Fair Division with money and prices: Bid & Sell versus Divide & Choose

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April 2025

Abstract

We divide efficiently a pile of indivisible goods in common property, using cash transfers to ensure fairness among agents with utility linear in money. We compare three cognitively feasible and privacy preserving division rules in terms of the guarantees (worst case utility) they offer to the participants.

In the first version of Divide & Choose to n agents, they bid for the role of Divider then everyone bids on the shares of the Divider's partition. In the second version each agent announces a partition and they all bid to select the most efficient one.

In the Bid & Sell rule the agents bid for the role of Seller: with two agents the *smallest* bid defines the Seller who then charges any price constrained only by her winning bid.

Both rules reward subadditive utilities and penalise superadditive ones, and B&S more so than both D&C-s. B&S is also better placed to collect a larger share of the surplus when agents play safe.

Key words: Bid and Sell, Divide & Choose, worst case, guarantees, safe play

Acknowledgement 1 *Moulin's research was supported in part by a Blaise Pascal Chair of the Region Ile-de-France, 2020-21. The critical and helpful comments of two referees are gratefully acknowledged.*

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1 Introduction

The fair allocation of indivisible objects is greatly facilitated if the agents who get few good objects or many bad ones accept compensations in cash or any other transferable and divisible commodity (workload, stocks, caviar, bitcoin). Examples of this common practice include the classic rent division problem ([18], Spliddit.org), the dissolution of a partnership ([13], the Texas Shoot Out clause to terminate a joint venture[1]), and the NIMBY problem (the allocation of a noxious facility between several communities [22]).

The familiar assumption that utilities are quasi-linear – each agent can attach to each bundle of objects a personal "price" and switching from one bundle to another is exactly compensated by the difference in their prices – yields a versatile fair division model that the economic literature, so far, discussed with any depth only in the special case of the *assignment* problem where each of the n agents must receive at most one object (references in section 2).

We discuss the fair division of a finite number of indivisible *goods* (freely disposable objects) and money. Utilities are weakly increasing over subsets of goods but externalities across goods are arbitrarily complex, exactly like in the combinatorial auction problem ([14]): if we distribute the m goods in a set A, a full description of an agent's utility measured in money is a vector of dimension $2^{|A|} - 1$.

We focus on two division rules, dubbed *Divide and Choose* and *Bid and Sell*, in which the message sent by each participant is of much smaller dimension than $2^{|A|}$: for the former it is a single partition of the objects and/or a set of transfers equalising one's utility between the components of that partition; for the latter a single bid followed by either selecting a price for each object or choosing to purchase some goods at a given price vector. Of course computing my optimal safe message in either rule relies on my entire utility functions, just like in the auction context. But the information exchanged when playing the rules remains cognitively simple, a critical requirement for their applicability (as argued in [29]). Privacy protection is a "dual" argument against eliciting a full report, even when the number of objects is small: revealing little of my preferences is an advantage in subsequent bargaining interactions.

[1]Both parties submit sealed bids and the party who makes the higher bid buys the company at that price.

Consider the "naive" division rule known as *Multi Auction* (MA): each agent i places a bid β_{ia} for each good a, the highest bidder i^* on a gets this object and pays $\frac{1}{n}\beta_{i^*a}$ to each of the $n-1$ other agents. Although MA is compelling if all utilities over the objects are additive, in our much more general domain of utilities its performance is very poor: this point is the object of section 11.1 of the Appendix.

Our two division rules of interest behave much better than MA. The first one adapts to our context with cash transfers and any number n of agents the time honored *Divide and Choose* (**D&C**^{1}) method: a round of bids determines the Divider agent, who picks a partition where each lot contains some objects (possibly none) and some cash transfer (possibly zero), after which each Chooser places bids summing to zero on the different lots. We also discuss in section 6 a similar but more efficient version denoted D&C^2.

The second rule is the new *Bid and Sell* rule (**B&S**) where each agent can have a role as Seller or Buyer. In the two agent case they bid first to assign these roles, and a bid is interpreted as the price the Seller can charge for *all* the goods. The agent with the *smallest* bid x takes that role. The Seller then chooses a price for every good so that their sum is x and the Buyer can buy at those prices any subset of goods, possibly all or none. The remaining goods go the the Seller, along with the cash from the Buyer's purchase.

We compare the performance of our two rules mostly in terms of the *ex ante guarantees* each agent secures by sending a *safe* message. A message by agent i is safe if it maximises i's worst case utility when this agent only knows the number of other agents but not their utility functions. Ensuring a high guarantee to each participant is the main interpretation of ex ante fairness, pioneered in Steinhaus' work on cake cutting ([31], [32]). As in that model, a natural guarantee is $\frac{1}{n}u_i(A)$ for agent i with utility u_i, that we call agent i's *Proportional Share* (PS). But unlike in that model, this guarantee is not unique, and does not follow when we use one of the D&C and B&C rules.

We argue that the PS guarantee, as the definition of ex ante fairness, is much too coarse in our rich domain of division problems: we want instead to reward agents with subadditive utilities and penalise those with superadditive utilities.

Example 0. *We divide $m \geq 2$ identical goods between two agents Frugal (female) and Greedy (male) with the following utilities*

$$u_F(S) = 1 \text{ for all } S, \varnothing \neq S \subseteq A \text{ ; } u_F(\varnothing) = 0$$

$$u_G(S) = 0 \text{ for all } S, \varnothing \subseteq S \subsetneq A \; ; \; u_G(A) = 1$$

Frugal is content with any single good – her utility is maximally sub-additive – while Greedy needs all goods to derive any utility – his utility is maximally super-additive.

We submit that it is not fair to offer ex ante the same PS guarantee $\frac{1}{2}$ to Frugal and Greedy. Under the veil of ignorance where we (as impartial observer) don't know person X who will share the goods with Frugal, we should take into account that together Frugal and X can produce at least as much utility surplus – and typically much more – than if X is paired with Greedy. The guarantee $\frac{1}{n}u_i(A)$ ignores this fact.

The *Responsiveness* property says that we should guarantee *strictly* more than her PS to Frugal, which implies that Greedy is guaranteed strictly less than his PS (because the sum of utils of F and G is 1 for any division of the goods and cash). The *Positivity* property, by contrast, protects Greedy: it requires to give him *some* positive guarantee because his equal rights to the goods should amount to something regardless of his uncompromising utility.

We compute first the guarantee offered by D&C to Greedy in Example 0. He must choose his bid x to perhaps become the Divider knowing that there is one other bidder, but clueless about the utility – and possible bids – of that agent. So he will compute the *worst case* utility that this bid can get him.

We write (S, t) for a share with the subset S of goods and the cash transfer t (of arbitrary sign). If x is the winning bid (that he first pays to the other agent) his safe move as the Divider is to offer Chooser a choice between the share $(A, -\frac{1}{2})$ (pay me $\frac{1}{2}$ and keep all the goods) and $(\varnothing, \frac{1}{2})$ (give me all of A and I will pay you $\frac{1}{2}$). In this way Greedy's utility from his allocation is for sure $\frac{1}{2}$, and his net utility is $\frac{1}{2} - x$. If x is the losing bid, he receives first at least x from the winner (whose bid is no less than x) then in the worst case faces a choice between two allocations $(S, 0)$ and $(A \diagdown S, 0)$ where both shares are non empty so that both allocations are worth zero to Greedy. *It happens here that Frugal will actually propose such a partition to optimise her worst case. But Greedy's worst case analysis uses no such information: he sees that in any other choice between (S, t) and $(A \diagdown S, -t)$, for any subset S and cash transfer t, he has a positive utility for at least one of the two allocations.*

Greedy's worst case utility if x loses and he Chooses is x. Not knowing if he wins or loses his worst net utility is the smallest of x and $\frac{1}{2} - x$, which is largest for $x = \frac{1}{4}$ and guarantees him a gain of $\frac{1}{4}$. Any other bid than $\frac{1}{4}$ *may* result in a smaller gain.

Turning to Frugal, we compute first her worst possible utility for each of the two possible roles after the bidding. As the Divider she secures the utility of 1 by offering a choice between $(S, 0)$ and $(A \setminus S, 0)$ (where S and $A \setminus S$ are both non empty). *As Chooser she guarantees a net gain of* $\frac{1}{2}$ *for any choice between* (S, t) *and* $(A \setminus S, -t)$: indeed if S is neither A nor $\varnothing$, one of the shares has a non negative transfer so it is worth at least a utility of 1; and if the choice is between $(A, -t)$ and $(\varnothing, t)$ she guarantees $\max\{1 - t, t\}$ which is at least $\frac{1}{2}$. So Frugal's bid of x in the first round secures the utility $1 - x$ if she wins and $\frac{1}{2} + x$ if she loses: the smallest of these two is $\frac{3}{4}$ for her safe bid $x = \frac{1}{4}$ (and strictly less for any other bid). The D&C rule guarantees to Frugal three times more utility than to Greedy.

In the Bid and Sell rule, the difference between Frugal's and Greedy's guarantees sensibly increases as the number m of goods grows so the contrast between their preferences increases. We check that Greedy's guaranteed utility is now $\frac{1}{m+1}$ versus $\frac{m}{m+1}$ for Frugal.

If x is Frugal's initial bid to become the Seller and she loses, it means that the (unknown) other agent's bid is *smaller*, and as Seller that agent must offer at least one good for a price at most $\frac{1}{m}x$ therefore Frugal can guarantee the net utility $1 - \frac{1}{m}x$ by buying just one such good. If Frugal becomes the Seller with the bid x, she will safely post the uniform price $\frac{1}{m}x$ for each good: her net utility is 1 if she sells nothing, x if she sells all the goods, and more than 1 if she sells some but not all goods: she gets at least $\min\{x, 1\}$. Choosing now x to maximise $\min\{1 - \frac{1}{m}x, \min\{x, 1\}\}$, Frugal picks $x = \frac{m}{m+1}$ and secures the net utility $\frac{m}{m+1}$.

Next consider Greedy with the initial bid y. His safe price as the Seller offering to an unknown Buyer is uniform at $\frac{1}{m}y$: he gets $\frac{1}{m}y$ by selling at least one good and 1 by selling nothing, which guarantees the utility $\min\{\frac{1}{m}y, 1\}$. As the Buyer, he will pay at most y for buying all the goods, which guarantees the utility $1 - y$. His safe bid $y = \frac{m}{m+1}$ maximises $\min\{\min\{\frac{1}{m}y, 1\}, 1 - y\}$: it is the same as Frugal's safe bid but only guarantees the utility $\frac{1}{m+1}$ to Greedy.

Contents After the literature review in section 2 and the basic definitions in section 3, we define guarantees in section 4. There we also describe simple auctions implementing the *fixed partition* guarantees, a key ingredient of the D&C rule.

Section 5 introduces two critical utility levels: the *MaxMin utility* that an agent can secure as the Divider is an upper bound on *all* guarantees; the *MinMax utility* that she can secure as a Chooser against an adversarial Divider is a lower bound on all reasonable guarantees: Proposition 1.

Section 6 defines two versions of the D&C rule and computes their (different) safe play and (identical) guarantees: Proposition 2. Section 7 does the same for the B&S rule: Proposition 3.

In section 8 we compare the PS, D&C and B&S guarantees. They share several regularity and monotonicity properties (Lemma 5) as well as computational complexity. Relative to the benchmark PS, the range of the B&S guarantee is much larger than that of the D&C one: Proposition 4. But the coarser messages in the D&C rule can have strongly unpalatable consequences: Example 3. Finally we compute explicitly our guarantees when the m goods are identical and utility are convex or concave (Lemma 6) or dichotomous (Lemma 7).

Section 9 evaluates some welfare consequences of implementing one of our individual guarantees. Does it distribute at least the total utility at the worst partition of the goods? Lemma 8 gives some partial answers and formulates a conjecture. Proposition 5 shows that, if agent 1's marginal utility for each good dominates that of every other agent, then the B&S safe play achieves full efficiency, i.e., gives all the goods to agent 1, whereas under D&C all but a $\frac{1}{n}$-th share of the efficient surplus can be lost.

The concluding section 10 includes reports on numerical experiments comparing the efficiency of safe play for our two main rules. With B&S the expected surplus is at least 95% of the efficient one, whether utilities are both superadditive, both subadditive, or mixed. The performance of the D&C rule is significantly weaker.

The Appendix (section 11) gathers several important proofs.

2 Relevant literature

Allowing cash compensations to smooth out the indivisibility of objects has been essentially ignored by the first four decades of the theoretical literature on fair division, if we except the cogent discussion by Steinhaus of what we call above the Multi Auction rule for additive utilities ([32] p. 317).

This changed with the microeconomic discussion of the assignment problem. Each agent wants at most one object and utilities are increasing in money but not necessarily quasi-linear; monetary compensations can restore fairness interpreted as Envy Freeness and even a version of the competitive equilibrium with equal incomes: [33] [1]. The quasi-linear case of the model is discussed in [3] selecting a canonical envy free allocation, in [13] for the dissolution of partnership, in [22] for adressing the NIMBY problem, and currently implemented on the user-friendly Spliddit platform [19].

In the assignment problem ex ante fairness is captured by the *unanimous utility*: the best equal utility in the hypothetical problem where everyone else shares my preferences ([25], [34]). This is unambiguously the best possible guarantee and it is compatible with Envy Freeness.

In our model the set of allocations and utilities are vastly more complex than in an assignment problem and the unanimity utility – that we call the MaxMin utility – is an upper bound on guarantees but not itself a feasible guarantee. Our newfound critique of Envy Freeness (Remarks 2, 3 in section 4.3, 5 respectively) complements the normative objections developed in [25].

