x_i) \geq (g(x_1) + \delta) + \sum_{i=2}^n g(x_i)$$

Taking the limit in t of $\mathcal{W}(x_1^t, x_{-1})$ and ignoring the middle term we see that we can increase g at x_1 without violating (4), a contradiction of our assumption $g \in \mathcal{G}^-$.

Step 2: Lemma 3.2. "If" is clear. For "only if" we fix $g \in \mathcal{G}^-$ and show that it meets property (*5*). For any $x_1 \in \mathcal{X}$ define

$$\delta(x_1) = \min_{x_{-1}\in\mathcal{X}^{[n-1]}}\{\mathcal{W}(x_1, x_{-1}) - \sum_{[n]} g^-(x_i)\}$$

[12] The proof is available upon request from the authors.

[13] We thank Fedor Sandomirskiy for pointing it out.

and note that this minimum is achieved at some $\overline{x}_{-1}$ because the function $x_{-1} \to \sum_{i=2}^{n} g^{-}(x_i)$ is u.h.c. (step 1). Moreover $\delta(x_1)$ is non negative.

If $\delta(x_1) = 0$ property (5) holds at $\overline{x}_{-1}$. If $\delta(x_1) > 0$ we can increase g at x_1 to $g(x_1) + \delta(x_1)$, everything else equal, to get a guarantee dominating g.

Step 3: a tight guarantee is lower-hemi-continuous. We fix $g \in \mathcal{G}^{-}$ and check that it is l.h.c.. By the continuity of $\mathcal{W}$ and compactness of $\mathcal{X}^{[n]}$ we have:

$$\forall \eta > 0, \exists \theta > 0, \forall x_1, x_1^*, x_{-1} : ||x_1 - x_1^*|| \leq \theta \Rightarrow \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^*, x_{-1}) + \eta$$

If g is not l.h.c. there is some x_1 and $\{x_1^t\}$ converging to x_1 and $\delta > 0$ s.t. $g(x_1^t) \leq g(x_1) - \delta$ for all t. Pick θ for which (14) holds with $\eta = \frac{1}{2}\delta$ and t large enough that $||x_1^t - x_1|| \leq \theta$: then for any x_{-1} we have

$$g(x_1) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^t, x_{-1}) + \frac{1}{2}\delta$$

Replacing $g(x_1)$ with $g(x_1^t) + \delta$ gives $g(x_1^t) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1^t, x_{-1}) - \frac{1}{2}\delta$ for any x_{-1}: this contradicts the contact property (5) for x_1^t.

9.2 statements ii) and iii) in Lemma 3.5

Proof *Statement iii)* Fix $\varepsilon = -$, an arbitrary $\widetilde{x}_1 \in \mathcal{X}$ and write $B(\widetilde{x}_1, r)$ for the closed ball of center $\widetilde{x}_1$ and radius r. Use the notation $\Delta(x) = \sum_1^n una(x_i) - \mathcal{W}(x)$ to define the function

$$\delta(x_1) = \max\{\Delta(x_1, x_{-1}) : \forall i \geq 2, x_i \in B(\widetilde{x}_1, d(x_1, \widetilde{x}_1))\}$$

It is clearly continuous, non negative because $\Delta(x_1, x_{-1}) = 0$ if $x_i = x_1$ for $i \geq 2$, and $\delta(\widetilde{x}_1) = 0$. Define $g = una - \delta$ and check that g is the desired lower guarantee of $\mathcal{W}$. At an arbitrary profile $x = (x_i)_1^n$ choose x_{i^*} s.t. $d(\widetilde{x}_1, x_{i^*})$ is the largest: this implies $\delta(x_{i^*}) \geq \Delta(x)$. Combining this with $\delta(x_i) \geq 0$ for $i \neq i^*$ gives $\sum_1^n \delta(x_i) \geq \Delta(x)$ which, in turn, is the LH inequality in (4) for g. As g is in $\boldsymbol{G}^{-}$, it is dominated by some $\widetilde{g}$ in $\mathcal{G}^{-}$(Lemma 3.1) and $\widetilde{g}(x_1) = una(\widetilde{x}_1)$ by inequality (8).

Second part of statement ii) We assume that $\boldsymbol{G}^{-}$ does not contain una and check that $\boldsymbol{G}^{-}$ is not a singleton. This assumption and the continuity of $\mathcal{W}$ imply that for an open set of profiles $x \in \mathcal{X}^{[n]}$ we have $\sum_{[n]} una(x_i) > \mathcal{W}(x)$. Fix such an x and (by statement i)) pick for each i a tight guarantee g_i equal to una at x_i: these n guarantees are not identical.

9.3 Lemma 4.1

Proof by contradiction. Fix the even $n = 2m$ and (without loss) $\mathcal{W}$ strictly supermodular. For some tight lower guarantee g of $\mathcal{W}$ *we have* $g(x_i) = una(x_i)$ at two types x_1 and x_2 with $x_1 < x_2$. This implies

$$mg(x_1) + mg(x_2) \le \mathcal{W}(\overset{m}{x_1}, \overset{m}{x_2}) \Longrightarrow \mathcal{W}(\overset{n}{x_1}) + \mathcal{W}(\overset{n}{x_2}) \le 2\mathcal{W}(\overset{m}{x_1}, \overset{m}{x_2}) \tag{23}$$

But by repeated application of supermodularity we have

$$\mathcal{W}(\overset{m}{x_2}, \overset{m}{x_2}) - \mathcal{W}(\overset{m}{x_1}, \overset{m}{x_2}) \ge \mathcal{W}(\overset{m}{x_2}, \overset{m}{x_1}) - \mathcal{W}(\overset{m}{x_1}, \overset{m}{x_1})$$

precisely the opposite inequality. This is a contradiction.

The proof for the odd $n = 2m + 1$ is similar. We sum first the two inequalities

$$(m+1)g(x_1) + mg(x_2) \le \mathcal{W}(\overset{m+1}{x_1}, \overset{m}{x_2}) \text{ and } mg(x_1) + (m+1)g(x_2) \le \mathcal{W}(\overset{m}{x_1}, \overset{m+1}{x_2})$$

$$\Longrightarrow \mathcal{W}(\overset{n}{x_1}) + \mathcal{W}(\overset{n}{x_2}) \le \mathcal{W}(\overset{m+1}{x_1}, \overset{m}{x_2}) + \mathcal{W}(\overset{m}{x_1}, \overset{m+1}{x_2})$$

and rewite the latter as

$$\mathcal{W}(\overset{m}{x_2}, \overset{m+1}{x_2}) - \mathcal{W}(\overset{m}{x_1}, \overset{m+1}{x_2}) \le \mathcal{W}(\overset{m}{x_2}, \overset{m+1}{x_1}) - \mathcal{W}(\overset{m}{x_1}, \overset{m+1}{x_1})$$

for another contradiction of strict supermodularity.

9.4 Theorem 4.1

Statement i) Without loss we fix $\mathcal{W}$ supermodular and check that g_L is a feasible lower guarantee. By Lemma 3.6 this implies that it is tight. Consider the following inequality Π_q for $q \in [n]$:

$$\mathcal{W}(x_1, x_2, \cdots; x_q, \overset{n-q}{L}) + \sum_{\ell=q+1}^{n} \mathcal{W}(x_\ell, \overset{n-1}{L}) \le \mathcal{W}(x) + (n-q)\mathcal{W}(\overset{n}{L})$$

Note that Π_n is a tautology and Π_1:

$$\sum_{[n]} \mathcal{W}(x_1; \overset{n-1}{L}) \le \mathcal{W}(x) + (n-1)\mathcal{W}(\overset{n}{L})$$

means that g_L is a lower guarantee (meets the LH of (4)). So it is enough to prove inductively that Π_{q+1} implies Π_q.

