Sequential implementation without commitment

Takashi Hayashi
University of Glasgow
E-mail: takashi.hayashi@glasgow.ac.uk

Michele Lombardi
University of Glasgow
E-mail: michele.lombardi@glasgow.ac.uk

May 31, 2016

Abstract

In a finite-horizon intertemporal setting, in which society needs to decide and enforce a socially optimal outcome in each period without being able to commit to future ones, the paper examines problems of implementing dynamic social choice processes. A dynamic social choice process is a social choice function (SCF) that maps every admissible state into a socially optimal outcome on the basis of past outcomes. A SCF is sequentially implementable if there exists a sequence of mechanisms (with observed actions and with simultaneous moves) such that for each possible state of the environment, each (pure strategy) subgame perfect (Nash-)equilibrium of games played sequentially by the same individuals in that state generates the outcome prescribed by the SCF for that state, *at every history.* The paper identifies necessary conditions for SCFs to be sequentially implemented, *sequential decomposability* and *sequential Maskin monotonicity*, and shows that they are also sufficient under auxiliary conditions when there are three or more individuals. It provides an account of welfare implications of the sequential implementability in the contexts of sequential trading and sequential voting.

Introduction

The goal of implementation theory is to study the relationship between outcomes in a society and the mechanisms under which those outcomes arise. Mechanisms are rules according to which outcomes are determined and enforced (constitutions, voting rules, contracts, etc.). The focus of the theory is thus to design mechanisms for which the strategic properties induce individuals to choose actions that lead to the desired outcomes.[1]

In this paper, we study implementation problems in a finite-horizon multi-period setting with complete information where:

- The information held by the individuals is summarized in the concept of a *state*. The true state is common knowledge among the individuals but is unknown to the planner. At each admissible state, each individual possesses a complete and transitive preference relation - not necessarily separable - over *sequences* of outcomes.
- The planner needs to decide and enforce a socially optimal outcome in each period which depends on private information held by various individuals and on past outcomes. Thus, the problem of the planner is to design a sequence of mechanisms to implement a *dynamic social choice process*. We assume that the planner does *not* learn the true state by observing past choices.
- Individuals are *unable* to make binding agreements about future outcomes.

Non-separability of preferences of individuals and their inability to make commitments are what distinguish our model from existing dynamic implementation models.

Such a lack of commitment by individuals can be found in many real life situations. Examples, in particular, are to be found in the political arena, where in each period we can typically cast a vote against the current tax policy without being able to vote for future ones. In the literature of positive political economy, it is known that such lack of commitment leads to inefficient outcomes. Similar examples can be found in situations where there is no supranational legal framework, such as in the context of international trade.

[1] For a thorough discussion, we refer to Jackson (2001) and Maskin and Sjöström (2002).

Moreover, such a lack of commitment can also be found in the standard model of sequential trading due to Radner (1972). Indeed, the basic Arrow-Debreu (1954)'s model assumes that commodities are traded only once and for all. In its dynamic variant, that assumption implies that trading takes place at the initial date, and there are no further trades in the future. This certainly does not resemble a realistic picture of trade because trade takes place, to a large extent, sequentially over time. Moreover, that feature of Arrow-Debreu's model relies on the hypothesis that individuals trust each other to honour their promises (Gale 1978; 1982). A better model for trading is thus the model of sequential trading, where individuals can make only one-period-ahead trading arrangements.[2] As of today, however, there has not been offered any strategic or mechanism-design-theoretic foundation for this model, to our knowledge. The primary reason is that, in the model of sequential trading, prices and allocations are defined only on an equilibrium path. Therefore, it is still unclear how those economic variables are formed when traders make mistakes. This paper is an attempt to address this issue as well.

Furthermore, individuals' preferences for sequences of social decisions are generally not time-separable, in the sense that preferences for future social decisions depend on current social decisions, and preferences for current social decisions depend on what social decisions are to be made in the future. Even when an individual's preference for sequences of consumptions is time-separable, her preference for sequences of net trades does need to be time-separable. For example, preferences for how much to save in future periods depend on how much to save in the current period, and vice versa. Likewise, preferences for future tax rates naturally depend on the choice of the current tax rate because the underlying saving behavior will change when we change the current tax rate, which in turn changes preferences for future tax rates, and vice versa.

Despite the fact that non-separability of players' preferences and players' lack of commitment are prominent in political economy, business and economics, they have received scant attention in implementation theory. This paper builds an implementation framework in which they are taken into account and in which:

[2] Arrow (1964) was the first to observe that sequential trading and trading at a single point in time are equivalent when markets are complete.