The search for a practical and appealing guarantee started the mathematical cake cutting literature ([31], [23]) and is a prominent theme in the vibrant 21st century algorithmic literature on fair division surveyed in [26], [5] and [35]. There the *standard model* has utilities additive over objects and no cash transfers or lotteries, so the definition of a convincing guarantee is complicated by the presence of "un-smoothable" indivisibilities. Our MaxMin and MinMax utilities are the counterpart of, respectively, the influential MaxMinShare due to Budish [11] and its dual MinMaxShare [9].[2] The MaxMinShare is *almost* a feasible guarantee (it is not feasible in extremely rare configurations [30]) while the dual MinMaxShare is strongly unfeasible. On the contrary in our model the profile of MinMax utilities is always feasible (Lemma 3 section 5) while the MaxMin profile is unfeasible; this holds as well when we divide a non atomic cake and utilities are continuous but otherwise arbitrary: see [8], [4].

In the standard model Envy Freeness is not feasible and one way to relax the EF requirement is to allow cash transfers provably small in a certain sense; these can (equivalently) come as non negative subsidies from the manager's pocket or as a set of balanced transfers bewteen agents. The initial positive result by [21] is strenghtened in [10], see also [12], [6].

In the first of our two n-person versions of the Divide & Choose rule

[2]Other definitions of guarantees are also discussed in the algorithmic literature, e. g. [7], as are guarantees adjusted to the granularity of the utilities in [17].

(section 6) the participants bid first for the role of Divider, which is similar to and inspired by the auction in [15] and [16] for implementing the egalitarian-equivalent division rule to distribute Arrow-Debreu commodities.

3 Basic definitions and notation

Objects and money The finite set A with cardinality $m \geq 2$ and generic elements $a, b, \cdots$, contains the indivisible objects that must **all** be distributed between the n agents in the set N with generic elements $i, j, \cdots$ and $n \geq 2$.

With the familiar notation $[n] = \{1, \cdots, n\}$ a n-partition π of A is a list $\pi = \{S_k\}_{k \in [n]}$ of **possibly empty** and pairwise disjoint subsets of A such that $A = \cup_{k \in [n]} S_k$: up to $n-1$ shares can be empty. If the relevant variable is unambiguous we write a partition simply as $\{S_k\}_{[n]}$.

The set of n-partitions is $\mathcal{P}(n; A)$ if the shares S_k are not assigned to specific agents, and $\mathcal{P}(N; A)$ if they are.

Money is available in unbounded quantities to perform balanced transfers between agents $t = (t_i)_{i \in N}$ that are balanced: $\sum_N t_i = 0$. The set of such transfers is $\mathcal{T}(N)$. An *allocation* is a pair $(\pi, t) \in \mathcal{P}(N; A) \times \mathcal{T}(N)$.

Utilities Each agent i is endowed with a quasi-linear utility $u_i \in \mathbb{R}^{2^A}$ over shares, with the important normalisation $u_i(\varnothing) = 0$: her utility from the allocation (π, t) is $u_i(S_i) + t_i$. The marginal utility of object a at $S \subseteq A$ for utility u is $\partial_a u(S) = u(S \cup a) - u(S \diagdown a)$. We assume throughout the paper that all objects are *goods*: $\partial_a u_i(S) \geq 0$ for all $S \subseteq A$; utility functions can be any (weakly) inclusion increasing non negative function on 2^A, and $\mathcal{M}^+$ is our notation for this domain.

The utility u is additive if for all $a \in A$ the marginal $\partial_a u(S) = u_a$ is independent of S; in this case we write $u_S = \sum_S u_a$ instead of $u(S)$.

We often use the following *cover* operation to generate examples in the domain $\mathcal{M}^+$.[3] Fix a subset $\{S_k; 1 \leq k \leq K\}$ of $2^A \diagdown \varnothing$ and K positive utilities v_k; the *cover* of the subset $\{(S_k, v_k)\}$ of $2^A \diagdown \varnothing \times \mathbb{R}_+$ is the smallest utility u in $\mathcal{M}^+$ such that $u(S_k) = v_k$ for all k:

$$u(S) = \max_{k: S_k \subseteq S} v_k \ ; \ u(S) = 0 \text{ if } S_k \nsubseteq S \text{ for all } k$$

[3] It corresponds to an XOR bid in Nisan's terminology of bidding languages ([27]).

For instance the Greedy utility u_G in section 1 is the cover of $\{(A, 1)\}$ while the Frugal utility u_F is the cover of $\{(a, 1); a \in A\}$.

We call $u \in \mathcal{M}^+$ *subadditive* if $u(S) + u(T) \leq u(S \cup T)$ for all disjoint S, T in A, and *superadditive* if the opposite inequalities hold. We write $\mathcal{S}ub$ and $\mathcal{S}up$ the corresponding subsets of $\mathcal{M}^+$; their intersection is the set $\mathcal{A}dd$ of additive utilities.

Efficiency A N-profile of utilities is $\overrightarrow{u} = (u_i)_N \in (\mathcal{M}^+)^N$ and if $\pi \in \mathcal{P}(N; A)$ we write $\overrightarrow{u}(\pi) = \sum_N u_i(S_i)$. An important special case is agent i's *unanimity* profile where all agents have the same utility u_i that we write $(\overset{n}{u_i})$, so that $(\overset{n}{u_i})(\pi) = \sum_{[n]} u_i(S_k)$.

The notation $\overset{q}{z}$ for the q-vector with q identical coordinates z will be used repeatedly.

The *efficient surplus* at profile $\overrightarrow{u}$ is $\mathcal{W}(\overrightarrow{u}) = \max_{\pi \in \mathcal{P}(N;A)} \overrightarrow{u}(\pi)$. Recall an easy but critical consequence of the quasi-linearity assumption*: the allocation $(\pi^*, t) \in \mathcal{P}(N; A) \times \mathcal{T}(N)$ is efficient (Pareto optimal: PO) if and only if π^* maximises $\overrightarrow{u}(\pi)$ over $\mathcal{P}(N; A)$*. Pareto optimality is independent of the balanced cash transfers.

Implementation Given an arbitrary n-agent mechanism agent i's strategy is *safe* if it delivers to i the largest "worst case" utility against all other agents playing adversarially against i after seeing i's strategy. That utility is the *guarantee* offered by this mechanism to agent i: it only depends upon the mechanism, agent i's utility function, and the number of other agents.

Several mechanisms can implement the same guarantee: an example is the two versions of Divide & Choose in section 6. When computing guarantees we systematically omit many tie-breaking details from the description of rules, and the reader will find it easy to check that they (the details) never affect the guarantee they implement.

At a given profile of utilities, in *any* Nash equilibrium of the game induced by the mechanism each agent gets at least their guaranteed utility (otherwise this agent agent would benefit from deviating to a safe strategy). Therefore how close is the sum of individual guarantees to the efficient maximum is an upper bound on the price of anarchy: the worst loss of efficiency at any equilibrium.

4 Guarantees, Positive and Responsive

Definition 1: *An n-person guarantee is a mapping* $\mathcal{M}^+ \ni u \to \Gamma_n(u) \in \mathbb{R}_+$ *such that*

$$\sum_N \Gamma_n(u_i) \leq \mathcal{W}(\overrightarrow{u}) \text{ for all } \overrightarrow{u} \in (\mathcal{M}^+)^N \tag{1}$$

The set of n-guarantees on A is written $\mathcal{G}(A;n)$.

By inequality (1) it is feasible at any utility profile $\overrightarrow{u}$ to give to each agent i a share of surplus weakly larger than $\Gamma_n(u_i)$.

Guarantees are *anonymous* by construction: they do not discriminate between agents on the basis of their name. The three guarantees getting most of our attention, Proportional Share, Bid & Sell and Divide & Choose, are also *neutral*, i. e., oblivious to the name of the objects in A. So these guarantees only depend upon the numbers of objects and agents, and the utility function of the concerned agent.

To any partition $\pi = \{S_k\}_{k\in[n]} \in \mathcal{P}(n;A)$ we associate the π-*guarantee* denoted

$$\Gamma_n^\pi(u) = \frac{1}{n}(\overset{n}{u})(\pi) = \frac{1}{n}\sum_{[n]} u(S_k) \text{ for all } u \in \mathcal{M}^+$$

We check that Γ_3^π meets inequality (1) at an arbitrary profile $\overrightarrow{u} = (u_1, u_2, u_3)$; for a general n the argument is quite similar. By definition of the efficient surplus the three sums

$$u_1(S_1)+u_2(S_2)+u_3(S_3) \; ; \; u_1(S_2)+u_2(S_3)+u_3(S_1) \; ; \; u_1(S_3)+u_2(S_1)+u_3(S_2)$$

are bounded above by $\mathcal{W}(\overrightarrow{u})$. Taking the average of these three inequalities gives the desired one: $\Gamma_3^\pi(u_1) + \Gamma_3^\pi(u_2) + \Gamma_3^\pi(u_3) \leq \mathcal{W}(\overrightarrow{u})$.

We speak of a generic π-guarantee (when π is not specified) as a *fixed partition guarantee*. The fixed partition guarantee corresponding to the bundling partition $\pi^{PS} = \{A, \overset{n-1}{\varnothing}\}$ is the familiar *Proportional Share* (PS) $\Gamma_n^{PS}(u) = \frac{1}{n}u(A)$.

4.1 Implementing the π-guarantees

The simple **Bundle Auction (BA)** implements Γ_n^{PS}. Each agent i submits a non negative bid β_i that the rule *interprets* as this agent's utility for the entire set A; (one of) the highest bidder(s) i^*gets A and pays $\frac{1}{n}\beta_{i^*}$ to each of the $n-1$ other agents.

The only safe bid in BA is the truthful one $\beta_i = u_i(A)$: it guarantees to agent i her PS $\frac{1}{n}u_i(A)$ while any other bid risks delivering a smaller benefit: this is clear for a winning overbid, and for an underbid losing to a bid between $\beta_i$and $u_i(A)$.[4]

We generalise BA to the **π-auction** implementing the π-guarantee Γ_n^π for any partition π of A**.** Given $\pi = \{S_k\}_{[n]}$ and the set N, each agent i reports a vector $t^i = (t_k^i)_{[n]} \in \mathcal{T}(n)$ of balanced transfers over those shares. The mechanism *interprets* t^i as equalising agent i's utility accross the different shares:

$$\text{for all } k, \ell \in [n] : u_i(S_k) + t_k^i = u_i(S_\ell) + t_\ell^i = \Gamma_n^\pi(u_i) \tag{2}$$

which reveals the utilities $u_i(S_k)$ up to an additive constant.

An *assignment* of π is a bijection σ of N into $[n]$, and their set is $\mathcal{C}$. An assignment σ^* is optimal at $\overrightarrow{u}$ if it maximises $\sum_N u_{i\sigma(i)}$ over $\mathcal{C}$. If each utility u_i meets equation (2) this is the same as minimising the "slack" $\delta(\sigma) = \sum_N t_{\sigma(i)}^i$ over $\mathcal{C}$.

Because each t^i is balanced we have $\sum_{\mathcal{C}} \delta(\sigma) = 0$, therefore the minimal slack $\delta(\sigma^*)$ is negative or zero. After each agent j receives $t_{\sigma^*(j)}^j$ (a cash handout if $t_{\sigma^*(j)}^j > 0$, a tax if $t_{\sigma^*(j)}^j < 0$) the remaining cash surplus $|\delta(\sigma^*)|$ is divided equally between all agents. Agent i's final allocation is $(S_{\sigma^*(i)}, t_{\sigma^*(i)}^i + \frac{1}{n}|\delta(\sigma^*)|)$ for which her utility is $\Gamma_n^\pi(u_i) + \frac{1}{n}|\delta(\sigma^*)|$.

We illustrate the π-guarantees and their implementation with a three good, three agent example, on which we apply more concepts and results until section 7.

Example 1 *Three agents X, Y, Z share three goods a, b, c and their utilities are*

	a	b	c	ab	ac	bc	abc
X	9	6	0	15	12	15	15
Y	15	15	15	15	18	18	18
Z	6	3	0	6	6	6	21

(3)

Note that Y's utility is almost Frugal, while Z's is somewhat Greedy.

Consider the partition $\pi = \{ac, b, \varnothing\}$ with corresponding utilities $(12, 6, 0)$ for X. The report $t^X = (-6, 0, +6)$ of balanced transfers defined by (2) is X's unique safe report securing the utility $\Gamma_3^\pi(u_X) = 6$ for each of the three shares $(ab, -6), (b, 0), (\varnothing, +6)$. Lemma 1 below proves this for a general problem.

[4] The tie break rule is irrelevant. The safe strategy and guarantee do not change if the winner only pays $\frac{1}{n}$-th of the second highest price to each loser.

Computing similarly the balanced transfers equalising Y's (resp. Z's) utilities for the shares $(ab, t_{ac}), (b, t_b), (\varnothing, t_\varnothing)$ gives:

$$\begin{array}{cccc} & t_{ac} & t_b & t_\varnothing \\ \text{X} & -6 & 0 & +6 \\ \text{Y} & -7 & -4 & +11 \\ \text{Z} & -3 & 0 & +3 \end{array} \tag{4}$$

from which we get the individual guarantees

$$(\Gamma_3^\pi(u_X), \Gamma_3^\pi(u_Y), \Gamma_3^\pi(u_Z)) = (6, 11, 3) \tag{5}$$

Upon comparing in matrix (4) the slack of the six assignments of the shares ac, b and $\varnothing$ to the agents X,Y and Z, we find that σ^* giving ac to X, b to Y and nothing to Z is efficient: it generates the smallest slack $\delta(\sigma^*) = -6 - 4 + 3 = -7$. Then we rebate to each agent $\frac{1}{3}$ of $|\delta(\sigma^*)|$, that is $2\frac{1}{3}$. The final allocation and utility profile are

$$\text{X} : (ac, -3\frac{2}{3}),\ \text{Y} : (b, -1\frac{2}{3}),\ \text{Z} : (\varnothing, 5\frac{1}{3}) \tag{6}$$

$$(u_X, u_Y, u_Z) = (8\frac{1}{3}, 13\frac{1}{3}, 5\frac{1}{3})$$

This is the profile of utilities when each agent reports safely (hence truthfully). Here and in general this is much more than their guaranteed utility. Indeed the π-auction implements the most efficient assignment of π. In particular if all agents report safely (i. e., truthfully) the final allocation will be efficient over all partitions if and only if π happens to be an efficient partition. Agent i's lower utility $\Gamma_3^\pi(u_i)$ is reached only when the other two agents report "adversarial" transfers resulting in a null slack.