Supermodularity implies

$$\mathcal{W}(x_{q+1}, \overset{n-1}{L}) - \mathcal{W}(L, \overset{n-1}{L}) \le \mathcal{W}(x_{q+1}, x_1, \cdots; x_q, \overset{n-q-1}{L}) - \mathcal{W}(L, x_1, \cdots; x_q, \overset{n-q-1}{L})$$

We call D the right hand term of this inequality and increase in the inequality Π_q the left term

$\mathcal{W}(x_{q+1}, \overset{n-1}{L})$ to $D + \mathcal{W}(L, \overset{n-1}{L})$: after rearranging we get the inequality Π_{q+1} as desired.

Statement ii) If the tight guarantee g^- in $\mathcal{G}^-$ satisfies $g^-(L) = \frac{1}{n}(\overset{n}{L})$ we have for all x_i

$$g^-(x_i) + (n-1)g^-(L) \leq \mathcal{W}(x_i; \overset{n-1}{L}) \Longrightarrow g^-(x_i) \leq g_L(x_i)$$

so $g^- = g_L$ because it is tight.

Statement iii) Fix $g^- \in \mathcal{G}^-$ two types x_1, x_1^* s. t. $x_1 < x_1^*$ and a contact profile x_{-1} for g^- at x_1; then apply the opposite of (6) in Lemma 3.3:

$$g^-(x_1^*) - g^-(x_1) \leq \mathcal{W}(x_1^*, x_{-1}) - \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^*, \overset{n-1}{H}) - \mathcal{W}(x_1, \overset{n-1}{H}) = g_H(x_1^*) - g_H(x_1)$$

where the 2d inequality comes from the supermodularity of $\mathcal{W}$. Next pick x_{-1}^* a contact profile for g^- at x_1^* and apply similarly the opposite of (6):

$$g^-(x_1^*) - g^-(x_1) \geq \mathcal{W}(x_1^*, x_{-1}^*) - \mathcal{W}(x_1, x_{-1}^*) \geq g_L(x_1^*) - g_L(x_1)$$

Statement iv) Suppose $g^- \in \mathcal{G}^-$ is s. t. $g^-(L) < g_H(L)$. We combine this inequality with $g^-(x_i) - g^-(L) \leq g_H(x_i) - g_H(L)$ (statement *iii)*) to get that g_H dominates g^-. Now if we have $g^-(L) > g_L(L)$ we combine it with $g^-(x_i) - g^-(L) \geq g_L(x_i) - g_L(L)$ to have g^- dominating g_L. This proves (14), and a similar argument proves the last two inequalities. ■

9.5 Proposition 4.2

We prove the statement for the serial↑ rule (15). By Lemma 3.7 it is enough to check the inequality $g_L(x_i) \leq \varphi_i^{ser\uparrow}(x) \leq una(x_i)$ for all x.

Step 1. We show that $\varphi_i^{ser\uparrow}(x)$ increases (weakly) in all variables x_j such that $x_j \leq x_i$, i. e., for $j \leq i-1$. This generalises Lemma 1 in [21].

If $\mathcal{W}$ is differentiable in $[L, H]^n$ we check this by computing the derivative $\partial_q \varphi_i^{ser\uparrow}$ for $q \leq i-1$ in the LH of equation (15) and using the symmetry of $\mathcal{W}$:

$$\partial_q \varphi_i^{ser\uparrow}(x) = \frac{\partial_q \mathcal{W}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})}{n-i+1} - \frac{\partial_q \mathcal{W}(x_1, \cdots, x_{q-1}, \overset{n-q+1}{x_q})}{n-q} - \sum_{j=q+1}^{i-1} \frac{\partial_q \mathcal{W}(x_1, \cdots, x_{j-1}, \overset{n-j+1}{x_i})}{(n-j+1)(n-j)}$$

Recall that the coordinates of x are weakly increasing. Because $\partial_q \mathcal{W}$ increases weakly in $x_j, j \neq q$, the numerator of each negative fraction is not larger than that of the first fraction. The identity $\frac{1}{n-i+1} = \frac{1}{n-q} + \sum_{j=q+1}^{i-1} \frac{1}{(n-j+1)(n-j)}$ concludes the proof.

Without the differentiability assumption the only step that requires an additional argument is the following consequence of supermodularity: as the coordinates of x increase weakly the term $\mathcal{W}(x) - \frac{1}{n-q+1}\mathcal{W}(x_1, \cdots, x_{q-1}, \overset{n-q+1}{x_q})$ increases weakly in x_q for each $q \leq n-1$. We omit the details.

Step 2. By construction of $\varphi^{ser\uparrow}$ we have $\varphi_i^{ser\uparrow}(x) = \varphi_i^{ser\uparrow}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})$ and by Step 1 it is enough to check that $g_L(x_i)$ lower bounds $\varphi_i^{ser\uparrow}(x)$ at the profile $(\overset{i-1}{L}, \overset{n-i+1}{x_i})$ while *una* upper bounds it at $(\overset{n}{x_i})$. The latter follows from $\varphi_i^{ser\uparrow}(\overset{n}{x_i}) = una(x_i)$.

Applying (15) we see that $g_L \leq \varphi_i^{ser\uparrow}$ reduces to

$$\mathcal{W}(\overset{n-1}{L}, x_i) \leq \frac{1}{n-i+1}\mathcal{W}(\overset{i-1}{L}, \overset{n-i+1}{x_i}) + \frac{n-i}{n-i+1}\mathcal{W}(\overset{n}{L})$$

$$\Longleftrightarrow (n-i)(\mathcal{W}(\overset{n-1}{L}, x_i) - \mathcal{W}(\overset{n}{L})) \leq \mathcal{W}(\overset{i-1}{L}, \overset{n-i+1}{x_i}) - \mathcal{W}(\overset{n-1}{L}, x_i)$$

Finally we apply supermodularity to successively lower bound $\mathcal{W}(\overset{q}{L}, \overset{n-q}{x_i}) - \mathcal{W}(\overset{q+1}{L}, \overset{n-q-1}{x_i})$ by $\mathcal{W}(\overset{n-1}{L}, x_i) - \mathcal{W}(\overset{n}{L})$ for $q = (n-2), \cdots, (i-1)$ and sum up these inequalities.

9.6 The Average Returns and Shapley rules in the commons problem

While Proposition 4.2 shows that the two Serial sharing rules implement the two incremental guarantees for any modular function $\mathcal{W}$, we check that in the commons problem of section 5 the guarantees implemented by the Average Returns and Shapley value rules are not tight from either above or below.

We fix F strictly concave on $\mathbb{R}_+$ and such that $F(0) = 0$, and the set of types $[L, H]$ s. t. $0 \leq L < H$. So $\mathcal{W}$ is submodular.

Average Returns (AR): $\varphi_i^{ar}(x) = x_i AF(x_N)$, with the notation $AF(z) = \frac{F(z)}{z}$;[14]

Shapley value (Sha): $\varphi_i^{Sha}(x) = \mathbb{E}_S(F(x_i + x_S) - F(x_S))$, where the expectation is over the set S of agents preceding i, when the ordering of agents is uniformly distributed.