1. The objective is to implement *dynamic social choice processes.* A dynamic social choice process is a SCF that maps every admissible state into a socially optimal outcome on the basis of past outcomes.

2. A *sequential mechanism* is a finite sequence of mechanisms with observed action, each one being a mechanism with simultaneous moves. These mechanisms are played sequentially by the same players. The convention that each mechanism is played in a distinct period is adopted so that period-t mechanism is played in period t. The mechanism played in a given period depends on the past history of the mechanisms and on the players' corresponding actions. Period-t mechanism requires agents to report only the information pertaining to period-t problem. Sequential rationality is common knowledge between the players at every period.

3. The implementation condition is that the game induced by the sequential mechanism has pure strategy subgame perfect (Nash-)equilibria (SPE) such that each SPE strategy profile generates the social outcome prescribed by the SCF at every history. When such a mechanism exists, we say that the SCF is *sequentially implementable.*

Thus, we are *not* interested in implementing an outcome in a given period or a sequence of outcomes but rather we focus on those that are suggested by a dynamic social choice process (think, for example, of Radner equilibria). In addition, our framework allows us to analyze what society should do even after it makes a mistake as part of the social choice objective. Because of non-separability of preferences for intertemporal social decisions, we assume that what society should choose depends on past outcomes. Therefore, this paper adds to the literature of implementation in SPE as it enables us to define relevant economic variables in equilibrium as well as out of equilibrium (see, e.g. Moore and Repullo, 1988; Abreu and Sen, 1990; Herrero and Srivastava, 1992; Vartiainen, 2007). In contrast to Jackson and Palfrey (2001), in our framework players cannot ask to replay the mechanism played in a given period if they do not like the outcome that it recommends.

In the literature on repeated implementation, the available characterization results rely on the assumption that individuals possess time-separable preferences and on the assumption that each individual's discount factor is known to the planner (Kalai and Ledyard, 1998;

Chambers, 2004; Lee and Sabourian, 2011; Mezzetti and Renou, 2016). The first common assumption has been criticized in the literature (Jackson, 2001). Although we agree that results in repeated implementation set-ups have brought new insights on how to design dynamic resource allocation mechanisms, we pursue a research direction that complements this literature since the time-preference of individuals is part of the information which is unknown to the planner and needs to be elicited and since, moreover, individuals' preferences for social decisions are not necessarily separable.

For a finite horizon T and via backward induction, the paper shows that a SCF that is sequentially implementable satisfies two conditions, *sequential decomposability* and *sequential Maskin monotonicity*. Sequential decomposability decomposes the dynamic implementation problem into several "apparently static" implementation problems, one for each period, where every allowable individual's preference for outcome paths is decomposed into T marginal orderings on the basis of past outcomes, as well as of future choices, and where the SCF is decomposed into T marginal SCFs. Thus, sequential implementation must be done *as if* the planner solves one "static implementation problem" in each period.

Sequential Maskin monotonicity is basically an adaptation of the standard invariance condition due to Maskin (1999), now widely referred to as Maskin monotonicity, to each period-t implementation problems. It coincides with Maskin monotonicity if and only if there is only one period/stage. We also prove that if a SCF satisfies our properties and two auxiliary conditions that are reminiscent of the so-called no veto-power condition, then it can be sequentially implemented when there are three or more individuals. This result is obtained by devising a canonical mechanism for each period, where each individual chooses a profile of marginal orderings as part of her strategy choice.

Further, the paper provides an account of welfare implications of its sufficiency result in the context of sequential trading and sequential voting.

Firstly, we consider a borrowing-lending model with no liquidity constraints, in which individuals trade in spot markets and transfer wealth between any two periods by borrowing and lending. In this set-up, intertemporal pecuniary externalities arise because trades in the current period change the spot price of the next period, which, in turn, affects its associated equilibrium allocation. The quantitative implication of this is that every individual's marginal preference concerns not only her own consumption/saving behavior but also the consumption/saving behavior of all other individuals. We show that, under such a pecuniary externality, the standard dynamic competitive equilibrium solution is not sequentially implementable because it fails to satisfy sequential decomposability. However, we have also identified preference domains – which involve no pecuniary externalities – for which the no-commitment version of the dynamic competitive equilibrium solution is definable and sequentially implementable.