Lemma 1 *The π-auction implements the π-guarantee, and the unique safe play is to report the transfers equalising one's utility across the shares of π (as in (2)).*

Proof We fix $t^1 \in \mathcal{T}(n)$ and compute agent 1's worst utility after reporting t^1.

Check first that any σ in $\mathcal{C}$ can be selected as uniquely optimal for some reports of the other agents. Suppose that all other agents j report t^1 as well: then $\sum_{i \in N} t^1_{\tau(i)} = 0$ for any assignment τ so they are all equally optimal.

For each agent $j \neq 1$, assigned $S_{\sigma(j)}$ by the given σ, we modify j's report as follows

$$t^j_{\sigma(j)} = t^1_{\sigma(j)} - \varepsilon \ ; \ t^j_\ell = t^i_\ell + \frac{1}{n-1}\varepsilon \text{ for all } \ell \neq \sigma(j)$$

indicating that j likes the share $S_{\sigma(j)}$ relative to the other shares $\frac{n}{n-1}\varepsilon$ more than 1 does. The slack of assignment σ is now $\delta(\sigma) = \sum_N t^i_{\sigma(i)} = -(n-1)\varepsilon$, smaller than for any other assignment in which at least one corrective term is positive. So σ is selected as announced, and results in agent 1's final utility $u_1(S_{\sigma(1)}) + t^1_{\sigma(1)} + \frac{n-1}{n}\varepsilon$.

As σ and ε were arbitrary we see that i's utility could be as low as $\min_{k\in[n]} u_i(S_k)+t^i_k$. The unique choice of t^1 maximising the latter equalises i's utility across these shares as in (2), and secures the utility $\frac{1}{n}\sum_{k\in[n]} u_i(S_k) = \frac{1}{n}(\overset{n}{u_i})(\pi)$, while any other report is unsafe. ■

4.2 The averaging auction

The set $\mathcal{G}(A;n)$ of n-guarantees is clearly convex.

For an arbitray finite set $\{\pi^r\}$ of partitions in $\mathcal{P}(N;A)$ indexed by $r \in R$ we describe the canonical implementation of the average guarantee $\frac{1}{|R|}\sum_R \Gamma^r_n$, which we call the **averaging-auction**. This is a key component of the second Divide & Choose rule in section 6.

Each agent i reports balanced transfers $t^i = (t^i_r)_R \in \mathcal{T}(R)$ over those guarantees, interpreted as equalising the utilities $\Gamma^r_n(u_i)$:

$$\text{for all } r,s \in R\text{: } \Gamma^r_n(u_i) + t^i_r = \Gamma^s_n(u_i) + t^i_s = \frac{1}{|R|}\sum_R \Gamma^r_n(u_i) \qquad (7)$$

Then we select a guarantee $\Gamma^{r^*}_n$ at which the sum of the corresponding transfers is minimal:

$$r^* \in \arg\min_R \sum_N t^i_r = \arg\max_R \sum_N \Gamma^r_n(u_i)$$

Call $\theta(r) = \sum_N t^i_r$ the slack of partition r in R and note that $\sum_R \theta(r) = 0$ implies $\theta(r^*) \leq 0$. We divide the surplus $|\theta(r^*)|$ equally and the net transfer to agent i is $t^i_{r^*} + \frac{1}{n}|\theta(r^*)|$. Finally $\Gamma^{r^*}_n$ is implemented and agent i's net utility is at least $\Gamma^{r^*}_n(u_i) + t^i_{r^*} + \frac{1}{n}|\theta(r^*)|$.

From equation (7) and the fact that if we fix u_i some choices of the other agents' utilities generate the slack $\theta(r) = 0$ for all r, we conclude that i's guaranteed utility is exactly $\frac{1}{|R|}\sum_R \Gamma^r_n(u_i)$ as desired.

Example 1 *(continued)* We describe the implementation of the average $\frac{1}{2}\Gamma_3^{PS} + \frac{1}{2}\Gamma_3^{\pi}$ where $\pi = \{ac, b, \varnothing\}$ *as above.* From the earlier computation of Γ_3^{π} for this example, and $\Gamma_n^{PS}(u_i) = \frac{1}{3}u_i(A)$, we compute the two guarantees and corresponding transfer vectors given by equation (7):

	$\Gamma_3^{PS}(u_i)$	$\Gamma_3^{\pi}(u_i)$
X	5	6
Y	6	11
Z	7	3

$\Longrightarrow$

	$t_{\Gamma_3^{PS}}$	$t_{\Gamma_3^{\pi}}$
X	0.5	−0.5
Y	+2.5	−2.5
Z	−2	+2
$\theta(r)$	+1	−1

As $t_{\Gamma_3^{\pi}} < t_{\Gamma_3^{PS}}$ we see that the π-auction brings more surplus than the bundle auction. So X and Y compensate Z as shown in the column $t_{\Gamma_3^{\pi}}$, and an equal share of the slack, $\frac{1}{3}$, is rebated to everyone. Then we implement Γ_3^{π} and the final utilities are $(u_X, u_Y, u_Z) = (5\frac{5}{6}, 8\frac{5}{6}, 5\frac{1}{3})$. Comparing with (5) Z is much better off than under Γ_3^{π} whereas X,Y are worse off.

Lemma 2 *The averaging-auction implements the average guarantee* $\frac{1}{|R|}\sum_R \Gamma_n^r$. *The unique safe play is to report the transfers equalising one's utility across guarantees (as in (7)).*

The straightforward proof, similar to that of Lemma 1, is omitted.

Remark 1 It is just as easy to implement any convex combination of guarantees $\sum_R \lambda_r \Gamma_n^r$ where each λ_r is positive and $\sum_R \lambda_r = 1$. Each agent i reports a vector of λ-balanced transfers t^i, $\sum_R \lambda_r t_r^i = 0$, and the rule proceeds as before: it implements $\Gamma_n^{r^*}$ where r^* minimises $\sum_N t_r^i$ so the slack $\theta(r^*) = \sum_{i \in N} t_{r^*}^i$ is still non positive and i receives $t_{r^*}^i + \frac{1}{n}|\theta(r^*)|$. The safe strategy is to choose λ-balanced transfers t^i equalising utilities as in (7).

4.3 Positivity and Responsiveness

The next two properties generalise the argument developed in Example 0 in the Introduction. Recall the Frugal utility, $u_F : u_F(S) = 1$ for $S \neq \varnothing$, and Greedy one, $u_G : u_G(S) \equiv 0$ for $S \neq A$, $u_G(A) = 1$.

Definition 2 *The n-guarantee $\Gamma_n \in \mathcal{G}(n, A)$ is*
Positive if for all $u \in \mathcal{M}^+ : u(A) > 0 \Longrightarrow \Gamma_n(u) > 0$
Responsive if $\Gamma_n(u_F) > \frac{1}{n} > \Gamma_n(u_G)$

If Positivity fails at the utility u of agent i, the goods are the common property of all the agents and yet deliver no benefit to agent i to whom they are valuable: this normative position is untenable.

For Responsiveness we observe first that the inequality $\Gamma_n(u_F) > \frac{1}{n}$ implies $\Gamma_n(u_G) < \frac{1}{n}$, because the efficient surplus is 1 when $(n-1)$ Greedy agents share A with a single Frugal agent. So Responsiveness boils down to $\Gamma_n(u_F) > \frac{1}{n}$.

We justify the latter inequality by comparing the contributions to the efficient surplus of a Frugal versus a Greedy agent. Fix a $(n-1)$-profile $u_{-1} \in (\mathcal{M}^+)^{n-1}$ and note that $\mathcal{W}(u_F, u_{-1}) \geq \mathcal{W}(u_G, u_{-1})$. If this is an equality we pick a partition $\pi = \{S_i\}_N$ efficient at (u_G, u_{-1}) and we have

$$u_G(S_1) + \sum_{i \geq 2} u_i(S_i) = u_F(S_1) + \sum_{i \geq 2} u_i(S_i)$$

implying that S_1 is $\varnothing$ or A.

If $S_1 = \varnothing$ both versions of agent 1 contribute nothing to the efficient surplus, and if $S_1 = A$ all $(n-1)$ other agents are equally useless. Hence replacing a Greedy agent by a Frugal one always brings more surplus if there is at least one efficient allocation of the goods where Frugal shares the goods with the $(n-1)$ others, whoever they are.

Among the fixed partition guarantees, only Γ_n^{PS} is Positive. All fixed partition guarantees are Responsive, with the single exception of Γ_n^{PS}. Thus a convex mixture of Γ_n^{PS} with any other π-guarantees meets both properties.

Remark 2 The standard interpretation of ex post fairness in our model is Envy Freeness (EF): the allocation (π, t) is EF if $u_i(S_i) + t_i \geq u_i(S_j) + t_j$ for all $i, j \in N$. Surprisingly, Positivity and Responsiveness are not together compatible with Envy Freeness! If the n-guarantee Γ_n in $\mathcal{M}^+$ is Positive and Responsive, then a rule implementing it cannot choose an envy-free allocation at all utility profiles.

Proof by contradiction. We fix such a guarantee Γ_n implemented by a rule selecting at each utility profile an EF allocation. At the profile with $(n-1)$ Greedy agents and a single Frugal we assume first that some agent gets all of A, with identical value 1 for everyone. By EF that agent pays $\frac{1}{n}$ to everyone else and all end up with utility $\frac{1}{n}$: this contradicts Responsiveness for Frugal. If the goods are split between at least two agents, by Positivity every Greedy one gets some positive transfer, and by EF all get the same transfer t, so Frugal pays $(n-1)t$. But then Frugal envies at least one Greedy agent who gets some good.

5 MaxMin and MinMax utilities

The recent literature on fair division pays close attention to these two canonical utility levels inspired by Divide & Choose for cake-cutting, but playing a role in many other models. Recall the notation $(\overset{n}{u})$ for the unanimity profile where all n agents have utility u.

Definition 3 *Fix A,n and $u \in \mathcal{M}^+$.*
i) The MaxMin utility at u is $MaxMin_n(u) = \frac{1}{n}\max_{\pi\in\mathcal{P}(n;A)}(\overset{n}{u})(\pi)$: the largest utility agent u can secure by choosing an (anonymous) allocation $(\pi,t) \in \mathcal{P}(n;A)\times\mathcal{T}(n)$ and eating his worst share (S_k,t_k) of that allocation.
ii) The MinMax utility at u is $MinMax_n(u) = \frac{1}{n}\min_{\pi\in\mathcal{P}(n;A)}(\overset{n}{u})(\pi)$: the largest utility agent u can secure by picking her best share in the worst possible (anonymous) allocation $(\pi,t) \in \mathcal{P}(n;A)\times\mathcal{T}(n)$.

Given an n-partition $\pi = \{S_k\}_{[n]}$ of A, the π-auction guarantees the utility $\frac{1}{n}(\overset{n}{u_i})(\pi)$ to each agent i (Lemma 1) therefore i reaches her $MaxMin$ utility if she can choose π, and at least her $MinMax$ one if the choice of π is adversarial.

Example 1 *(continued)*
Consider agent X. The partition $\pi_1 = \{bc, a, \varnothing\}$ gives $(\overset{n}{u_X})(\pi_1) = 24$, and every other partition gives her less. By attaching balanced transfers to the shares agent X ensures that all three shares are worth $\frac{24}{3} = 8$, thus maximising her utility for the worst share: $MaxMin_3(u_X) = 8$. For $MinMax_3(u_X)$ note that the three partitions $\{abc,\varnothing,\varnothing\}$, $\{a,b,c\}$, $\{ab,c,\varnothing\}$ minimise $(\overset{n}{u_X})(\pi)$ at the level 15. The worst balanced transfers attached to any such partition make all the shares worth 5 to X and any other choice allows at least one share to give X more utility: $MinMax_3(u_X) = 5$.

Similar computations for Y and Z give

	$MaxMin_3$	$MinMax_3$
X	8	5
Y	11	6
Z	7	2

(8)

Lemma 3 *In the domain $\mathcal{M}^+$*
i) If A contains at least two goods, the mapping $u \to MaxMin_n(u)$ is not a n-guarantee (property (1) fails)

but it is an upper bound for every guarantee $\Gamma_n \in \mathcal{G}(A;n)$*:*

$$\Gamma_n(u) \leq MaxMin_n(u) \text{ for all } u \in \mathcal{M}^+$$

ii) The mapping $u \rightarrow MinMax_n(u)$ *is a n-guarantee:* $MinMax_n(\cdot) \in \mathcal{G}(A;n)$

Proof For $i)$ we fix an arbitrary guarantee Γ_n and utility u. Inequality (1) at the unanimity profile $(\overset{n}{u})$ is $n\Gamma_n(u) \leq \max_{\pi \in \mathcal{P}(n;A)} u(\pi)$ as desired.

To check that $MaxMin$ is not a guarantee we have

$$MaxMin_n(u_F) = \min\{1, \frac{m}{n}\} \text{ and } MaxMin_n(u_G) = \frac{1}{n}$$

because if $n \leq m$ Frugal can choose the partition with n shares containing a single object, but if $n > m$ she can only offer m such shares. The only valuable partition to Greedy bundles A as a single share.