Lemma 9.1 *For the Average Returns and Shapley rules on* $[L, H]$:

$$g_{AR}^{-}(x_i), g_{Sha}^{-}(x_i) < una(x_i) = \frac{1}{n}F(nx_i)$$

for $x_i \in [L, H[$, *with equality at* H.

[14] At the profile $(\overset{n}{0})$ the definition needs adjusting, e. g. to equal split, but this does not affect the computations of worst and best cases.

If $L=0$ *we have* $g^+_{AR}=g^+_{Sha}=g_L$*. If* $L>0$ *we have*

$$g^+_{AR}(x_i), g^+_{Sha}(x_i) > g_L(x_i) = F(x_i+(n-1)L) - \frac{n-1}{n}F(nL)$$

for $x_i \in]L,H]$ *with equality at* L*.*

Proof For the AR rule the average return AF decreases strictly so that $g^-_{AR}(x_i) = x_i AF(x_i + (n-1)H) < x_i AF(nx_i)$ on $[0,H[$. Similarly on $[L,H]$ we have $g^+_{AR}(x_i) = F(x_i+(n-1)L) = g_L(x_i) + \frac{n-1}{n}F(nL)$ so that g^+_{AR} is only tight if $L=0$ and in that case it is g_L.

For the Shapley rule the strict concavity of F implies, for $x_i < H$

$$\begin{aligned} g^-_{Sha}(x_i) &= \frac{1}{n}\sum_{q=0}^{n-1}(F(x_i+qH)-F(qH)) \\ &< \frac{1}{n}\sum_{q=0}^{n-1}(F(x_i+qx_i)-F(qx_i)) = \frac{1}{n}F(nx) \end{aligned}$$

$$g^+_{Sha}(x_i) = \frac{1}{n}\sum_{q=0}^{n-1}(F(x_i+qL)-F(qL))$$

If $L=0$ this gives $g^+_{Sha}=g_L$. If $L>0$ and $x_i>L$ we sum up, for $1\le q\le n-1$, the inequalities

$$F(x_i+(q-1)L)-F(qL) > F(x_i+(n-1)L)-F(nL)$$

$$\Longrightarrow g^+_{Sha}(x_i) > \frac{n-1}{n}(F(x_i+(n-1)L)-F(nL)) + \frac{1}{n}F(x_i+(n-1)L) = g_L(x_i)$$

9.7 Proposition 5.1

We assume without loss that F is convex.

Step 1 The function $g_{\ell,h}$ *defined by (16) is a lower guarantee:* $g_{\ell,h}\in \mathbf{G}^-$. By Lemma 3.6 this implies that $g_{\ell,h}$ is tight.

We set $Z=\ell L+hH$ for easier reading. The feasibility inequality (4) applied to $g_{\ell,h}$ reads

$$\sum_{[n]} F(x_i+Z) \le F(x_N) + \ell F(Z+L) + hF(Z+H) \text{ for } x\in[L,H]^{[n]} \tag{24}$$

We proceed by induction on n. There is nothing to prove if $n=2$. For $n=3$ we already know that $g_{2,0}$ and $g_{0,2}$ are in $\mathcal{G}^-$; for $g_{1,1}$ the inequality (24) is

$$\sum_{i=1}^{3} F(x_i + L + H) \leq F(x_{123}) + F(2L + H) + F(L + 2H) \tag{25}$$

Suppose $x_{12} \geq L + H$: then the convexity of F implies

$$F(x_3 + L + H) - F(2L + H) \leq F(x_{123}) - F(x_{12} + L)$$

Replacing $F(x_3 + L + H)$ in (25) by this upper bound and rearranging gives a more demanding inequality

$$F(x_1 + L + H) + F(x_2 + L + H) \leq F(x_{12} + L) + F(L + 2H)$$

following again from the convexity of F. So we are done if $x_{ij} \geq L + H$ for any pair i, j.

Suppose next $x_{ij} \leq L + H$ for all three pairs. Then we have for $i = 1, 2, 3$

$$x_{123}, 2L + H \leq x_i + L + H \leq L + 2H$$

and the uniform distribution on the triple $x_{123}, 2L + H, L + 2H$ is a mean-preserving spread of that on $(x_i + L + H)_{i=1}^{3}$, which proves (25).

For the inductive argument we fix $n \geq 4$ and $g_{\ell,h}$ s. t. $\ell + h = n - 1$ and $\ell \geq 1$. We assume that (24) holds for $n - 1$ agent problems and prove it for (ℓ, h).

Suppose $x_{N \smallsetminus \{n\}} \geq Z$ for some agent labeled n without loss of generality. Then the convexity of F implies

$$F(x_n + Z) - F(Z + L) \leq F(x_N) - F(x_{N \smallsetminus \{n\}} + L)$$

As before we replace $F(x_n + Z)$ by this upper bound and rearrange (24) to the more demanding

$$\sum_{[n-1]} F(x_i + Z) \leq F(x_{N \smallsetminus \{n\}} + L) + (\ell - 1)F(Z + L) + hF(Z + H)$$

which for the convex function $\widetilde{F}(y) = F(y + L)$ and $\widetilde{Z} = (\ell - 1)L + hH$ is exactly (24) at x_{-n} for the guarantee $g_{(\ell-1),h}$.

We are left with the case where $x_{N \smallsetminus \{i\}} \leq Z$ for all i for which the different terms under F in (24) are ranked as follows:

$$x_N, Z + L \leq x_i + Z \leq Z + H$$

and the distribution $(\frac{1}{n}, \frac{\ell}{n}, \frac{h}{n})$ on the support $x, Z + L, Z + H$ is a mean-preserving spread of the uniform distribution on the n inputs $x_i + Z$. So $g_{\ell,h}$ meets (24).

If $h \geq 1$ the symmetric proof starts by assuming $x_{N \setminus \{n\}} \leq Z$ and using the convexity inequality

$$F(x_{N \setminus \{n\}} + H) - F(x_N) \leq F(Z + H) - F(x_n + Z)$$

to obtain a more demanding inequality that is in fact (24) for $g_{\ell,h-1}$ and the function $\widehat{F}(y) = F(y + H)$.

Statement ii) The derivative of the gap function is $\frac{dF}{dx}(nx_i) - \frac{dF}{dx}(x_i + Z)$ which changes from negative to positive at $\frac{1}{n-1}Z$ so Z achieves the smallest gap. The equality $g_{\ell,h}(Z) = una(Z)$ is rearranged as$_i$:

$$F(Z) = \frac{1}{n}F(nZ) + \frac{\ell}{n}F(Z + L) + \frac{h}{n}F(Z + H))$$

This contradicts the strict convexity of F if ℓ, h are both positive.

9.8 Proposition 5.2

We assume without loss that F is convex so that $\mathcal{W}$ is supermodular.

Case 1 We already noted that θ_a is in $\mathbf{G}^-$. For tightness we fix a type x_i and look for a vector x_{-i} such that $x_i + x_{N \setminus i} = na$. This implies $\sum_{[n]} \theta_a(x_j) = F(na)$ by the definition of θ_a, so (x_i, x_{-i}) is a contact profile of θ_a at x_i and we are done by Lemma 3.5. The desired vector x_{-i} exists if and only if $x_i + (n-1)L \leq na \leq x_i + (n-1)H$, precisely as we assume.