Secondly, we consider a bi-dimensional policy space where an odd number of individuals vote sequentially on each dimension and where an ordering of the dimensions is exogenously given. We assume that each voter's type space is unidimensional, that a majority vote is organized around each policy dimension and that the outcome of the first majority vote is known to the voters at the beginning of the second voting stage. This sequential resolution is common in political economy models (see, e.g., Persson and Tabellini, 2000). In this environment, we show that the simple majority solution, which selects the Condorcet winner in each voting stage, is sequentially implementable. In this process, we explicitely state the conditions on the utility function of each voter that are needed for this SCF to be well-defined and show that this is the case. As established by De Donder et al (2012) for the case where there is a continuum of voters, the assumption that both dimensions are strategic complements, as well as the requirement that the marginal utility of both dimensions is increasing in the type of the voter, are particularly important for guaranteeing the existence of a Condorcet winner in each voting stage.

The remainder of the paper is organized as follows. Section 2 sets out the theoretical framework and outlines the basic implementation model. Necessary and sufficient conditions are presented in section 3. Section 4 covers sequentially implementable SCFs in the context of trading and voting problems. Section 5 concludes. Appendix includes proofs not in the main body.

2. Basic framework

Let us imagine that a set of individuals indexed by $i \in \mathcal{I} \equiv \{1, \cdots, I\}$ have to decide what outcome is best in each time period indexed by $t \in \mathcal{T} \equiv \{1, 2, \cdots, T\}$. Let us denote the universal set of period-t outcomes by X^t, with x^t as a typical outcome. Thus, the universal set of outcome paths available to individuals is the space:

$$\mathcal{X} \subseteq \prod_{t \in \mathcal{T}} X^t,$$

with x as a typical outcome path. The t-head x^{-t} is obtained from the path $x \in \mathcal{X}$ by omitting the last t components, that is, $x^{-t} \equiv (x^1, \cdots, x^{t-1})$, the t-tail is obtained from x by omitting the first $t-1$ components, that is, $x^{+t} \equiv \left(x^t, \cdots, x^T\right)$, and we identify (x^{-t}, x^{+t}) with x. The same notational convention will be followed for any profile of outcomes. We will refer to the t-head x^{-t} as the past outcome history x^{-t}.

The feasible set of period-$t+1$ outcomes available to individuals depends upon past outcome history $x^{-(t+1)}$, that is, $X^{t+1}\left(x^{-(t+1)}\right) \subseteq X^{t+1}$ for every period $t \neq T$.

We write $\mathcal{F}^t$ for the collection of functions defined as follows:

$$\mathcal{F}^t \equiv \left\{f^t | f^t : \mathcal{X}^{-t} \to X^t \text{ such that } f^t\left[x^{-t}\right] \in X^t\left(x^{-t}\right)\right\}, \quad \text{for all } t \neq 1.$$

We also write $\mathcal{F}$ for the product space $X^1 \times \mathcal{F}^2 \times \cdots \times \mathcal{F}^T$.

The information held by the individuals is summarized in the concept of a state, which is a complete description of the variable characterizing the environment. Write Θ for the domain of possible states, with θ as a typical state. For every period $t \geq 2$, the description of the variable characterizing the environment after the outcome history x^{-t} is denoted by $\theta|x^{-t}$. Moreover, for every $t \geq 2$ we write $\theta|x^{-t}, x^{+(t+1)}$ for a complete description of the variable characterizing the environment in period t after the outcome history x^{-t} and the future sure outcome path $x^{+(t+1)}$.

In the usual fashion, individual i's preferences in state θ are given by a complete and transitive binary relation, subsequently an ordering, $R_i(\theta)$ of elements of $\mathcal{X}$. The corresponding strict and indifference relations are denoted by $P_i(\theta)$ and $I_i(\theta)$, respectively. The statement $xR_i(\theta)\,y$ means that agent i judges x to be at least as good as y. The statement $xP_i(\theta)\,y$ means that agent i judges x better than y. Finally, the statement $xI_i(\theta)\,y$ means that agent i judges x and y as equally good, that is, she is indifferent between them.

2.1 Implementation model

Dynamic social objectives

The goal of the central designer is to implement a social choice function (SCF) $f : \Theta \to \mathcal{F}$ that assigns to each state θ a dynamic "socially optimal" process

$$f[\theta] = \left(f^1[\theta], f^2[\theta|\cdot], \cdots, f^T[\theta|\cdot]\right),$$

where:

- $f^1[\theta] \in X^1$ is the period-1 socially optimal outcome and
- $f^t[\theta|\cdot] \in \mathcal{F}^t$ is the period-t socially optimal process that selects the socially optimal outcome $f^t[\theta|x^{-t}]$ in period $t \geq 2$ at the state θ after the past outcome history $x^{-t} \in \mathcal{X}^{-t}$.