At the n-profile $\overrightarrow{u}$ with one u_F and $n-1$ others u_G we have $\mathcal{W}(\overrightarrow{u}) = 1$ therefore inequality (1) fails. Note that this failure is not a knife edge situation: the set of profiles where the corresponding profile of MaxMin utilities is not feasible is open in $\mathbb{R}_+^{2^A}$.

For $ii)$ pick any partition π and check the inequality $\Gamma_n^\pi(u) = \frac{1}{n}(\overset{n}{u})(\pi) \geq MinMax_n(u)$ for all u. ■

We note that both statements in Lemma 3 hold in the cake-cutting model with very general preferences ([8]) but there the proof of $ii)$ is much harder!

Our next result, technically very simple, shows an important benefit of choosing a guarantee in the "duality interval" $[MinMax_n(u), MaxMin_n(u)]$.

Recall from section 3 (second paragraph) the notation $\mathcal{S}ub$ and $\mathcal{S}up$ for the sets of sub- and super-additive functions. For instance in Example 1 Y's utility is subadditive, Z's is superadditive and X's is neither.

Proposition 1 *Suppose the guarantee* $\Gamma_n \in \mathcal{G}(A;n)$ *is such that*

$$\Gamma_n(u) \in [MinMax_n(u), MaxMin_n(u)] \text{ for all } u \in \mathcal{M}^+ \tag{9}$$

Then $\Gamma_n(u) = \frac{1}{n}u(A)$ *if* $u \in \mathcal{A}dd$*;* $\Gamma_n(u) \geq \frac{1}{n}u(A)$ *if* $u \in \mathcal{S}ub$*; and* $\Gamma_n(u) \leq \frac{1}{n}u(A)$ *if* $u \in \mathcal{S}up$*.*

Proof If $u \in \mathcal{S}ub$ (resp. $u \in \mathcal{S}up$) we have $u(A) = \min_{\pi \in \mathcal{P}(n;A)}(\overset{n}{u})(\pi)$ (resp. $u(A) = \max_{\pi \in \mathcal{P}(n;A)}(\overset{n}{u})(\pi)$) hence $\frac{1}{n}u(A) = MinMax_n(u)$ (resp. $MaxMin_n(u)$) therefore (9) implies the desired inequalities. ■

If u is additive $MaxMin_n(u) = \frac{1}{n}u(A)$, so by statement $i)$ in Lemma 3 the Proportional Share is the best possible guarantee and the compelling interpretation of ex ante fairness.

Note that property (9) is not very restrictive: it is clearly satisfied by the Proportional Share, the fixed partitions guarantees Γ_n^π, the D&C and B&S guarantees defined shortly, and their convex combinations.

In subsection 11.1 of the Appendix we show that the guarantee of the naive Multi Auction rule (auctioning objects one by one, see 5-th paragraph in section 1) falls below the duality interval: it is often much smaller than the $MinMax$ guarantee. We dismiss MA for this very reason.

Remark 3 There is a precise connection between the duality interval in (9) and Envy Freeness, confirming the trade-off between ex ante and ex post fairness in Remark 2 above. At an envy free allocation, it is clear that every agent i gets at least her $MinMax_n(u_i)$ utility. Conversely if the single-valued rule $(\mathcal{M}^+)^N \ni \overrightarrow{u} \to (\pi, t) \in \mathcal{P}(N; A) \times \mathcal{T}(N)$ is efficient and envy-free, it must implement precisely the $MinMax$ guarantee: we check that for each utility function u_i we can complete a profile (u_i, u_{-i}) at which the rule gives to agent i precisely his $MinMax_n(u)$ utility.

Fix $u_1 \in \mathcal{M}^+$, $\pi = (S_k)_{k=1}^n$ achieving $\min_{\pi \in \mathcal{P}(n;A)}[u_1](\pi)$, and a positive number δ. Construct a profile where the common utility v of the $n-1$ other agents is the cover of the sequence $\{(S_k, u_1(S_k) + \alpha); k \in [n]\}$. If α is very large any assignment of the shares S_k to the agents is efficient (and any other efficient partition distributes the same utilities pre-transfers). By the construction of utility v, at an envy free and efficient allocation the transfers make agent 1 indifferent between all the shares so her utility is $minMax_n(u)$.

6 Two Divide&Choose rules

The two rules have the same guarantee and their building blocks are the π-auction and averaging-auction in section 4.

Definition 4 *Divide&Choose*$_n^{\mathbf{1}}$
Stage 1: run a simple auction for the role of Divider; the winner i^* *is (one of) the highest bidder(s) with a bid* $\beta_i \geq 0$*;*
Stage 2: agent i^* *pays* $\frac{1}{n}\beta_i$ *to every other agent and picks a partition* $\pi^* = \{S_k\}_{k=1}^n$ *in* $\mathcal{P}(n, A)$*;*
Stage 3: run the π^**-auction between all agents.*

Definition 5 *Divide&Choose*$_n^{\mathbf{2}}$

Stage 1: each agent i picks a partition π^i in $\mathcal{P}(n,A)$;
Stage 2: run the averaging auction between the guarantees $\Gamma_n^{\pi^i}, i \in N$.

The D&C$_n^2$ rule takes longer to run than D&C$_n^1$ because the averaging auction will first identify a partition $\pi^{\widehat{i}}$ maximising $\sum_{i\in N} \Gamma_n^{\pi^j}(u_i)$ over j before running the $\pi^{\widehat{i}}$-auction.

Proposition 2
*In the D&C$_n^1$ rule, agent i's play is safe **if and only if** he bids $\beta_i = MaxMin_n(u_i) - MinMax_n(u_i)$ in stage 1; chooses if he wins a partition π^* maximising $(\overset{n}{u}_i)(\pi)$ in stage 2, and reports truthful equalising transfers across the shares of π^* in stage 3.*
*In the D&C$_n^2$ rule, agent i's play is safe **if and only if** she proposes a partition π^i maximising $(\overset{n}{u}_i)(\pi)$ in stage 1, then reports truthful equalising transfers across the guarantees $\Gamma_n^{\pi^j}(u_i), j \in N$, and finally reports truthful transfers in the final $\pi^{\widehat{i}}$-auction.*
Both rules implement the guarantee

$$\Gamma_n^{DC}(u) = \frac{1}{n} MaxMin_n(u) + \frac{n-1}{n} MinMax_n(u)$$
$$= \frac{1}{n^2} \max_{\pi\in\mathcal{P}(n,A)} u(\pi) + \frac{n-1}{n^2} \min_{\pi\in\mathcal{P}(n,A)} u(\pi) \qquad (10)$$

The guarantees Γ_n^{DC} is Positive, Responsive, and in the duality interval (9).

Proof *For D&C$_n^1$.* In the π-auction agent i guarantees the utility $\frac{1}{n}(\overset{n}{u})(\pi)$ (Lemma 1). So as the Divider her best choice of π guarantees the utility $\max_\pi \frac{1}{n}(\overset{n}{u})(\pi) = MaxMin_n(u_i)$ (Definition 3). As a Chooser, the worst possible choice of π by the Divider gives $\min_\pi \frac{1}{n}(\overset{n}{u})(\pi) = MinMax_n(u_i)$ to our agent. So the worst drop in guaranteed utility between the roles of Divider and Chooser is $\delta_i = MaxMin_n(u_i) - MinMax_n(u_i)$.

Her bid x_i in stage 1 secures the utility $MaxMin_n(u_i) - \frac{n-1}{n}x_i$ if it wins, and $MinMax_n(u_i) + \frac{1}{n}x_i$ if it loses: bidding δ_i maximises the smallest of these two, and her final guarantee is as announced in (10).

For D&C$_n^2$. Agent i's guaranteed utility in stage 2 is $\frac{1}{n}\sum_{j\in[n]} \frac{1}{n}(\overset{n}{u}_i)(\pi^j)$ (Lemma 2) so the worst case is when $(\overset{n}{u}_i)(\pi^j) = \min_{\pi\in\mathcal{P}(n;A)}(\overset{n}{u}_i)(\pi)$ for each $j \neq i$. Therefore proposing in stage 1 an optimal partition π^i securing $\max_\pi \frac{1}{n}(\overset{n}{u})(\pi)$ delivers the same guarantee (10).

We omit the easy proof that no other play is safe in either version of D&C$_n$. ■

Example 1 *(continued)* **for D&C$_3^1$**

From the $MaxMin$ and $MinMax$ values in (8) we have

$$\begin{array}{ccc} \Gamma_3^{DC}(u_X) & \Gamma_3^{DC}(u_Y) & \Gamma_3^{DC}(u_Z) \\ 6 & 7\frac{2}{3} & 3\frac{2}{3} \end{array}$$

We compute the allocation reached by the safe play of all three agents.

In D&C^1 the bids in stage 1 are $(3, 5, 5)$ for X,Y and Z respectively. The way we break ties between Y and Z is now critical. If Z is chosen as the Divider, he pays $1\frac{2}{3}$ to X and to Y, then picks the bundle partition π^{PS} where his safe bid of 21 wins and he gives an extra 7 to X and to Y. Final allocation and utilities are

$$\begin{array}{ccc} X & Y & Z \\ (\varnothing, 8\frac{2}{3}) & (\varnothing, 8\frac{2}{3}) & (A, -17\frac{1}{3}) \\ 8\frac{2}{3} & 8\frac{2}{3} & 3\frac{2}{3} \end{array}$$

where Z gets nothing more than his guaranteed utility.

If instead Y wins stage 1, in stage 2 she pays $1\frac{2}{3}$ to X and to Z then can safely divide A either as $\pi^* = \{ac, b, \varnothing\}$ or $\pi^{**} = \{bc, a, \varnothing\}$. Say she chooses π^*. We computed in subsection 4.1 the corresponding (safe and truthful) transfer reports (4) and the resulting allocation (6). To the latter we add the payments in stage 2. Final allocation and utilities

$$\begin{array}{ccc} X & Y & Z \\ (ac, -2) & (b, -5) & (\varnothing, 7) \\ 10 & 10 & 7 \end{array}$$

a serious Pareto improvement over the choice of Z as winner in stage 1, reflecting the fact that π^* delivers 6 more units of total surplus than π^{PS}.

If agent Y after winning stage 1 chooses instead the efficient (unbeknownst to her) partition π^{**} we let the reader check the final result

$$\begin{array}{ccc} X & Y & Z \\ (bc, -3) & (a, -5) & (\varnothing, 8) \\ 12 & 10 & 8 \end{array}$$

yet another Pareto improvement over the previous choice of Y.

Example 1 *(continued)* **for D&C$_3^2$**

To agent Z the partition with the best guarantee is the bundle π^{PS}. For X the best choice is $\pi^{**} = \{bc, a, \varnothing\}$ with a guarantee of 8, but Y has a choice between $\pi^* = \{ac, b, \varnothing\}$ and π^{**}.

Assuming that Y picks π^* in stage 1, we compute the 3×3 matrix of guarantees $\Gamma_3^\pi(u_i) = \frac{1}{3}(\overset{3}{u_i})(\pi)$ and the corresponding safe balanced transfers:

$$[\frac{1}{3}(\overset{3}{u_i})(\pi)] : \begin{array}{cccc} & \pi^{**} & \pi^* & \pi^{PS} \\ X & 8 & 6 & 5 \\ Y & 11 & 11 & 6 \\ Z & 4 & 3 & 7 \end{array} \Longrightarrow [t_i^\pi] : \begin{array}{cccc} & \pi^{**} & \pi^* & \pi^{PS} \\ X & -1\frac{2}{3} & +\frac{1}{3} & +1\frac{1}{3} \\ Y & -1\frac{2}{3} & -1\frac{2}{3} & +3\frac{1}{3} \\ Z & +\frac{2}{3} & +1\frac{2}{3} & -2\frac{1}{3} \end{array}$$

The surplus maximising partition is π^{**}: the sum of its column in the right (resp. left) matrix is minimal at $-2\frac{2}{3}$ (resp. maximal at 23). So before running the π^{**}-auction, we perform transfers τ as in the π^{**} column, augmented by a share $\frac{1}{3}|2\frac{2}{3}|$ of the slack for each agent: $\tau = (-\frac{7}{9}, -\frac{7}{9}, +1\frac{5}{9})$. Then the π^{**}-auction delivers the allocation X: $(bc, -4\frac{2}{3})$; Y: $(a, -1\frac{2}{3})$; Z: $(\varnothing, 6\frac{1}{3})$ which we finally combine with τ:

$$\begin{array}{ccc} X & Y & Z \\ (bc, -5\frac{4}{9}) & (a, -2\frac{4}{9}) & (\varnothing, 7\frac{8}{9}) \\ 9\frac{5}{9} & 12\frac{5}{9} & 7\frac{8}{9} \end{array}$$

We conclude that safe reporting in the two versions of D&C delivers significantly different allocations, all the more so if some agents have several choices of optimal partitions.

Finally we comment on an unappealing feature of D&C^{1,2}. In the reporting stages common to both rules each agent only reveals the relative utilities between the shares of certain partitions but the level of his absolute utility remains private: this increases privacy but is detrimental to efficiency.

For instance if utility u is additive the safe bid in D&C^1 is zero and any partition is a safe proposal in both rules (because $(\overset{n}{u})(\pi) = u(A)$ for any π). Then if u is so much higher than other utilities that efficiency requires to give this agent all the goods, her bid in D&C^1 is still zero and some agent 2 with non additive utility will become the Divider; if 2 does not bundle all goods in one share, the final allocation is for sure inefficient.

On the contrary in the Bid&Sell rule to which we now turn, individual messages are related to the absolute utilities and avoid this type of inefficiencies: this is formally proven by Proposition 5 in section 9.

7 The Bid&Sell rule

For a non negative price vector $p \in \mathbb{R}_+^A$ we use the same notation $p_S = \sum_{a \in S} p_a$ as if p described an additive utiltity. We write $\Delta(x)$ for the simplex of prices such that $p_A = x$. Because the recursive definitions of the B&S rule and its guarantee work over shrinking subsets of objects, we make explicit their dependence on the set A.