Case 2 At a profile x where $x_i \leq na - (n-1)L$ for all i, we just saw that $g_a = \theta_a$ meets the LH of (4). We check now this inequality for a profile x where the first t types are above $na - (n-1)L$, $t \geq 1$, and the other $n - t$ types (possibly zero) are below that bound. For $i \leq t$ we can write $g_a(x_i) = F(x_i + (n-1)L) + C_i$ where C_i is a constant w r t x, and similarly $g_a(x_j) = \partial\theta_a \times x_j + C_j$ if $j > t$. Then the desired LH inequality of (4) is

$$\sum_{i \leq t} F(x_i + (n-1)L) + \sum_{j > t} \partial\theta_a \times x_j + C \leq F(x_N) \tag{26}$$

for some constant C.

For $i \leq t$ the difference $F(x_i + x_{N \setminus i}) - F(x_i + (n-1)L)$ is smallest for $x_i = na - (n-1)L$ so it is enough to prove (26) if this is the case. For $j > t$ we check similarly that the difference $\Delta = F(x_j + x_{N \setminus j}) - \partial\theta_a \times x_j$ is smallest if $x_j = L$. Note that $t \geq 1$ implies $x_{N \setminus j} \geq na - (n-1)L + (n-2)L = na - L$, therefore the derivative of $F(x_j + x_{N \setminus j})$ w r t x_j at any $x_j > L$ is weakly larger than $\partial\theta_a$ which proves the claim. So it is enough to prove (26) if $x_j = L$ for $j > t$ and $x_i = na - (n-1)L$ for $i \leq t$. In this case we have $x_N = tna - (t-1)nL$, $x_N \geq na$ and (26) is

$$tg_a(na - (n-1)L) + (n-t)g_a(L) \leq F(tna - (t-1)nL)$$

$$\iff \partial\theta_a \times (t-1)n(a-L) \leq F(tna-(t-1)nL) - F(na)$$

which follows at once from the convexity of F.

Checking tightness. At a type $x_i \leq na-(n-1)L$ we have

$$x_i + (n-1)L \leq na \leq x_i + (n-1)(na-(n-1)L)$$

(replace x_i by L in the last term and rearrange). This implies the existence of a profile x_{-i} entirely inside $[L, na-(n-1)L]$ and s. t. $x_i + x_{N \smallsetminus i} = na$: as in Case 1 this is a contact profile. And at a type $x_i \geq na-(n-1)L$ the definition of g_a shows directly that $(x_i, \overset{n-1}{L})$ is a contact profile.

We omit the symmetric proof of Case 3. ■

9.9 Lemma 6.1

Fix $\mathcal{W}$ defined by (18) and the continuous functions w_q.

Proof of "only if" We assume that $\mathcal{W}$ is supermodular. For any $(n-2)$-profile $x_{-12} \in [L,H]^{[n]\smallsetminus\{1,2\}}$ and any 4-tuple of types s. t. $y_1 < x_1$ and $y_2 < x_2$ this implies

$$\mathcal{W}(x_1,x_2;x_{-12}) - \mathcal{W}(y_1,x_2;x_{-12}) \geq \mathcal{W}(x_1,y_2;x_{-12}) - \mathcal{W}(y_1,y_2;x_{-12}) \tag{27}$$

Fix x_1, y_1 s. t. $L < y_1 < x_1 < H$ and pick any rank q except n. If we pick y_2, x_2 so that $L < y_2 < y_1 < x_1 < x_2 < H$ then we can choose x_{-12} so that x_1 and y_1 are of rank q in the profiles on the RH of (27), whereas after increasing y_2 to x_2 they are of rank $q+1$ in the profiles on the LH. Then (27) amounts to

$$w_{q+1}(x_1) - w_{q+1}(y_1) \geq w_q(x_1) - w_q(y_1)$$

Because $\mathcal{W}$ is continuous this desired inequality holds for any x_1, y_1 s. t. $x_1 \leq y_1$.

Proof of "if" We are given the continuous w_q such that w_q grows weakly slower (resp faster) than w_{q+1} for all $q \leq n-1$. We show that the rank separable function $\mathcal{W}$ given by (18) is supermodular, i. e. we prove (27) for any x, y_1, y_2 s. t. $y_1 \leq x_1$ and $y_2 \leq x_2$. As $\mathcal{W}$ is continuous it is enough to prove it when y_1, y_2 and all the coordinates of x are diffferent.

Next it is without loss to assume that in the jump up from y_i to x_i either the rank of this coordinate does not change, or it goes down but exactly one. Indeed if y_1 is ranked q in $(y_1, x_2; x_{-12})$ and x_1 ranked $q-3$ in $(x_1, x_2; x_{-12})$ we decompose the jump in three small jumps each decreasing the rank by one, and sum up the corresponding inequalities (27). The same argument applies to the jump from y_2 to x_2.

In the profile $(y_1, y_2; x_{-12})$ we call q_i the rank of y_i, so $q_1 > q_2$. For clarity we omit the fixed term x_{-12} in the computations below and define

$$\Delta_1 = \mathcal{W}(x_1,y_2) - \mathcal{W}(y_1,y_2) \; ; \; \Delta_2 = \mathcal{W}(y_1,x_2) - \mathcal{W}(y_1,y_2) \; ; \; \Delta_0 = \mathcal{W}(x_1,x_2) - \mathcal{W}(y_1,y_2)$$

so that (27) is equivalent to $\Delta_0 \geq \Delta_1 + \Delta_2$.

We assume $y_1 < y_2$ without loss and consider first the case $x_1 < y_2$. Then $\Delta_1 = w_{q_1}(x_1) - w_{q_1}(y_1)$ if agent 1's rank does not change in the move from y_1 to x_1, or $\Delta_1 = w_{q_1-1}(x_1) + w_{q_1}(x_j) - w_{q_1-1}(x_j) - w_{q_1}(y_1)$ if agent 1 jumps above agent j of rank $q_1 - 1$ (that cannot be agent 2). Next $\Delta_2 = w_{q_1}(x_2) - w_{q_1}(y_2)$ and we see that $\Delta_0 = \Delta_1 + \Delta_2$.

Assume from now on $x_1 > y_2$ which implies $q_1 = q_2 + 1$, and agent 1's rank at (x_1, y_2) is q_2 (because each rank upgrade is at most one). We distinguish three cases. If $y_1 < y_2 < x_2 < x_1$ then only agents 1 and 2 swap ranks and

$$\Delta_1 = w_{q_2}(x_1) + w_{q_2+1}(y_2) - w_{q_2}(y_2) - w_{q_2+1}(y_1) \; ; \; \Delta_2 = w_{q_2}(x_2) - w_{q_2}(y_2)$$

$$\Delta_0 = w_{q_2}(x_1) + w_{q_2+1}(x_2) - w_{q_2}(y_2) - w_{q_2+1}(y_1)$$

so that $\Delta_0 - \Delta_1 + \Delta_2 = (w_{q_2+1}(x_2) - w_{q_2+1}(y_2)) - (w_{q_2}(x_2) - w_{q_2}(y_2))$ non negative by assumption.

If $y_1 < y_2 < x_1 < x_2$ and agent 2's rank at x_2 is still q_2, then Δ_1 and Δ_2 are just as in the previous case, and

$$\Delta_0 = w_{q_2}(x_2) + w_{q_2+1}(x_1) - w_{q_2}(y_2) - w_{q_{2+1}}(y_1)$$

implies $\Delta_0 - \Delta_1 + \Delta_2 = (w_{q_2+1}(x_1) - w_{q_2+1}(y_2)) - (w_{q_2}(x_1) - w_{q_2}(y_2))$ and the conclusion.

We omit for brevity the similar computations of the last case where agent 2's rank at x_2 is $q_2 - 1$.

9.10 Theorem 6.1

9.10.1 Statement $i) \implies$ statement $ii)$

We fix $\mathcal{W}$ rank separable and supermodular.