To save writing, for every period $t \neq 1$ and every past outcome history x^{-t}, we write $f^{+t}[\theta|x^{-t}]$ for the t-tail path of socially optimal outcomes in state θ that follows the past outcome history x^{-t}, whose period-τ element is the value of the composition $f^\tau \circ f^{\tau-1} \circ \cdots \circ f^t$ at $\theta|x^{-t}$; that is:

$$f^{+t}\left[\theta|x^{-t}\right] \equiv \left(f^\tau\left[\theta|x^{-t}\right]\right)_{\tau \geq t}$$

where $f^\tau[\theta|x^{-t}] \equiv (f^\tau \circ f^{\tau-1} \circ \cdots \circ f^t)[\theta|x^{-t}]$ for every period $\tau \geq t$. The image or range of the period-t function f^t of the SCF f at the past outcome history x^{-t} is the set:

$$f^t\left[\Theta|x^{-t}\right] \equiv \left\{f^t\left[\theta|x^{-t}\right]|\theta \in \Theta\right\}, \quad \text{for every } x^{-t} \in \mathcal{X}^{-t} \text{ with } t \neq 1.$$

The image or range of the period-1 function f^1 of the SCF f is the set $f^1[\Theta] \equiv \{f^1[\theta] | \theta \in \Theta\}$.

Sequential mechanisms

The central designer delegates the choice to individuals according to a sequential (or multi-period) mechanism (or game form) with observed actions and simultaneous moves and then commits to that choice. In other words, we assume that the actions of every individual are perfectly monitored by every other individual as well as that every individual chooses an action in period t without knowing the period t action of any other individual.

More formally, in the first period all individuals $i \in \mathcal{I}$ choose actions from nonempty choice sets $A_i(h^1)$, where $h^1 \equiv \varnothing$ denotes the initial history. Thus, the period-1 action space is the product space:

$$A\left(h^1\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(h^1\right),$$

with $a(h^1) \equiv (a_1^1(h^1), \cdots, a_I^1(h^1))$ as a typical period-1 action profile.

In the second period, individuals know the history $h^2 \equiv a^1$, and the actions that every individual $i \in \mathcal{I}$ has available in period 2 depends on what has happened previously. Then, let $A_i(h^2)$ denote the period-2 nonempty action space of individual i when the history is h^2 and let $A(h^2)$ denote the corresponding period-2 nonempty action space, which is defined by:

$$A\left(h^2\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(h^2\right),$$

with $a(h^2) \equiv (a_1(h^2), \cdots, a_I(h^2))$ as a typical period-2 action profile.

Continuing iteratively, we can define h^t, the (nontrivial) history at the beginning of period $t > 1$, to be the list of $t-1$ action profiles,

$$h^t \equiv \left(a^1, a^2, \cdots, a^{t-1}\right),$$

identifying actions played by individuals in periods 1 through $t-1$. We let $A_i(h^t)$ be individual i's nonempty action set in period t when the history is h^t and let $A(h^t)$ be the corresponding period-t action space, which is defined by:

$$A\left(h^t\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(h^t\right),$$

with $a\left(h^t\right) \equiv \left(a_1\left(h^t\right), \cdots, a_I\left(h^t\right)\right)$ as a typical profile of actions.

We assume that in each period t, every individual knows the history h^t, this history is common knowledge at the beginning of period t, and that every individual $i \in I$ chooses an action from the action set $A_i\left(h^t\right)$. We also assume that in each period t, all individuals $i \in I$ choose actions simultaneously.

We let H^t be the set of all period-t histories, where we define H^1 to be the null set, and let

$$H \equiv \bigcup_{t \in \mathcal{T}} H^t$$

be the set of all possible histories.

For any nontrivial history $h^t \equiv \left(a^1, a^2, \cdots, a^{t-1}\right) \in H$, define a subhistory of h^t to be a sequence of the form $\left(a^1, \cdots, a^m\right)$ with $1 \leq m \leq t-1$, and the trivial history consisting of no actions is denoted by $\varnothing$.