Definition 6 *Bid&Sell for two agents*: B&S$_2(A)$
stage 1: each agent i bids x_i (a non negative real number) to become the Seller; (one of) the lowest bidder(s) with bid x becomes the Seller;
stage 2: the Seller chooses a price p in $\Delta(x)$;
stage 3: the Buyer can buy any share S of objects (possibly $\varnothing$ or A) at price p; the Seller cashes the revenue and enjoys the unsold goods.
Final allocation: Buyer $(S, -p_S)$; Seller $(A \diagdown S, p_S)$.

To understand how to bid safely we compute first the safe utility $W_2(u; x|A)$ an agent with utility u becoming the Seller after bidding x can secure by choosing optimally the price offered to the Buyer and expecting the worst purchase from that agent:

$$W_2(u; x|A) = \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (u(T) + p_{A \diagdown T}) = x + \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (u(T) - p_T) \quad (11)$$

We compare it with the safe utility $L_2(u; x|A)$ this agent can secure if her bid x loses *by a hair* (to a bid just below x) so she becomes the Buyer and is offered the worst possible price such that the whole bundle A costs x:

$$L_2(u; x|A) = \min_{p \in \Delta(x)} \max_{\varnothing \subseteq S \subseteq A} (u(S) - p_S) \quad (12)$$

Clearly $W_2(u; x|A)$ increases in x while $L_2(u; x|A)$ decreases hence the safe bid in stage 1 is x^* such that $W_2(u; x^*|A) = L_2(u; x^*|A)$, which we show below is well defined. This common value is the Bid & Sell guarantee $\Gamma_2^{BS}(u|A)$.

Even with three goods and two agents the computation of the bid functions W_2 and L_2 is a linear program harder to solve than computing the $MaxMin$ and $MinMax$ partitions as in section 5.

In the next computation and in Example 2 after Definition 7 we use the familiar notation

$$(z)_+ = \max\{z, 0\} \ ; \ (z)_- = \min\{z, 0\}$$

Example 1 *(continued)*

For agent X involved in a **two person** division of A the guaranteed utility after a winning bid x is

$$W_2(u_X; x|A) = x + \max_{p\in\Delta(x)} \min\{0, 9-p_a, 6-p_b, -p_c, 15-p_{ab}, 12-p_{ac}, 15-p_{bc}, 15-x\}$$

We can drop the two dominated terms $15 - p_{ab}$ and $15 - p_{bc}$, then check that for $p = (\frac{3}{5}x, \frac{2}{5}x, 0)$ the max min term is $(15 - x)_-$ and that this price is optimal. Therefore $W_2(u_X; x|A) = \min\{x, 15\}$.

Next we compute $L_2(u_X; x|A)$, the guaranteed utility after a losing bid x:

$$L_2(u_X; x|A) = \min_{p\in\Delta(x)} \max\{0, 9-p_a, 6-p_b, -p_c, 15-p_{ab}, 12-p_{ac}, 15-p_{bc}, 15-x\}$$

where we can only drop the term $15 - x$.

For $x \leq 3$ the price $p = (x, 0, 0)$ is optimal and $L_2(u_X; x|A) = 15 - x$. For $x \geq 3$ the optimal price solves $15 - p_{ab} = 12 - p_{ac} = 15 - p_{bc}$ and $L_2(u_X; x|A) = (14 - \frac{2}{3}x)_+$. Finally the two functions intersect at the safe bid $x^* = 8\frac{2}{5}$, guaranteeing to agent X the utility $u_X = 8\frac{2}{5}$.

Similar computations, omitted for brevity, give for Y:

$$W_2(u_Y; x|A) = \min\{x, 18\}$$

$$L_2(u_Y; x|A) = 18 - x \text{ on } [0, 3] \ ; \ = 16\frac{1}{2} - \frac{1}{2}x \text{ on } [3, 9] \ ; \ = (18 - \frac{2}{3}x)_+ \text{ above } 9$$

and these two functions intersect at the safe bid $x^* = 11\frac{1}{4}$ guaranteeing the utility $11\frac{1}{4}$.

For agent Z we find similarly

$$W_2(u_Z; x|A) = x \text{ on } [0, 6] \ ; \ = \frac{1}{2}x + 3 \text{ on } [6, 18] \ ;$$

$$= \frac{1}{3}x + 6 \text{ on } [18, 45] \ ; \ = 21 \text{ above } 45$$

$$L_2(u_Z; x|A) = (21 - x)_+$$

so that Z's safe bid is $x^* = 12$ for the guaranteed utility 9.

We find that the B&S guarantees improve those of the D&C$_2$ rule in a **two person** division of A (Proposition 2) for all three agents

	Γ_2^{DC}	Γ_2^{BS}
X	$6\frac{1}{2}$	$8\frac{2}{5}$
Y	$8\frac{1}{2}$	$11\frac{1}{4}$
Z	$4\frac{1}{2}$	9

This pattern is of course not a general feature of the comparison between D&C and B&S.

For a larger number n of agents, the rule $B\&S_n(A)$ is defined recursively, through at most $n-1$ rounds of bidding: in each round one agent is the Buyer and the remaining other agents are Sellers; the Buyer leaves after buying some goods (perhaps none) from all the Sellers. Naturally the computational difficulty increases sharply.

Definition 7 $\mathrm{B\&S}_n(A)$: Bid&Sell for $n \geq 3$
Suppose the rule $\mathrm{B\&S}_v(B)$ is already defined for $|B| \leq n-1$ and define $\mathrm{B\&S}_n(A)$ as follows.

Stage 1: each agent i bids x_i to become Seller or Buyer; (one of) the highest bidder(s) becomes the Buyer;
Stage 2: each of the $n-1$ Sellers j chooses a price p_j in $\Delta(x_j)$;
*Stage 3: the Buyer buys a share S of goods by paying $p_j(S)$ to **each** Seller and leaves; the rule stops if $S = A$, otherwise we go to*
Stage 4: the remaining agents play $\mathrm{B\&S}_{n-1}(A \diagdown S)$.

The worst utility $W_n(u; x|A)$ from becoming a Seller after bidding x is now

$$W_n(u; x|A) = \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (\Gamma^{BS}_{n-1}(u|T) + p_{A \diagdown T}) = x + \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (\Gamma^{BS}_{n-1}(u|T) - p_T) \tag{13}$$

and the worst utility as a Buyer after bidding x is

$$L_n(u; x|A) = \min_{p \in \Delta((n-1)x)} \max_{\varnothing \subseteq S \subseteq A} (u(S) - p_S) = \min_{p \in \Delta(x)} \max_{\varnothing \subseteq S \subseteq A} (u(S) - (n-1)p_S) \tag{14}$$

because the worst case is when the $n-1$ other bids are just below x.

Lemma 4 *For any non null utility $u \in \mathcal{M}^+$ the recursive programs (13),(14), together with the initial pair (11), (12), define unambiguously the function $W_n(u; x|A)$ concave and strictly increasing in x from 0 to $u(A)$; the function $L_n(u; x|A)$ convex and strictly decreasing in x from $u(A)$ to 0; and the guarantee $\Gamma^{BS}_n(u)$ at their intersection: $W_n(u; x^*|A) = L_n(u; x^*|A) = \Gamma^{BS}_n(u|A)$.*

These properties imply: $0 < \Gamma^{BS}_n(u|A) < u(A)$. In particular the buyer in stage 3 buys at least one good.

Proposition 3 *The guarantee Γ^{BS}_n is Positive, Responsive, and in the duality interval (*9*).*

The proof of the key Lemma 4 and its corollary Proposition 3, in subsection 11.2 of the Appendix, is a non trivial application of the minimax theorem.

We illustrate the recursion defining the B&S$_3$ rule in a simpler instance than Example 1.

Example 2 *Three agents F, H, K share three identical goods and their utilities are*

# of goods	1	2	3
F	5	5	5
H	0	4	6
K	1	3	6

So F is a Frugal agent who needs not more than one good, K is superadditive and H is neither sub- nor super-additive.

As the goods are identical, we use the fact that the optimal price p in (13), (14) can be taken symmetric over the goods (Lemma 10 in section 11.3).

Before computing the two functions W_3, L_3 for a utility u we must retrieve the two person guarantees $\Gamma_2^{BS}(u|k)$ when only k goods are available, $k = 1, 2, 3$. We computed this for the Frugal agent in Example 0 section 1: after scaling up 5 times those earlier results we have

$$\Gamma_2^{BS}(F|3) = 3\frac{3}{4} \ ; \ \Gamma_2^{BS}(F|2) = 3\frac{1}{3} \ ; \ \Gamma_2^{BS}(F|1) = 1\frac{2}{3}$$

then we can apply (13), (14):

$$W_3(F; x|3) = \min\{x, \frac{5}{3} + \frac{2}{3}x, \frac{10}{3} + \frac{1}{3}x, \frac{15}{4}\} = \min\{x, 3\frac{3}{4}\}$$

$$L_3(F; x|3) = \max\{0, 5 - \frac{2}{3}x, 5 - \frac{4}{3}x, 5 - 2x\} = (5 - \frac{2}{3}x)_+$$

Agent F's safe bid in Stage 1 of B&S$_3$, at the intersection of these two functions, is $x_F^* = 3$. Her guaranteed surplus is also $\Gamma_3^{BS}(F) = 3$.

The same computations for agent H start with the two person problems with 1, 2 or 3 goods. For instance the two functions

$$W_2(H; x|3) = \min\{x, \frac{2}{3}x, 4 + \frac{1}{3}x, 6\} = \min\{\frac{2}{3}x, 6\}$$

$$L_2(H; x|3) = \max\{0, 4 - \frac{2}{3}x, 6 - x\} = (6 - x)_+$$

intersect at $x = \frac{18}{5}$ and $\Gamma_2^{BS}(H|3) = \frac{12}{5}$. We find similarly $\Gamma_2^{BS}(H|2) = \frac{4}{3}$, $\Gamma_2^{BS}(H|1) = 0$. Then we compute

$$W_3(H;x|3) = \min\{\frac{2}{3}x, \frac{4}{3} + \frac{1}{3}x, \frac{12}{5}\} = \min\{\frac{2}{3}x, \frac{12}{5}\}$$

$$L_3(H;x|3) = \max\{0, 4 - \frac{4}{3}x, 6 - 2x\} = (6-2x)_+$$

and conclude that H's safe bid in stage 1 of B&S$_3$ is $x_H^* = 2\frac{1}{4}$ guaranteeing $\Gamma_3^{BS}(H) = 1\frac{1}{2}$.

Agent K two person guarantees are computed as $\Gamma_2^{BS}(K|3) = 3$, $\Gamma_2^{BS}(K|2) = 1\frac{1}{2}$, $\Gamma_2^{BS}(K|1) = \frac{1}{2}$, and her safe bid is $x_K^* = 2\frac{1}{16}$ guaranteeing $\Gamma_3^{BS}(K) = 1\frac{7}{8}$.

The largest bid in stage 1 is $x_{F^5}^* = 3$ so F is the first buyer. In stage 2 agents H,K choose equal unit prices for the 3 goods, respectively $p^H = \frac{1}{3}x_H^* = \frac{3}{4}$ and $p^K = \frac{1}{3}x_K^* = \frac{11}{16}$. In stage 3 agent F pays $1\frac{7}{16}$ for one good and her final utility is $5 - 1\frac{7}{16} = 3\frac{9}{16}$.

In stage 4 agents H and K play B&S$_2$ for the two remaining goods. Agent H bids $2\frac{2}{3}$, larger than K's bid $\frac{1}{2}$ so H is the next buyer: he buys both goods and pays $1\frac{1}{2}$ to K. The final allocation is efficient: one good to F and two to H, for the final utilities

$$F : u = 3\frac{9}{16} \ ; \ H : u = 3\frac{1}{4} \ ; \ K : u = 2\frac{3}{16}$$

where F' and K's share of surplus are less than 20% larger than their respective guaranteed shares (respectively $\Gamma_3^{BS}(F) = 3$ and $\Gamma_3^{BS}(K) = 1\frac{30}{16}$) whereas H more than doubles his guarantee $\Gamma_3^{BS}(H) = 1\frac{1}{2}$.

8 Comparing the B&S, D&C, and PS guarantees

8.1 More common properties

We already know that all three guarantees are Positive and in the duality interval, and that Γ^{BS} and Γ^{DC} (the same guarantee for both D&C rules) are Responsive.

Lemma 5

i) The guarantees $\Gamma_n^{BS}, \Gamma_n^{DC}$ and Γ_n^{PS} are continuous and weakly increasing in the individual utility u.

ii) They are also scale invariant ($\Gamma_n(\lambda u) = \lambda\Gamma_n(u)$ *for* $\lambda > 0$)*, weakly increasing in* A *and weakly decreasing in* n*. For all* A, n, u *we have*

$$\Gamma_n(u|A) \leq \Gamma_n(u|A \cup a) \text{ and } \Gamma_{n+1}(u|A) \leq \Gamma_n(u|A)$$

Proof Statement $i)$ is clear for Γ_n^{DC} and Γ_n^{PS}. For Γ_n^{BS} both functions $W_n(u;\cdot)$ and $L_n(u;\cdot)$ increase weakly in u, so their intersection does too.