Step 1.*For any c the function g_c defined by (10) is in $\mathcal{G}^-$.* By Lemma 3.6 it is enough to show $g_c \in \mathbf{G}^-$.

Because $g_c(x_i)$ and $\mathcal{W}(x_i; c)$ are continuous in x_i, c it is enough to prove the LH inequality (4) for strictly decreasing sequences $\{x_\ell\}_1^n$ and $\{c_q\}_1^{n-1}$ s. t. $H > c_1$ and $c_{n-1} > L$ and $x_\ell \neq c_q$ for all ℓ, q. This is always assumed in the rest of the proof.

Step 1.1 Call the ordered sequence of types x *regular (w r t c)* if

$$x_1 > c_1 > x_2 > c_2 > \cdots > c_{q-1} > x_q > c_q > \cdots > c_{n-1} > x_n \tag{28}$$

then check that x is a contact profile of g_c:

$$\sum_1^n g_c(x_q) = \sum_1^n \mathcal{W}(x_q, c) - \sum_1^{n-1} \mathcal{W}(c_q, c) = \sum_1^{n-1} (w_q(x_q) - w_q(c_q)) + \mathcal{W}(x_n, c) = \mathcal{W}(x)$$

Recall that (x_i, c) is also a contact profile (Definition 3.3).

Step 1.2 For any three sequences x, x' and c we say that x' is reached from x by an *elementary jump up above* c_q if there is some ℓ such that $x_{-\ell} = x'_{-\ell}$; c_q is adjacent to x_ℓ in x from above and adjacent to x'_ℓ in x' from below. In other words $x'_\ell > c_q > x_\ell$ and there is no other element of x or c between x_ℓ and x'_ℓ. The definition of an elementary jump down below c_q is symmetrical.

Starting from for an arbitrary profile $\widetilde{x}$ we construct the canonical path of profiles $\sigma = \{\widetilde{x} = {}^1x, \cdots, {}^\ell x, \cdots, {}^T x = x^*\}$ from $\widetilde{x}$ to a regular profile x^* such that 1) each step from ${}^\ell x$ to ${}^{\ell+1}x$ is an elementary jump up or down of some ${}^t x_\ell$ over some c_q and 2) $\ell \leq q$ if ${}^t x_\ell$ jumps up above c_q, and $\ell \geq q+1$ if ${}^t x_\ell$ jumps down below c_q.

Case 1: $\widetilde{x}_1 > c_1$. Then $\widetilde{x}_1$ never moves and $\widetilde{x}_1 = x_1^*$; if $\widetilde{x}_2, \cdots, \widetilde{x}_\ell$ are above c_1 then $\ell - 1$ successive elementary jumps down of $\widetilde{x}_\ell$, then $\widetilde{x}_{\ell+1}$, etc.. below c_1 defines the first $\ell - 1$ steps of the desired path, and we are left with the shorter sequences $\widetilde{x}_{-1}$ and c_{-1}.

Case 2: $c_1 > \widetilde{x}_1$. Then the successive elementary jumps up of $\widetilde{x}_1$ over the closest c_q then $c_{q-1}, \cdots, c_1$ define the first q steps of the desired path until ${}^{q+1}x = x_1^*$ that never moves again; then as above we use the induction for the sequences $\widetilde{x}_{-1}$ and c_{-1}.

Step 1.3 We pick an arbitrary profile $\widetilde{x}$, construct a sequence σ from $\widetilde{x}$ to some regular x^*, and check that in each step of the sequence the sum $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ cannot decrease, which together with Step 1.1 concludes the proof that $g_c \in \mathbf{G}^-$. This sum develops as

$$\overbrace{(\sum_{\ell=1}^{n} \mathcal{W}(x_\ell, c))}^{B} - \overbrace{\mathcal{W}(x)}^{C} - \overbrace{\sum_{q=1}^{n-1} \mathcal{W}(c_q, c)}^{D}$$

Consider an elementary jump up of ${}^t x_\ell$ above c_q: ${}^{t+1}x_\ell > c_q > {}^t x_\ell$. The net changes to the three terms in the sum are

$$\begin{array}{rcl} \Delta B &=& w_q({}^{t+1}x_\ell) - w_{q+1}({}^t x_\ell) + w_{q+1}(c_q) - w_q(c_q) \\ \Delta C &=& w_\ell({}^{t+1}x_\ell) - w_\ell({}^t x_\ell) \ ; \ \Delta D = 0 \end{array}$$

With the notation $\Delta(f; a \to b) = f(b) - f(a)$ and some rearranging this gives

$$\Delta B - \Delta C + \Delta D = \Delta(w_q - w_\ell; c_q \to {}^{t+1}x_\ell) + \Delta(w_{q+1} - w_\ell; {}^t x_\ell \to c_q)$$

where both final Δ terms are non negative because $\ell \leq q$ and by Lemma 6.1 $w_q - w_\ell$ and $w_{q+1} - w_\ell$ increase weakly.

The proof for an elementary jump down is quite similar by computing the variation of $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ to be $\Delta(w_\ell - w_q)(c_q \to x_\ell^t) + \Delta(w_\ell - w_{q+1})(x_\ell^{t+1} \to c_q)$ and recalling that in this case we have $\ell \geq q+1$.

Step 2 *A tight guarantee $g \in \mathcal{G}^-$ of $\mathcal{W}$ takes the form g_c in (10) (Definition 3.3).*

Recall the notation $\mathcal{C}(g)$ for the set of contact profiles of g defined by (5). For each $q \in [n]$ its projection $\mathcal{C}_q(g)$ is the set of those $x_i \in [L, H]$ appearing in some profile $x \in \mathcal{C}(g)$ with the rank q; it is closed because $\mathcal{C}(g)$ is closed and we call its lower bound c_q. The sequence $\{c_q\}$ decreases weakly because in a contact profile where c_q has rank q the type x_{q+1} ranked $q+1$ is below c_q. Also $c_n = L$ because g is tight so c_n is in one of its contact profiles.

We show first that $\mathcal{C}_1(g) = [c_1, H]$ and that in this interval g "follows" w_1, i. e., $g - w_1$ is a constant. The critical tool is Lemma 3.3 and we also keep in mind that g is continuous (Lemma 3.1). Pick a profile $\overline{x} \in \mathcal{C}(g)$ s. t. $\overline{x}^1 = c_1$ and apply inequality (6) to c_1 and an arbitrary $\widehat{x}_1$ in $[c_1, H]$ to get $g(\widehat{x}_1) - g(c_1) \geq w_1(\widehat{x}_1) - w_1(c_1)$. Combine this with the contact equation for $\overline{x}$:

$$g(c_1) - w_1(c_1) = \sum_2^n (w_q(\overline{x}_q) - g(\overline{x}_q)) \leq g(\widehat{x}_1) - w_1(\widehat{x}_1)$$

and recall that g is a lower guarantee: the latter inequality must be an equality therefore $\widehat{x}_1$ is in $[c_1, H]$; as $\widehat{x}_1$ was arbitrary this shows $[c_1, H] = \mathcal{C}_1(g)$ and that $g - w_1$ is a constant in $[c_1, H]$.