Statement ii) for Γ_n^{DC}

Scale invariance is clear. For the monotonicity in A one checks easily that both $MaxMin_n(u|A)$ and $MinMax_n(u|A)$ increase weakly in A. For the monotonicity in n we fix $\pi^* \in \mathcal{P}(n+1;A)$ and pick a share S in π^* such that $u(S) \leq \frac{1}{n+1}(\overset{n+1}{u})(\pi^*|A)$ (e. g. an empty share, if any). This implies $\frac{n}{n+1}(\overset{n+1}{u})(\pi^*|A) \leq (\overset{n+1}{u})(\pi^*|A) - u(S) \leq \max_{\mathcal{P}(n;A)}(\overset{n}{u})(\pi|A)$ and that $MaxMin_n(u)$ decreases weakly in n. Pick next $\widehat{\pi} \in \mathcal{P}(n;A)$ such that $(\overset{n}{u})(\widehat{\pi}|A) = \min_{\mathcal{P}(n;A)}(\overset{n}{u})(\pi|A)$ and note that $(\overset{n}{u})(\widehat{\pi}|A) = (\overset{n+1}{u})(\widetilde{\pi}|A)$ for the partition $\widetilde{\pi}$ adding an empty share to $\widehat{\pi}$, therefore

$$nMinMax_n(u|A) = (\overset{n}{u})(\widehat{\pi}|A) = (\overset{n+1}{u})(\widetilde{\pi}|A) \geq (n+1)MinMax_{n+1}(u|A)$$

implying that $MinMax_n(u)$ is also weakly decreasing in n.

Statement ii) for Γ_n^{BS}

Checking Scale Invariance is routine. The monotonicity in A is proven in subsection 11.2 in the second paragraph of the proof of Lemma 4.

For the monotonicity in n: taking $T = A$ in the minimisation part of program (13) gives $W_n(u;x) \geq \Gamma_{n-1}^{BS}(u|A)$ for all x, and this holds in particular at the x^* optimal in the problem with n agents. ■

For an additive utility u the three rules share the guarantee $\frac{1}{n}u(A)$ (Proposition 1). For B&S$_n$, just like for BA$_n$, the only safe bid is $x^*(u) = \frac{1}{n}u(A) = \Gamma_n^{BS}(u)$; and if this makes you the Seller the price $p_a = \frac{1}{n}u(a)$ is uniquely safe. The omitted proof checks by induction that $W_n(u;x) = \min\{x, u(A)\}$ and $L_n(u;x) = (u(A) - (n-1)x)_+$.

Recall from the discussion at the end of section 6 that, on the contrary, in D&C$_n^1$ the safe bid is zero so that any partition is a safe choice for both versions of the rule.

Computational complexity The recursive computation of $\Gamma_n^{BS}(u|A)$ from Γ_{n-1}^{BS} solves the two LPs (13), (14) of size 2^m. For a general n we solve a pair of LPs for each agent to go from $n-1$ to n and this may happen in each of the $n-1$ steps of the full recursive algorithm. Therefore the number of LPs grows as n^2 so the complexity remains polynomial as long as the number of objects is fixed. We already noticed that it is exponential in the number of goods unless the goods are identical, as follows from the general result in [28]. The same conclusions apply to either D&C rule, where the only hard step is to identify the partitions π minimising or maximising the utilities $(\overset{n}{u})(\pi)$.

The easy case of identical goods is discussed in subsections 8.4 below and 11.3 in the Appendix.

8.2 Divergence from the Proportional Share

We turn to a different effect already illustrated in Examples 0 and 1: as the utility function becomes more sudadditive or more superadditive, the B&S$_n$ guarantee deviates more from the Proportional Share than the D&C$_n$ does.

Proposition 4 *For all n and all $u \in \mathcal{M}^+$ we have*

$$\frac{n}{(n-1)m+1} \leq \frac{\Gamma_n^{BS}(u)}{\frac{1}{n}u(A)} \leq \frac{n \times m}{n+m-1}$$

$$\frac{1}{n} \leq \frac{\Gamma_n^{DC}(u)}{\frac{1}{n}u(A)} \leq \frac{\min\{m,n\}+n-1}{n}$$

In both cases the bounds are achieved at u_G and u_F respectively.

We see that the upper bound of $\frac{\Gamma_n^{BS}}{\Gamma_n^{PS}}$ is strictly larger than that of $\frac{\Gamma_n^{DC}}{\Gamma_n^{PS}}$, with a single exception at $n=m=2$. And the lower bound of $\frac{\Gamma_n^{BS}}{\Gamma_n^{PS}}$ is strictly lower than that of $\frac{\Gamma_n^{DC}}{\Gamma_n^{PS}}$ if $m \geq n+2$, strictly larger if $m \leq n$, and equal if $m = n+1$.

Moreover the ratio $\frac{\Gamma_n^{DC}}{\Gamma_n^{PS}}$ is always below 2, while $\frac{\Gamma_n^{BS}}{\Gamma_n^{PS}}$ can be arbitrarily large.

Proof We apply Lemma 5 twice. Every utility u in $\mathcal{M}^+$ s. t. $u(A)=1$ satisfies $u_G \leq u \leq u_F$ and $\Gamma_n^{BS}, \Gamma_n^{DC}$ increase weakly in u, therefore

$$\Gamma_n^M(u_G) \leq \Gamma_n^M(u) \leq \Gamma_n^M(u_F) \text{ where } M \text{ is } D\&C \text{ or } B\&S$$

By scale invariance it is enough to show that u_G *and* u_F achieve the announced bounds for the two rules. If $M = D\&C$ it follows by Proposition

2 after checking $\max_\pi(\overset{n}{u_G})(\pi) = 1$, $\min_\pi(\overset{n}{u_G})(\pi) = 0$ and

$$\max_\pi(\overset{n}{u_F})(\pi) = \min\{m, n\} \ ; \ \min_\pi(\overset{n}{u_F})(\pi) = 1$$

For $M = B\&C$ we use the more general result about dichotomous utilities in Lemma 7 two subsections below. ■

8.3 A revealing example

Here the D&C guarantee is unpalatable because it ignores important aspects of the externalities across objects. This critique is more subtle than –but similar to –that of the Proportional Share by the way it treats Greedy and Frugal.

Example 3 Two agents, Abstemious and Choosy, share 4ℓ goods partitioned as four subsets, each with ℓ objects: $A = R \cup R^* \cup L \cup L^*$. Think of two types of right gloves and two types of left gloves.

Abstemious is happy with any pair of one right and one left glove: her utility is the cover of $\{((r,\ell), 1)\}$ over the whole set $(R \cup R^*) \times (L \cup L^*)$. Choosy wants no less than all gloves in $R \cup L$ or all in $R^* \cup L^*$: his utility is the cover of $\{(R \cup L, 1), (R^* \cup L^*, 1)\}$.

For both agents $MinMax_2(u) = 0$, $MaxMin_2(u) = 1$, so Γ_2^{DC} gives $\frac{1}{2}$ to *both* agents: the D&C guarantee is shockingly coarse, the more so as ℓ grows. By contrast we check that the optimal bid in the B&S$_2$ rule is $x^* = \frac{2\ell}{\ell+1}$ for *both* agents (almost twice larger than $u(A)$). To Abstemious this guarantees $\frac{\ell}{\ell+1}$ because her worst case as Seller is to sell exactly $R \cup R^*$ or exactly $S \cup S^*$ for a net utility $\frac{x}{2}$; and as Buyer there will be at least one pair costing at most $\frac{2x}{m}$.

And to Choosy the B&S$_2$ guarantee is $\frac{1}{\ell+1}$ because his worst case as Seller is to sell exactly one glove in $R \cup R^*$ and one in $S \cup S^*$ for a net utility $\frac{2x}{m}$; and as Buyer he will have to pay $\frac{x}{2}$ to get any benefit.

8.4 Identical goods

Here the utility $u(S)$ depends only on the cardinality s of the subset S of goods: it is an increasing function $s \to u_s$ from $[m]$ into $\mathbb{R}_+$. The *median* of u, written u_{med} is $u_{\frac{m}{2}}$ if m is even, and $u_{med} = \frac{1}{2}(u_{\frac{m-1}{2}} + u_{\frac{m+1}{2}})$ if m is odd.

In this rich class of utilities computing the D&C$_2^1$ safe bid and guarantee is fairly simple because $MaxMin_2(u)$ and $MinMax_2(u)$ are respectively the maximum and minimum of $u_s + u_{m-s}$ over s. More work is needed to compute them for the B&S$_2$ rule without restrictions on the sequence $(u_s)_{[m]}$: the still simple programs are described in Lemma 11, section 11.3 of the Appendix. Here we apply this result to describe Γ_2^{BS} for convex or concave utility functions, and compare it to Γ_2^{DC}.

Lemma 6
i) Suppose u is either convex or concave. Then the optimal bid and guarantee in the D&C$_2$ rule are

$$x^* = |\frac{1}{2}u_m - u_{med}| \ ; \ \Gamma_2^{DC}(u) = \frac{1}{2}u_{med} + \frac{1}{4}u_m$$

ii) If u is convex the optimal bid and guarantee in the B&S$_2$ rule are

$$x^* = \max_{0\leq s\leq m}\{\frac{m}{m+s}(u_m - u_{m-s})\} \ ; \ \Gamma_2^{BS}(u) = \frac{m}{m+s^*}u_{m-s^*} + \frac{s^*}{m+s^*}u_m$$

iv) If u is concave they are

$$x^* = \max_{0\leq s\leq m}\frac{m}{m+s}u_s \ ; \ \Gamma_2^{BS}(u) = \frac{m}{m+s^*}u_{s^*}$$

Statement $i)$ is clear once we check that for a convex u: $MaxMin_2(u) = \frac{1}{2}u_m$ and $MinMax_2(u) = u_{med}$; and vice versa if u is concave.

Finally we generalise Example 0 to the sequence of dichotomous utilities connecting the Frugal and Greedy ones: we compute explicitly the D&C and B&S guarantees and bids for arbitrary n and m.

For each integer $\theta \in [m]$ the dichotomous utility u^θ is satisfied with no less and no more than θ goods:

$$u^\theta(S) = 1 \text{ if } |S| \geq \theta \ ; \ u^\theta(S) = 0 \text{ if } |S| < \theta$$

Here $u^1 = u_F$ and $u^m = u_G$.

Lemma 7 *For the dichotomous utilities above*
i) The optimal bid and guarantee in the D&C$_n^1$ rule are[5]

$${}^\theta x^* = 1 - \frac{1}{n} \ ; \ \Gamma_2^{DC}(u^\theta) = \frac{1}{n}(2 - \frac{1}{n}) \text{ if } \theta \leq \frac{m}{n}$$

[5] We omit for easy reading the case $\frac{m}{n} < \theta < \frac{m}{n} + 1$ where ${}^\theta x^* = \frac{1}{n}(\lfloor \frac{m}{\theta} \rfloor - 1)$ and $\Gamma_2^{DC}(u^\theta) = \frac{1}{n^2}(\lfloor \frac{m}{\theta} \rfloor + n - 1)$.

$$^{\theta}x^* = \frac{1}{n}\lfloor\frac{m}{\theta}\rfloor \; ; \; \Gamma_2^{DC}(u^\theta) = \frac{1}{n^2}\lfloor\frac{m}{\theta}\rfloor \text{ if } \theta \geq \frac{m}{n} + 1$$

ii) The optimal bid and guarantee in the B&S$_n$ rule are

$$x^* = \frac{m}{m+1+(n-2)\theta} \; ; \; \Gamma_n^{BS}(u^\theta) = \frac{m+1-\theta}{m+1+(n-2)\theta}$$

For instance with five agents and twenty goods Γ_5^{BS} dominates Γ_5^{DC} for all values of θ except 18, 19 and 20 when they are both less than a quarter of the Proportional guarantee. The ratio $\frac{\Gamma_5^{BS}}{\Gamma_5^{PS}}$ decreases regularly from $4\frac{3}{20}$ while $\frac{\Gamma_5^{DC}}{\Gamma_5^{PS}}$ is never above $1\frac{4}{5}$.

The proof of statement *i*) is routine once we compute

$$MaxMin_5(u^\theta) = \min\{1, \frac{1}{n}\lfloor\frac{m}{\theta}\rfloor\}$$

$$MinMax(u^\theta) = \frac{1}{n} \text{ if } \theta < \frac{m}{n} + 1 \; ; \; = 0 \text{ if } \theta \geq \frac{m}{n} + 1$$

That of statement *ii*) is in section 11.3 of the Appendix.

9 Guaranteed collective welfare

9.1 Reducing the bargaining gap

At any n-profile of utilities $\overrightarrow{u}$ the n-person guarantee Γ_n ensures a collective welfare not smaller than $\sum_N \Gamma_n(u_i)$. We can evaluate the bite of our guarantee by measuring the difference between the efficient surplus $\mathcal{W}(\overrightarrow{u}) = \max_\pi \overrightarrow{u}(\pi)$ and that sum, relative to the largest efficiency loss resulting from a misallocation of the objects.

We call the interval $[\min_{\mathcal{P}(N;A)} \overrightarrow{u}(\pi), \max_{\mathcal{P}(N;A)} \overrightarrow{u}(\pi)]$ the *bargaining gap* of the problem $(A, N, \overrightarrow{u})$ and we say that the guarantee Γ_n *reduces the bargaining gap* if

$$\min_{\pi\in\mathcal{P}(N;A)} \overrightarrow{u}(\pi) \leq \sum_N \Gamma_n(u_i) \leq \max_{\pi\in\mathcal{P}(N;A)} \overrightarrow{u}(\pi) \text{ for all } \overrightarrow{u} \in (\mathcal{M}^+)^N \qquad (15)$$

The right hand inequality is just the definition (1) of a guarantee, but the left hand inequality is not necessarily true.

Recall from Proposition 1 that Γ^{BS}, Γ^{DC} and Γ^{PS} are not less than $MinMax_n(u_i)$, therefore they guarantee the collective welfare $\sum_N MinMax_n(u_i)$. This lower bound is not logically related to $\min_{\pi\in\mathcal{P}(N;A)} \overrightarrow{u}(\pi)$.[6] However, the PS guarantee$\frac{1}{n}u(A)$ clearly meets (15): by Proposition 1 again, so do our guarantees Γ^{BS} and Γ^{DC} if the utilities are subadditive.

Lemma 8

i) With two agents, $n = 2$, the Bid & Sell and Divide & Choose guarantees reduce the bargaining gap.
ii) With three or more agents, the Divide & Choose guarantee may not reduce the bargaining gap.

Proof

Statement i)
Step 1: for B&S. Fix a profile u, v where u's optimal bid x^* wins against v's larger optimal bid y^*. Let $p \in \Delta(x^*)$ be such that $\Gamma_2^{BS}(u) = L(u; x^*) = \max_{\varnothing\subseteq S\subseteq A}(u(S) - p_S)$. We increase p to some $q \in \Delta(y^*)$ so that $\Gamma_2^{BS}(u) \geq \max_{\varnothing\subseteq S\subseteq A}(u(S) - q_S)$.

Also $\Gamma_2^{BS}(v) = W(v; y^*) = \min_{\varnothing\subseteq S\subseteq A}(v(S) - q_S) + y^* = v(\overline{S}) - q_{\overline{S}} + y^*$ for some $\overline{S}$. Then $\Gamma_2^{BS}(u) \geq u(\overline{S}^c) - q_{\overline{S}^c}$ so $\Gamma_2^{BS}(u) + \Gamma_2^{BS}(v) \geq u(\overline{S}^c) + v(\overline{S})$.
Step 2: for D&C. Fix a profile u, v and let S, T be such that $\min_\pi(\overset{n}{u})(\pi) = u(S) + u(S^c)$ and $\min_\pi[v](\pi) = v(T) + v(T^c)$. The computation of Γ_2^{DC} (Proposition 1) implies

$$\Gamma_2^{DC}(u) \geq \frac{1}{4}(u(T) + u(T^c) + u(S) + u(S^c))$$

and a similar lower bound for $\Gamma_2^{DC}(v)$. Summing up these inequalities and rearranging gives the desired inequality (15).

Statement ii) Recall the dichotomous utilities u^θ and their guarantees in Lemma 7. A simple three person profile violating (15) for Γ_3^{DC} has three goods and the profile $\overrightarrow{u} = (u^2, u^2, u^1)$:

$$\Gamma_3^{DC}(u^2) = \frac{1}{9}\ ,\ \Gamma_3^{DC}(u^1) = \frac{5}{9} \text{ but } \min_{\pi\in\mathcal{P}(N;A)} \overrightarrow{u}(\pi) = 1$$

■

[6] If $A = \{a, b, c, d\}$, u_1 is the cover of $\{ab, bc, cd, ad\}$ and u_2 is the cover of $\{ac, ad, bc, bd\}$ (all with value 1), then $MinMax_2(u_i) = 0$ for $i = 1, 2$, but $\min_\pi \overrightarrow{u}(\pi) = 1$ for all π. If $A = \{a, a', b, b'\}$, u_1 is the cover of $\{a, a'\}$ and u_2 is the cover of $\{b, b'\}$, (all with value 1) then $\overrightarrow{u}(\pi) = 0$ if each agent gets useless goods, but $MinMax_2(u_i) = \frac{1}{2}$ for $i = 1, 2$.

We **conjecture** that the B&S guarantee reduces the bargaining gap for any n.

Our intuition comes again from the equation $\Gamma^{BS}(u^\theta) = \frac{m+1-\theta}{m+1+(n-2)\theta}$ in Lemma 7. At a profile $(u^{\theta_i})_N$ the equality $\min_\pi \overrightarrow{u}(\pi) = 1$ holds if and only if $\sum_{i=1}^n \theta_i \leq m+n-1$. Then the left inequality in (15) follows from the convexity of $\theta \to \Gamma^{BS}(u^\theta)$ and is tight.

9.2 When an agent's utility dominates

For any two $u, v \in \mathcal{M}^+$ we say that u *dominates* v (resp. dominates strictly) if we have

$$\max_{\varnothing \subseteq S \subseteq A} \partial_a v(S) \leq \min_{\varnothing \subseteq S \subseteq A} \partial_a u(S) \text{ (resp. a strict inequality) for all } a \in A$$

If in the profile $\overrightarrow{u} = (u_i)_{i=1}^n$ utility u_1 dominates each u_i, $i \geq 2$, it is efficient to give all the goods to agent 1, strictly so if each domination is strict. This follows by repeated application of the inequality $u_1(S)+u_i(T) \leq u_1(S \cup a) + u_i(T \diagdown a)$ when S, T are disjoint and $a \in T$.

Our last result reveals another serious advantage of the Bid & Sell rule over the Divide&Choose rules.

Proposition 5 *Fix a profile $\overrightarrow{u} = (u_i)_{i=1}^n$ where utility u_1 dominates strictly u_i for each $i \geq 2$.*
i) The B&S_n division rule where all agents play safely implements the efficient outcome where agent 1 eats all the goods.
ii) The outcome of safe play in the D&C rules may only collect $\frac{1}{n}$-th of the efficient surplus.

Proof of statement *i*) in subsection 11.4.

For statement *ii*) suppose for simplicity $m = n$, all goods are identical, agent 1 has the additive utility $u_1(S) = |S|$ and all utilities u_i for $i \geq 2$ have marginals $\partial_a u_i(|S|)$ below ε, with $\varepsilon < 1$, and strictly decreasing in $|S|$. In D&C_n^1 agent 1 bids zero and the others bids are positive. The Divider will offer the partition of A in $n = m$ singletons and the surplus collected will be $1 + (n-1)\varepsilon$, but the efficient surplus is n.

10 Conclusion

Our interpretation of ex ante fairness in terms of an individual guarantee for each agent u_i inside the benchmark interval $[MinMax_n(u_i), MaxMin_n(u_i)]$

delivers the desired correction to the coarse Proportional Share: a reward to a subadditive utility and a penalty to a superadditive one (Proposition 1).

The Bid & Sell and Divide & Choose rules implement such guarantees (Propositions 2 and 3), with B&S responding more strongly than D&C to sub- and super-additivity (Proposition 4). It strongly outperforms D&C when one agent values each subset of goods much more than any other agent (Proposition 5).

All agents playing safe in either rule does not in general extract the efficient surplus because the messages reveal much less than full utilities.Yet the price message in B&S reveals more of the shape of the Seller's utility than the partition in D&C does of the Divider's. This suggests that the safe play in B&S captures more of the efficient surplus than D&C, at least on average.

Numerical experiments with two agents sharing up to seven goods in [2] confirm this intuition. The B&S rule captures on average between 95-99% of the efficient surplus whether both utilities are superadditive, both subadditive, or one of each type. The corresponding range for the D&S rule (version 1 or 2) is 80-90% for two subadditive agents, 65-75% for two superadditive ones, and 45-60% for a mixed pair.

11 Appendix: missing proofs

11.1 The Multi Auction rule

Recall from section 1 that the MA simply runs m independent Bundles Auctions, one for each good. Each agent i submits a profile of bids $\beta^i \in \mathbb{R}^A_+$; for each good a (one of) the highest bidder(s) on a, agent i^*, gets a and pays $\frac{1}{n}\beta_{i^*a}$ to every other agent.

If utility u_i is additive under MA the truthful bid u_{ia} on each a is the unique safe play and implement the PS guarantee. If all utilities are additive the safe play by all picks an efficient allocation (that is even Envy Free).

If the rule MA is used for general utilities in $\mathcal{M}^+$ the marginal utility of adding a to a subset of goods varies, so there is no "truthful" bid on a. For any utility $u \in \mathcal{M}^+$ the safe vector of bids solves the program:

$$\Gamma_n^{MA}(u) = \max_{\beta \in \mathbb{R}^A_+} \{ \min_{\varnothing \subseteq S \subseteq A} (u(S) - \frac{n-1}{n}\beta_S + \frac{1}{n}\beta_{S^c}\} = \max_{\beta \in \mathbb{R}^A_+} \min_{\varnothing \subseteq S \subseteq A} (u(S) - \beta_S) + \frac{1}{n}\beta_A \tag{16}$$

If our agent wins the auctions for the goods in S and those only, she pays $\frac{n-1}{n}\beta_a$ for each a in S, and gets in the worst case $\frac{1}{n}\beta_a$ for each a outside S.

The guarantee Γ_n^{MA} is neither Responsive nor Positive: $\Gamma_n^{MA}(u_G) = 0 < \frac{1}{n} = \Gamma_n^{MA}(u_F)$. Indeed if Greedy's bid β is not zero, pick a such that $\beta_a = \min_{b \in A} \beta_b$, suppose Greedy wins all auctions except a and check that his worst utility is negative or zero. Next Frugal's safe bid of $\frac{1}{n}$ on every good secures utility $\frac{1}{n}$ in the worst cases where she wins all auctions or none of them.

Moreover Γ_n^{MA} is dominated by the *MinMax* guarantee, often strictly so. To check the first claim pick $u \in \mathcal{M}^+$, a partition $\pi = \{S_k\}_{k \in [n]}$ of A and an optimal bid β of u in (16). Then $\Gamma_n^{MA}(u) \leq (u(S_k) - \beta_{S_k}) + \frac{1}{n}\beta_A$ for all k and the sum of these inequalities is $\Gamma_n^{MA}(u) \leq \frac{1}{n}(\overset{n}{u})(\pi)$.

An example where domination is strict is the utility $u = u_F + u_G$ when $m \geq 3$. One checks easily that $MaxMin_2(u) = MinMax_2(u) = 1$ but $\Gamma_n^{MA}(u) = \frac{m}{2(m-1)}$.

11.2 Proof of Lemma 4 and Proposition 3

Fixing A and a single utility $u \in \mathcal{M}^+$, our first step is to rewrite the programs (11) (12) in a more compact though less transparent format using a well known combinatorial concept.

A vector $\delta = (\delta_S) \in \mathbb{R}_+^{2^A \smallsetminus \varnothing}$ is a *balanced (set of) weights* if for all $a \in A$ we have $\sum_{S: S \ni a} \delta_S = 1$. We call δ *minimal* if it is an extreme point of the convex compact set of balanced weights, and write $\mathcal{B}_m$ the set of minimal balanced weights for m goods.[7] The simplest elements of $\mathcal{B}_m$ come from the *true partitions* $\{S_k\}$ of A: those where each S_k is non empty, $\delta_{S_k} = 1$ for each k, and all other weights are 0. Let $\mathcal{B}_m^*$ be $\mathcal{B}_m$ minus the balanced weights coming from the trivial partition $\{A\}$ ($\delta_A = 1$ and other $\delta_S = 0$).

Write the total weight of δ as $\overline{\delta} = \sum_{\varnothing \neq S \subseteq A} \delta_S$. Then $\overline{\delta} > 1$ for each δ in $\mathcal{B}_m^*$. The smallest of these sums is $\overline{\delta} = \frac{m}{m-1}$ when $\delta_{A \smallsetminus a} = \frac{1}{m-1}$ for all a (all other weights are zero), and the largest one is $\overline{\delta} = m$ when $\delta_a = 1$ for all a. Both claims follow from the identity $\sum_{S \subsetneq A} |S| \times \delta_S = m$.

Lemma 9

[7] The size of $\mathcal{B}$ grows astronomically fast with m: $|\mathcal{B}| = 2$ for $m = 2$, $= 6$ for $m = 3$, $= 27$ for $m = 4$ and more than $15,000$ for $m = 5$: see [20].

The programs (11) (12) can be rewritten as follows:

$$W_2(u;x) = \min\{x, u(A), \min_{\mathcal{B}_m^*} \frac{1}{\overline{\delta}}(\delta \cdot u - x) + x\} \tag{17}$$

$$L_2(u;x) = \max\{0, u(A) - x, \max_{\mathcal{B}_m^*} \frac{1}{\overline{\delta}}(\delta \cdot u - x)\} \tag{18}$$

Proof.
We write $\nabla(Z)$ for the set of convex weights on Z, and first rewrite the MaxMin expression in (11):

$$W_2(u;x) - x = \max_{p\in\Delta(x)} \min_{\varnothing\subseteq T\subseteq A}(u(T) - p_T) = \max_{p\in\Delta(x)} \min_{\xi\in\nabla(2^A)} \sum_{T\in 2^A} \xi_T(u(T) - p_T)$$

where ξ has two coordinates ξ_A and $\xi_\varnothing$.

Note that the mapping $\nabla(2^A) \ni \xi \to \zeta \in \mathbb{R}^A : \zeta_a = \sum_{T:a\in T} \xi_T$ is onto $[0,1]^A$, and apply the minimax theorem to rewrite the last maxmin term above as

$$\min_{\xi\in\nabla(2^A)} \max_{a\in A} \sum_{T\in 2^A} \xi_T u(T) - x\zeta_a \Longrightarrow W_2(u;x) = \min_{\xi\in\nabla(2^A)} \sum_{T\in 2^A} \xi_T u(T) + x(1 - \min_a \varsigma_a) \tag{19}$$

We check now that for an optimal ξ in the minimisation program above, ζ_a is independent of a. Assume $\zeta_a > \min_b \zeta_b$ where the minimum is achieved by some good b^*. We can choose S containing a but not b^* and such that $\xi_S > 0$: if this was impossible $\zeta_a \leq \zeta_{b^*}$ would follow. For ε small enough we construct $\xi' \in \nabla(2^A)$ identical to ξ except for $\xi'_S = \xi_S - \varepsilon$, $\xi'_{S\diagdown\{a\}} = \xi_{S\diagdown\{a\}} + \varepsilon$. By construction the net change in the objective is $-\varepsilon u(S) + \varepsilon u(S\diagdown\{a\}) \leq 0$; moreover ζ' and ζ coincide everywhere except at a where $\zeta'_a = \zeta_a - \varepsilon$. We can now choose ε such that either $\zeta'_a = \min_b \zeta_b$ or $\xi_S = 0$ and still $\zeta'_a > \min_b \zeta_b$. Then we repeat the construction until all coordinates of ζ coincide.