We show next that $[c_2, c_1] \subseteq \mathcal{C}_2(g)$ and g follows w_2 in this interval. Pick any $\widehat{x}_2 \in [c_2, c_1[$ and $\overline{x} \in \mathcal{C}(g)$ s. t. $\overline{x}^2 = c_2$ and apply again (6) to c_2 and $\overline{x}$:

$$g(\widehat{x}_2) - g(c_2) \geq \mathcal{W}(\widehat{x}_2, \overline{x}_{-2}) - \mathcal{W}(c_2, \overline{x}_{-2}) \tag{29}$$

The rank of $\widehat{x}_2$ in $(\widehat{x}_2, \overline{x}_{-2})$ is at least 2 because $\widehat{x}_2 < c_1$. If it is 2 the RH term above is $w_2(\widehat{x}_2) - w_2(c_2)$. We combine again this inequality with the contact equation for c_2 to get $\sum_{q \neq 2}(w_q(\overline{x}_q) - g(\overline{x}_q)) \leq g(\widehat{x}_2) - w_2(\widehat{x}_2)$ and deduce exactly as before that $\overline{x}_{-2}$ is a contact profile of $\widehat{x}_2$ so that $\widehat{x}_2 \in \mathcal{C}_2(g)$ moreover $g - w_2$ is constant for those types $\widehat{x}_2$ in $[c_2, c_1]$ of rank 2 in $(\widehat{x}_2, \overline{x}_{-2})$.

If the rank of $\widehat{x}_2$ in $(\widehat{x}_2, \overline{x}_{-2})$ is 3 inequality (29) becomes

$$g(\widehat{x}_2) - g(c_2) \geq w_2(\overline{x}_3) + w_3(\widehat{x}_2) - w_2(c_2) - w_3(\overline{x}_3)$$

After using the contact equation of $(c_2, \overline{x}_{-2})$ to replace $w_2(c_2) - g(c_2)$ by $\sum_{q \neq 2}(g(\overline{x}_q) - w_q(\overline{x}_q))$ we obtain

$$g(\widehat{x}_2) + \sum_{q \neq 2} g(\overline{x}_q) \geq (\sum_{q \neq 2,3} w_q(\overline{x}_q)) + w_2(\overline{x}_3) + w_3(\widehat{x}_2)$$

Because g is a lower guarantee this is an equality, proving that $(\widehat{x}_2, \overline{x}_{-2}) \in \mathcal{C}(g)$ and $\widehat{x}_2 \in \mathcal{C}_2(g)$; moreover $g - w_2$.is constant for this subset of types in $[c_2, c_1]$.

The similar argument when the rank q of $\widehat{x}_2$ in $(\widehat{x}_2, \overline{x}_{-2})$ is larger than 3 should now be clear: it concludes the proof that $[c_2, c_1] \subseteq \mathcal{C}_2(g)$, moreover identifies at most $n - q + 1$ closed subsets of the interval in which $g - w_2$.is constant: this function is continuous therefore it is constant on the whole interval.

There is no additional difficulty to prove for all $q \in [n]$ the inclusion $[c_q, c_{q-1}] \subseteq \mathcal{C}_q(g)$ and the fact that $g - w_q$ is constant in this interval. Now the guarantee g_c for $c = (c_q)_{q=1}^{n-1}$ also follows w_q as well in $[c_q, c_{q-1}]$: both g and g_c are continous so they differ by a constant, that must be zero because they are both tight.

9.10.2 Statement $ii) \implies$ statement $i)$

Fixing $\mathcal{W}$ supermodular and s. t. each function $g_c, c \in [L.H]^{n-1}$ is a tight lower guarantee of $\mathcal{W}$, we prove that $\mathcal{W}$ is rank separable.

Step 1 *The function $\mathcal{W}$ meets the equation*

$$\mathcal{W}(x) = \sum_{q=1}^{n} \mathcal{W}(x_q, c) - \sum_{\ell=1}^{n-1} \mathcal{W}(c_\ell, c) \tag{30}$$

for any $(n-1)$-sequence c and profile x, intertwined as in (28):

$$x_1 \geq c_1 \geq x_2 \geq c_2 \geq \cdots \geq c_{q-1} \geq x_q \geq c_q \geq \cdots \geq c_{n-1} \geq x_n \tag{31}$$

By the Definition 3.3 of g_c ((10)) the RH term of (30) is just $\sum_{q=1}^{n} g_c(x_q)$ hence bounded above by $\mathcal{W}(x)$ by assumption. We check the opposite inequality by induction on n.

For $n = 2$ the supermodularity and symmetry of $\mathcal{W}$ imply at once $\mathcal{W}(x_1, x_2) + \mathcal{W}(c, c) \leq \mathcal{W}(x_1, c) + \mathcal{W}(x_2, c)$ whenever $x_1 \geq c \geq x_2$. For a general n repeated application of supermodularity implies

$$\mathcal{W}(c_{n-1}, c) - \mathcal{W}(x_n, c) \leq \mathcal{W}(c_{n-1}, x_{-n}) - \mathcal{W}(x_n, x_{-n}) \tag{32}$$

The desired inequality in (30) holds if it still holds after we add to its LH the RH of (32) and the LH of (32) to its RH. This operation produces, after rearranging, the inequality

$$\mathcal{W}(c_{n-1}, x_{-n}) \leq \sum_{q=1}^{n-1} \mathcal{W}(x_q, c) - \sum_{\ell=1}^{n-2} \mathcal{W}(c_\ell, c)$$

precisely (30) for the supermodular symmetric function of $(n-1)$ variables $x_{-n} \to \widetilde{\mathcal{W}}(x_{-n}) = \mathcal{W}(c_{n-1}, x_{-n})$.

Step 2 *Solving the functional equation*

For $n = 2$ we apply (30) to $x_1 \geq c \geq x_2 = L$:

$$\mathcal{W}(x_1, L) = \mathcal{W}(x_q, c) + \mathcal{W}(L, c) - \mathcal{W}(c, c) \Longleftrightarrow \mathcal{W}(x_q, c) = \mathcal{W}(x_1, L) + w_2(c)$$

where $w_2(c) = \mathcal{W}(c, c) - \mathcal{W}(L, c)$. As x_1 and c can be freely chosen in $[L.H]$ provided $x_1 \geq c$, the rank separability of $\mathcal{W}$ follows.

Assuming the result is true until $n-1$ we apply (30) for $\mathcal{W}$ with domain $[L,H]^n$ to a pair x,c meeting (31) and s. t. $x_n = c_{n-1} = L$:

$$\mathcal{W}(x_{-n},L) = \{\sum_{q=1}^{n-1} \mathcal{W}(x_q, c_{-(n-1)}, L)\} + \mathcal{W}(c_{-(n-1)}, \overset{2}{L}) - \{\sum_{\ell=1}^{n-2} \mathcal{W}(c_\ell, c_{-(n-1)}, L)\} - \mathcal{W}(c_{-(n-1)}, \overset{2}{L})$$

For the function $\widetilde{\mathcal{W}}(x_{-n}) = \mathcal{W}(x_{-n}, L)$ over $[L,H]^{n-1}$ this is exactly equation (30) so by the inductive assumption $\widetilde{\mathcal{W}}$ is rank separable. For some continuous functions $\theta_1, \cdots, \theta_{n-1}$ on $[L,H]$ we have $\mathcal{W}(x_{-n},L) = \sum_{q=1}^{n-1} \theta_q(x_q)$.for any decreasing sequence $x_1, \cdots, x_{n-1}$.

We now apply (30) to the pair x,c meeting (31) as well as $x_q = c_q$ for $2 \le q \le n-1$, and $x_n = L$:

$$\mathcal{W}(x_1, c_{-1}, L) = \mathcal{W}(x_1, c) + \sum_{q=2}^{n-1} \mathcal{W}(c_q, c) + \mathcal{W}(c, L) - \sum_{\ell=1}^{n-1} \mathcal{W}(c_\ell, c)$$

Taking advantage of separability of $\mathcal{W}(\cdot, L)$ in the first $n-1$ variables, this reduces to

$$\theta_1(x_1) - \theta_1(c_1) = \mathcal{W}(x_1, c) - \mathcal{W}(c_1, c)$$

In this equation the weakly decreasing sequence $x_1, c_1, \cdots, c_{n-1}$ is arbitrary, which shows that $\mathcal{W}$ separates its largest variable from the $n-1$ others: for some continous functions τ, T we have $\mathcal{W}(x) = \tau(x_1) + T(x_{-1})$ if x_1 is a largest coordinate.

The argument leading to the earlier decomposition of $\mathcal{W}(x_{-n}, L)$ is easily replicated for the function $\mathcal{W}(H, x_{-1})$ (apply (30) when $x_1 = c_1 = H$) to show the decomposition $\mathcal{W}(H, x_{-1}) = \sum_{q=2}^{n} \lambda_q(x_q)$ for some continuous λ-s. We also know $\mathcal{W}(H, x_{-1}) = \tau(H) + T(x_{-1})$ therefore T is additively separable as well over decreasing sequences and we are done.

9.11 Lemma 7.2

Statement i) is clear because $\mathcal{W}$ is symmetric. In *Statement ii*) upper-hemi-continuity of γ is clear because $\mathcal{W}$ and g are both continous (step 1 in the proof of Lemma 3.5 above).

To check that γ is convex valued we fix $(x_1, x_2), (x_1, x_2') \in \Gamma(\gamma)$ and z s. t. $x_2 < z < x_2'$, and check that $\Gamma(\gamma)$ contains (x_1, z) too. Pick some $w \in \gamma(z)$: if $w > x_1$ we see that $\Gamma(\gamma)$ contains (x_1, x_2) and (w, z) s.t. $(x_1, x_2) \ll (w, z)$ which is a contradiction by Lemma 7.1. If $w < x_1$ we use instead (w, z) and (x_1, x_2') to reach a similar contradiction, and we conclude $w = x_1$.

The proof below that γ is single-valued a. e. will complete that of statement *ii*).

Statement iii) If $x_1 < x_1'$ in $\mathcal{X}$ and $\gamma^-(x_1) < \gamma^+(x_1')$ we again contradict the strict supermodularity of $\mathcal{W}$ (Lemma 7.1) . So $x_1 < x_1' \implies \gamma^-(x_1) \ge \gamma^+(x_1')$ and γ^- and γ^+ are weakly decreasing.

If $\gamma(x_1)$ is not a singleton, $\gamma^+(x_1) > \gamma^-(x_1)$, then γ^+ jumps down at x_1; a weakly decreasing function can only do this a countable number of times. That the u.h.c. closure of γ^+ contains $[\gamma^-(x_1), \gamma^+(x_1)]$ follows from $\gamma^-(x_1) \geq \gamma^+(x_1 + \delta)$ for any $\delta > 0$.

Statement iv) If $\gamma(L)$ does not contain H we pick some x_1 in $\gamma(H)$: by statement *i)* $\gamma(x_1)$ contains H therefore $x_1 > L$; we reach a contradiction again from Lemma 7.1 because $\Gamma(\gamma)$ contains $(L, \gamma^+(L))$ and the strictly larger (x_1, H).

Statement v) Kakutani's theorem implies that at least one fixed point exists. If $\Gamma(\gamma)$ contains both (a, a) and (b, b) we contradicts again Lemma 7.1. Check finally that the inequalities $\gamma^-(a) < a < \gamma^+(a)$ are not compatible. Pick $\delta > 0$ s.t. $\gamma(a)$ contains $a - \delta$ and $a + \delta$: then $\Gamma(\gamma)$ contains $(a, a + \delta)$ and $(a - \delta, a)$ (by symmetry) and we invoke Lemma 7.1 again.

9.12 Differentiability of tight guarantees

Lemma 9.2 *Inheritance of differentiability*

Suppose $\mathcal{X} = [L, H]$ is the interval $L \leq x \leq H$ in $\mathbb{R}^A$. We fix $x_i \in \mathcal{X}$, a tight guarantee $g \in \mathcal{G}^\varepsilon$ for $\varepsilon = +, -$ *and a contact profile $x = (x_i, x_{-i})$ of g at x_i. If $g(\cdot)$ and $\mathcal{W}(\cdot, x_{-i})$ are both differentiable at x_i, we have*

if $L < x_i < H$

$$\frac{dg}{dx_i}(x_i) = \frac{\partial \mathcal{W}}{\partial x_i}(x_i, x_{-i}) \tag{33}$$

if $x_i = L$ and $g \in \mathcal{G}^-$, or $x_i = H$ and $g \in \mathcal{G}^+$

$$\frac{dg}{dx_i}(x_i) \leq \frac{\partial \mathcal{W}}{\partial x_i}(x_i, x_{-i})$$

if $x_i = H$ and $g \in \mathcal{G}^-$, or $x_i = L$ and $g \in \mathcal{G}^+$

$$\frac{dg}{dx_i}(x_i) \geq \frac{\partial \mathcal{W}}{\partial x_i}(x_i, x_{-i})$$

Proof *Equation (33).* Pick an arbitrary contact profile x of $\mathcal{W}$ and $g \in \mathcal{G}^-$. Inequality (6) in Lemma 3.3 implies $g(x_i^*) - g(x_i) \leq \mathcal{W}(x_i^*, x_{-i}) - \mathcal{W}(x_i, x_{-i})$ for all x_i^* in some neighborhood of x_i. If $L < x_i < H$ and both $g(\cdot)$ and $\mathcal{W}(\cdot, x_{-i})$ are differentiable at x_i we develop this inequality as

$$(\frac{dg}{dx_i}(x_i) + o(x_i^* - x_i)) \times (x_i^* - x_i) \leq (\frac{\partial \mathcal{W}}{\partial x_i}(x) + o'(x_i^* - x_i)) \times (x_i^* - x_i)$$

where both $o(\cdot)$ and $o'(\cdot)$ are continuous and vanish at zero. As $x_i^* - x_i$ can take both signs, this implies the equality (33). The two inequalities follow similarly when the sign of $x_i^* - x_i$ is constant in the neighborhood of x_i. ■

Note that if $\mathcal{W}(\cdot, x_{-i})$ is differentiable at x_i the function $z_i \to \mathcal{W}(z_i, x_{-i})$ is K-Lipschitz for some K in a neighborhood V of x_i. By the Corollary to Lemma 3.3, so is $g(\cdot)$ at x_i in V. In turn this implies that g has bounded variation on any interval, hence is differentiable in x_i almost everywhere in $[L, H]$.

Corollary *Suppose $\mathcal{W}$ is differentiable in $[L, H]^{[n]}$. Then for $\varepsilon = +, -$ the tight guarantees in $\mathcal{G}^\varepsilon$ are characterised by their contact set $\mathcal{C}(g)$: for any two different $g, h \in \mathcal{G}^\varepsilon$ we have $\mathcal{C}(g) \neq \mathcal{C}(h)$.*

Moreover any (true) convex combination of g, h stays in $\boldsymbol{G}^\varepsilon$ but leaves $\mathcal{G}^\varepsilon$: $]g, h[\cap \mathcal{G}^\varepsilon = \varnothing$.