If ξ is deterministic on $\varnothing$ or on A, we get the first two terms in (17). For any other ξ we can assume in (19) that ξ puts no weight on $\varnothing$ or on A, and write $\zeta \in [0,1]$ for the common value ζ_a. Setting $\delta = \frac{1}{\zeta}\xi$ defines a balanced set of weights and $\sum_T \xi_T u(T) + x(1-\zeta) = \zeta(\delta \cdot u) + (1-\zeta)x$. Without loss we can minimise over minimal balanced weights. Finally $\overline{\delta} = \frac{1}{\zeta}$ and the proof of (17) is complete.

The similar argument for (18) starts with

$$L_2(u,x) = \min_{p\in\Delta(x)} \max_{\xi\in\nabla(2^A)} \sum_{T\in 2^A} \xi_T(u(T)-p_T) = \max_{\xi\in\nabla(2^A)} \{\sum_{T\in 2^A} \xi_T u(T) - x(\max_{a\in A} \zeta_a)\}$$

The critical argument that we can take ζ_a independent of a assumes $\zeta_a < \max\zeta = \zeta$, picks S s. t. $\xi_S > 0$ and containing b^* but not a and changes ξ by $\xi'_S = \xi_S - \varepsilon$, $\xi'_{S\cup a} = \xi_{S\cup a} + \varepsilon$: for ε small enough the $\max\zeta$ does not change and the net change on the objective is at least$-\varepsilon u_S + \varepsilon u_{S\cup a} \geq 0$. ■

Proof of Lemma 4. Equation (17) defines a concave function. Each term in x increases strictly because $\overline{\delta} > 1$ and reaches $u(A)$ for x large enough, therefore $W_2(u;x)$ increases strictly up to $u(A)$. Similarly in (18) $L_2(u;x)$ is convex and strictly decreasing as long as all terms in x are positive, which terminates for x large enough. So the intersection of $W_2(u;\cdot)$ and $L_2(u;\cdot)$ as $\Gamma_2^{BS}(u|A)$ is well defined.

We proceed now by induction after checking that the function $S \to \Gamma_2^{BS}(u|S)$ is in $\mathcal{M}^+(A)$. Going back to the definition (11) we see that $W_2(u;x|S)$ increases weakly in S because agent u can choose in the problem augmented to $S \cup a$ a price s. t. $p_a = 0$; and so does $L_2(u;x|S)$ by (12) because in the augmented problem the agent can choose only among subsets not containing a. Both $W_2(u;x|S)$ and $L_2(u;x|S)$ increase weakly in S, so their intersection in x increases too.

The induction step applies Lemma 9 to $\Gamma_{n-1}^{BS}(u|\cdot) \in \mathcal{M}^+(A)$ and gives $W_n(u;x|A), L_n(u;x|A)$ by the two programs

$$W_n(u;x|A) = \min\{x, u(A), \min_{\mathcal{B}_m} \frac{1}{\overline{\delta}}(\delta \cdot \Gamma_{n-1}^{BS}(u|\cdot) - x) + x\} \tag{20}$$

$$L_n(u;x|A) = \max\{0, u(A) - (n-1)x, \max_{\mathcal{B}_m} \frac{1}{\overline{\delta}}(\delta \cdot u - (n-1)x)\} \tag{21}$$

with the properties announced in Lemma 5, and their intersection $\Gamma_n^{BS}(u|\cdot)$ as a function in $\mathcal{M}^+(A)$.

Proof of Proposition 3. If $u(A) > 0$ both functions $W_n(u;x), L_n(u;x)$ are strictly positive for x small enough, proving Positivity. For Responsiveness we compute formally $\Gamma_2^{BS}(u_F)$ (more rigorously than in the Introduction). First (17) gives $W_2(u_F;x) = \min\{x, 1\}$ because the smallest $\overline{\delta}$ in $\mathcal{B}_m^*$ is $\frac{m}{m-1}$ and $L_2(u_F;x) = \max\{0, 1 - \frac{1}{m}x\}$ because the largest $\overline{\delta}$ in $\mathcal{B}_m^*$ is m. This shows $\Gamma_2^{BS}(u_F|S) = \frac{|S|}{|S|+1}$

We omit the straightforward induction argument giving $\Gamma_n^{BS}(u_F|S) = \frac{|S|}{|S|+n-1}$.

It remains to check $\Gamma_n^{BS}(u) \geq MinMax_n(u)$ for all u and n. This is true for $n = 1$. Assume next it holds for Γ_{n-1}^{BS} and pick any $u \in \mathcal{M}^+(A)$ with optimal bid x^* where W_n and intersect. Choose $p \in \Delta(x^*)$ optimal in program (14) so that $L_n(u; x^*|A) = \max_{\varnothing\subseteq S\subseteq A}(u(S) - (n-1)p_S)$. Then (13) and the inductive argument imply

$$W_n(u; x^*) \geq \min_{\varnothing\subseteq S\subseteq A}(\Gamma_{n-1}^{BS}(u|S) - p_S) + x^* = \Gamma_{n-1}^{BS}(u|T) - p_T + x^* \text{ for some } T$$

$$\Longrightarrow W_n(u; x^*) \geq \frac{1}{n-1}(\overset{n}{u})(\pi) - p_T + x^* \text{ where } \pi \text{ is some } (n-1)\text{-partition of } T$$

We can now combine this lower bound for $(n-1)W_n(u; x^*)$ with $L_n(u; x^*) \geq u(T^c) - (n-1)p_{T^c}$ to get $n\Gamma_n^{BS}(u) \geq (\overset{n}{u})(\pi) + u(T^c)$ which completes the proof.

11.3 Proofs for identical goods

We say that the goods a, b are symmetric in u if we have

$$u(S - b + a) = u(S) \text{ for all } S \text{ s. t. } a \notin S \ni b$$

Lemma 10 *If two goods a, b are symmetric in u their optimal (safe) prices in $W_n(x; u|A)$ and their worst prices in $L_n(x; u|A)$ can be taken equal when we compute the safe bids in B&S$_n$.*

Proof For brevity we give the argument for $n = 2$ and omit the obvious induction argument.

Fix $u \in \mathcal{M}^+$ and assume u is symmetric in the goods a, b. In the program (*12*) defining $L_2(u; x)$ assume the worst price p has $p_a < p_b$. Let q obtains from p by averaging p_a and p_b and changing nothing else. Then $\max_{\varnothing\subseteq S\subseteq A}(u(S) - p_S)$ differs from $\max_{\varnothing\subseteq S\subseteq A}(u(S) - q_S)$ only in pairs of terms of the form $u(S) - p_S$, $u(S - b + a) - p_{S-b+a}$. Replacing p by q lowers the largest of these two terms, so q is still optimal in the program (*12*). The argument for $W_2(x; u)$ is identical. ■

In order to compute now Γ_2^{BS} when the all the goods are identical (without assuming convexity or concavity of the utility) we use the notation $\partial_k u_\ell = u_{\ell+k} - u_\ell$.

Lemma 11 *Fix a utility u for identical goods. Agent u's optimal bid in the B&S$_2$ rule is*

$$x^* = \max\{\frac{m}{m+k}\partial_k u_\ell | 0 \leq k, \ell \leq m \text{ and } 0 \leq \ell + k \leq m\} \tag{22}$$

If $x^* = \frac{m}{m+k^*}\partial_{k^*} u_{\ell^*}$ *his guarantee is*

$$\Gamma_2^{BS}(u) = \frac{\ell^* + k^*}{m+k^*} u_{\ell^*} + \frac{m-\ell^*}{m+k^*} u_{\ell^*+k^*} \tag{23}$$

Proof By Lemma 9 in section 11.2 the programs (11) and (12) simplify to

$$W_2(u;x) = \min_{0 \leq s \leq m}\{u_s + \frac{m-s}{m}x\} \; ; \; L_2(u;x) = \max_{0 \leq s \leq m}\{u_s - \frac{s}{m}x\}$$

The optimal bid x^* solves $W_2(u;x^*) = L_2(u;x^*)$. Because W_2 increases and L_2 decreases, both strictly, the inequality $x \geq x^*$ is equivalent to $W_2(u;x) \geq L_2(u;x)$. If $s' \leq s$ the inequality $u_s + \frac{m-s}{m}x \geq u_{s'} - \frac{s'}{m}x$ is automatic, therefore $x \geq x^*$ amounts to

$$u_\ell + \frac{m-\ell}{m}x \geq u_{\ell+k} - \frac{\ell+k}{m}x \text{ for all } k, \ell \geq 0 \text{ s. t. } \ell + k \leq m$$

$$\Longleftrightarrow x \geq \max_{0 \leq \ell+k \leq m} \frac{m}{m+k}(u_{\ell+k} - u_\ell)$$

which proves (22) and in turn (23). ■

Lemma 6 follows at once from this result.

Proof of statement *ii*) **in Lemma 7**

For $t \in [m]$ write $\Gamma_n(\theta|t) = \Gamma_n^{BS}(u^\theta|T)$ the n person B&S-guarantee when there are only t (identical) goods to distribute. Note that $\Gamma_n(\theta|t) = 0$ if $t < \theta$. We compute first $\Gamma_2(\theta|t)$ for $t \geq \theta$:

$$W_2(\theta;x|t) = \min\{1, \frac{t-\theta+1}{t}x\} \; ; \; L_2(\theta;x|t) = \max\{0, 1 - \frac{t}{m}x\}$$

$$\Longrightarrow \Gamma_2(\theta|t) = \frac{t-\theta+1}{t+1} \text{ for } \theta \leq t \leq m$$

For $n = 3$ equation (*14*) is simply: $L_3(\theta;x|m) = \max\{0, 1 - \frac{2\theta}{m}x\}$. By the concavity of $t \to \Gamma_2(\theta|t)$ (*13*) becomes

$$W_3(\theta;x|m) = \min_{\theta-1 \leq t \leq m}\{\Gamma_2(\theta|t) + \frac{m-t}{m}x\} = \min\{\frac{m-\theta+1}{m}x, \frac{1}{\theta+1} + \frac{m-\theta}{m}x\}$$

after which one checks that the graph of L_3 intersects that of W_3 on the line $x \to \frac{m+1-\theta}{m}x$, and finally $\Gamma_3(\theta|m) = \frac{m+1-\theta}{m+1+\theta}$ with the optimal bid $x^* = \frac{m}{m+1+\theta}$. The general inductive step works in exactly the same way with the recursive equations (13),(14).

11.4 Proof of Proposition 5 statement $i)$

We assume $n = 2$ and omit for brevity the straightforward induction argument extending the result to any n.

Fix any $u \in \mathcal{M}^+$; from the proof of Lemma 4 (and Lemma 9 in that proof) we know that $W_2(u;x)$ reaches $u(A)$ at some finite value denoted $\widetilde{x}(u)$: $W_2(u;\cdot)$ increases strictly up to $\widetilde{x}(u)$ after which it is flat. Agent u's optimal bid $x^*(u)$ in B&S is strictly below $\widetilde{x}(u)$ (because $W_2(u;\widetilde{x}(u)) = u(A) > L_2(u;\widetilde{x}(u))$.

Write now for brevity $\partial_a^+ u = \max_{\varnothing\subseteq S\subseteq A} \partial_a u(S)$ and $\partial_a^- u = \min_{\varnothing\subseteq S\subseteq A} \partial_a u(S)$.

Step 1 Fix u and $x \leq \widetilde{x}(u)$ and suppose that in the program (11) an optimal price is $p \in \Delta(x)$. Then $p_a \leq \partial_a^+ u$ for all a.

Proof by contradiction: we assume $p_a > \partial_a^+ u$ for some a and define a new price p' s. t. $p'_a = p_a - \varepsilon$ and $p'_b = p_b$ otherwise; we choose $\varepsilon > 0$ small enough that $p'_a > \partial_a^+ u$ still holds. For some $T \in 2^A$ we have $\min_{\varnothing\subseteq S\subseteq A}(u(S) - p'_S) = u(T) - p'_T$. This implies $a \in T$ otherwise adding a to T would contradict the optimality of T. We compute now

$$W_2(u;x-\varepsilon) \geq u(T) - p'_T + (x-\varepsilon) = u(T) - p_T + x$$

$$\geq \min_{\varnothing\subseteq S\subseteq A}(u(S) - p_S) + x = W_2(u;x)$$

We see that $W_2(u;\cdot)$ is flat before x therefore $x > \widetilde{x}(u)$ contradicting the choice of x.

Step 2 Assume u_1 dominates u_2 strictly.

A first consequence is $L_2(u_1;x) > L_2(u_2;x)$ for all $x \leq x^*(u_2)$. Indeed $u_1(S) - p_S > u_2(S) - p_S$ for all non empty S and $p \in \Delta(x)$, and $L_2(u_2;x)$ is positive therefore for any $p \in \Delta(x)$ the maximum of $u_2(S) - p_S$ is achieved at some non empty S.

Next we pick $p \in \Delta(x^*(u_2))$ optimal in (11) for u_2. By step 1 and inequality $x^*(u_2) < \widetilde{x}(u_2)$ we have $p_a \leq \partial_a^+ u_2 < \partial_a^- u_1$ for all a, implying $u_1(S) > p_S$ for all non empty S. Therefore

$$W_2(u_1;x^*(u_2)) \geq \min_{\varnothing\subseteq S\subseteq A}(u(S) - p_S) + x^*(u_2) = x^*(u_2)$$

Because $W_2(u_1; y) \leq y$ for all y we see that $W_2(u_1; y) = y \geq W_2(u_2; y)$ for all $y \leq x^*(u_2)$.

Gathering the first and last statements in this step we conclude that $L_2(u_1; \cdot)$ and $W_2(u_1; \cdot)$ intersect beyond $x^*(u_2)$ so agent u_1's safe bid makes her the Seller in stage 2. We showed a few lines ago $u_1(S) > p_S$ for any S and any possible price charged by agent u_2 therefore agent u_1 will buy all the goods and the proof is complete. ■

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