Proof. By Lemma 9.2 if $\mathcal{C}(g) = \mathcal{C}(h)$ we get $\frac{dg}{dx} = \frac{dh}{dx}$ in the interval $]L, H[$ so g and h differ by a constant, and if the constant is not zero one of g, h is not tight.

For the second statement suppose that $\mathcal{G}^-$ contains g, h and $\frac{1}{2}(g+h)$, all different. Fix $x_i \in]L, H[$ and a contact profile $(x_i, \widetilde{x}_{-i})$ of $\frac{1}{2}(g+h)$ at x_i. Clearly $\widetilde{x}_{-i}$ is also a contact profile of g and of h at x_i. Again by Lemma 9.2 this implies $\frac{dg}{dx_i}(x_i) = \frac{dh}{dx_i}(x_i) = \partial_i \mathcal{W}(x_i, \widetilde{x}_{-1})$ almost surely in $]L, H[$. We conclude that $g - h$ is a constant and get a contradiction of $g \neq h$. The argument for larger convex combinations with general weights is entirely similar. ■

9.13 Theorem 7.1

Step 0: the integral in (22) is well defined.

For any correspondence γ as in Lemma 7.2 the integral $\int_a^{x_1} \partial_1 \mathcal{W}(t, \gamma(t)) dt$ is the value of $\int_a^{x_1} \partial_1 \mathcal{W}(t, f(t)) dt$ for any single-valued selection f of γ: this is independent of the choice of f because γ is multi-valued only at a countable number of points and every single-valued selection of $\gamma(x_1)$ is a measurable function.

Statement ii) Fix $g \in \mathcal{G}^-$ and its contact correspondence γ. As discussed in the previous subsection g is differentiable a. e. in $[L, H]$. Its derivative $\frac{dg}{dx}$ is $\frac{dg}{dx}(x_1) = \partial_1 \mathcal{W}(x_1, x_2)$ for any $x_2 \in \gamma(x_1)$. Therefore we can write the RH of (22) as $\partial_1 \mathcal{W}(x_1, \gamma(x_1))$ without specifying a particular selection of $\gamma(x_1)$.

Note that $g(a) = una(a)$ because $(a, a) \in \Gamma(\gamma)$. Now integrating the differential equation above with this initial condition at a gives the desired representation (22).

Statement *i*)

Step 1 Lemma 7.2 implies that $\Gamma(\gamma)$ is a one-dimensional line connecting (L, H) and (H, L) that we can parametrise by a smooth mapping $s \to (\xi_1(s), \xi_2(s))$ from $[0, 1]$ into $[L, H]^2$ s.t. $\xi_1(\cdot)$ increases weakly from L to H and $\xi_2(\cdot)$ decreases weakly from H to L. We can also choose this mapping so that $\xi_1(\frac{1}{2}) = \xi_2(\frac{1}{2}) = a$, the fixed point of γ.[15]

We fix an arbitrary selection γ^* of γ, an arbitrary $\overline{x}_1$ in $[L, H]$, and check the identity

$$\int_a^{\overline{x}_1} \partial_1 \mathcal{W}(t, \gamma(t)) dt + \int_a^{\gamma^*(\overline{x}_1)} \partial_1 \mathcal{W}(t, \gamma(t)) dt = \mathcal{W}(\overline{x}_1, \gamma^*(\overline{x}_1)) - \mathcal{W}(a, a) \quad (34)$$

[15]If a is 0,or 1 we check that (22) defines the two canonical incremental guarantees in Theorem 4.1.

We change the variable t to s by $t = \xi_1(s)$ in the former and by $t = \xi_2(s)$ in the latter. Next $\overline{s}$ is the parameter at which $(\xi_1(\overline{s}), \xi_2(\overline{s})) = (\overline{x}_1, \gamma^*(\overline{x}_1))$ and we rewrite the LH of (34) as

$$\int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_1}{\partial s}(s) ds + \int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_2(s), \xi_1(s)) \frac{\partial \xi_2}{\partial s}(s) ds$$

where in each term $\partial_1 \mathcal{W}(t, \gamma(t))$ we can pick a proper selection of the (possible) interval because $(\xi_1(s), \xi_2(s)) \in \Gamma(\gamma)$. As $\mathcal{W}(x_1, x_2)$ is symmetric in x_1, x_2, we can replace the second integral by $\int_{\frac{1}{2}}^{\overline{s}} \partial_2 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_2}{\partial s}(s) ds$ and conclude that the sum is precisely

$$\mathcal{W}(\xi_1(\overline{s}), \xi_2(\overline{s})) - \mathcal{W}(\xi_1(\frac{1}{2}), \xi_2(\frac{1}{2})) = \mathcal{W}(\overline{x}_1, \gamma^*(\overline{x}_1)) - \mathcal{W}(a, a)$$

Step 2 We show that (22) defines a bona fide guarantee g: $g(x_1) + g(x_2) \leq \mathcal{W}(x_1, x_2)$ for $x_1, x_2 \in [L, H]$.

The identity (34) amounts to $g(x_1) + g(\gamma^*(x_1)) = \mathcal{W}(x_1, \gamma^*(x_1))$ for all x_1. If we prove that $g \in \boldsymbol{G}^-$ this will imply it is tight. Compute

$$g(x_1) + g(x_2) = \mathcal{W}(x_1, \gamma^*(x_1)) + g(x_2) - g(\gamma^*(x_1)) = \mathcal{W}(x_1, \gamma^*(x_1)) + \int_{\gamma^*(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \gamma(t)) dt$$

We are left to show

$$\int_{\gamma^*(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \gamma(t)) dt \leq \mathcal{W}(x_1, x_2) - \mathcal{W}(x_1, \gamma^*(x_1)) \tag{35}$$

We assume without loss $x_1 \leq x_2$ and distinguish several cases by the relative positions of a and x_1, x_2 .

Case 1: $a \leq x_1 \leq x_2$, so that $\gamma^*(x_1) \leq a$. For every $t \geq \gamma^*(x_1)$ property *iii)* in Lemma 7.2 implies $\gamma^+(t) \leq \gamma^-(\gamma^*(x_1))$ and $\gamma(\gamma^*(x_1))$ contains x_1: therefore submodularity of $\mathcal{W}$ implies $\partial_1 \mathcal{W}(t, \gamma(t)) \leq \partial_1 \mathcal{W}(t, x_1)$ and

$$\int_{\gamma^*(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \gamma(t)) dt \leq \int_{\gamma^*(x_1)}^{x_2} \partial_1 \mathcal{W}(t, x_1) dt = \mathcal{W}(x_2, x_1) - \mathcal{W}(\gamma^*(x_1), x_1)$$

Case 2: $x_1 \leq a \leq \gamma^*(x_1) \leq x_2$. Similarly for $t \geq \gamma^*(x_1)$ we have $\gamma^+(t) \leq \gamma^-(\gamma^*(x_1))$ and conclude as in Case 1.

Case 3: $x_1 \leq x_2 \leq a$, so that $\gamma^*(x_1) \geq a$. For all $t \leq \gamma^*(x_1)$ we have $\gamma^-(t) \geq \gamma^+(\gamma^*(x_1))$ and $\gamma(\gamma^*(x_1))$ contains x_1: now submodularity of $\mathcal{W}$ gives $\partial_1 \mathcal{W}(t, z) \geq \partial_1 \mathcal{W}(t, x_2)$ for z between x_2 and $\gamma^*(x_1)$ and the desired inequality because the integral in (35) goes from high to low.

Case 4: $x_1 \leq a \leq x_2 \leq \gamma^*(x_1)$. Same argument as in Case 3.

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