The delegation to individuals is made by means of a sequential mechanism $\Gamma \equiv (\mathcal{I}, H, A\left(H\right), g)$, where H is the set of all possible histories, $A\left(H\right)$ is the set of all profiles of actions available to individuals, defined by

$$A\left(H\right) \equiv \bigcup_{h \in H} A\left(h\right),$$

and $g \equiv \left(g^1, \cdots, g^T\right)$ is a sequence of outcome functions, one for each period $t \in \mathcal{T}$, with the property that: a) the outcome function g^1 assigns to period-1 action profile $a\left(h^1\right) \in A\left(h^1\right)$ a unique outcome in X^1, and b) for every period $t \neq 1$ and every nontrivial history $h^t \equiv \left(a^1, a^2, \cdots, a^{t-1}\right) \in H^t$, the outcome function g^t assigns to each period-t action profile $a\left(h^t\right) \in A\left(h^t\right)$ a unique outcome in $X^t\left(g^{-t}\left(h^t\right)\right)$.

The sequential submechanism of a sequential mechanism Γ that follows the history h^t is the sequential mechanism

$$\Gamma\left(h^t\right) \equiv \left(\mathcal{I}, H|h^t, A\left(H|h^t\right), g^{+t}\right),$$

where $H|h^t$ is the set of histories for which h^t is a subhistory for every $h \in H|h^t$,

$$A\left(H|h^t\right) \equiv \bigcup_{h \in H|h^t} A\left(h\right)$$

is the set of all profiles of actions available to individuals from period t to period T, and g^{+t} is t-tail of the sequence g that begins with period t after the history h^t such that for every $h^T \equiv \left(a^1, \cdots, a^{T-1}\right) \in H^T|h^t$ and every $a\left(h^T\right) \in A\left(h^T\right)$ it holds that $g\left(h^T, a\left(h^T\right)\right) = \left(g^{-t}\left(h^T, a\left(h^T\right)\right), g^{+t}\left(h^T, a\left(h^T\right)\right)\right)$.

Sequential implementation

A sequential mechanism Γ and a state θ induce a sequential game (Γ, θ) (with observed actions and simultaneous moves within each period). The sequential subgame of the sequential game (Γ, θ) that follows the history $h^t \in H$ is the sequential game $(\Gamma(h^t), \theta)$.

Let $A_i \equiv \bigcup_{h \in H} A_i(h)$ be the set of all actions for individual $i \in \mathcal{I}$. A (pure) strategy for individual i is a map $s_i : H \rightarrow A_i$ with $s_i(h) \in A_i(h)$ for every history $h \in H$. Individual i's space of strategies, S_i, is simply the space of all such s_i.

A strategy profile $s \equiv (s_1, \cdots, s_I)$ is a list of strategies, one for each individual $i \in \mathcal{I}$. The strategy profile s_{-i} is obtained from s by omitting the ith component, that is, $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_I)$, and we identify (s_i, s_{-i}) with s.

For any strategy s_i of individual i and any history h^t in the sequential mechanism Γ, the strategy that s_i induces in the sequential subgame $(\Gamma(h^t), \theta)$ is denoted by $s_i|h^t$. Individual i's space of strategies that follows history h^t is denoted by $S_i|h^t$. The period-t strategy of individual i is sometimes denoted by s_i^t.

For every sequential game (Γ, θ), the strategy profile s^* is a Nash equilibrium of (Γ, θ) if for every individual $i \in \mathcal{I}$ it holds that:

$$g\left(s_i^*, s_{-i}^*\right) R_i(\theta)\, g\left(s_i, s_{-i}^*\right) \text{ for every } s_i \in S_i.$$

Let $NE(\Gamma, \theta)$ denote the set of Nash equilibrium strategy profiles of (Γ, θ).

Moreover, for every sequential game (Γ, θ) and every nontrivial history $h^t \in H$, the strategy profile $s^*|h^t$ is a Nash equilibrium of $(\Gamma(h^t), \theta)$ if for every individual $i \in \mathcal{I}$ and every past outcome history $x^{-t} \in \mathcal{X}^{-t}$ it holds that:

$$\left(x^{-t}, g^{+t}\left(s_i^*|h^t, s_{-i}^*|h^t\right)\right) R_i(\theta) \left(x^{-t}, g^{+t}\left(s_i|h^t, s_{-i}^*|h^t\right)\right) \text{ for every } s_i|h^t \in S_i|h^t.$$

Let $NE(\Gamma(h^t), \theta)$ denote the set of Nash equilibrium strategy profiles of $(\Gamma(h^t), \theta)$.

A strategy profile s^* is a *subgame perfect equilibrium* (SPE) of a sequential game (Γ, θ) if it holds that:

$$s^*|h^t \text{ is a Nash equilibrium of } \left(\Gamma\left(h^t\right), \theta\right), \qquad \text{for every history } h^t \in H.$$

Let $SPE(\Gamma, \theta)$ denote the set of SPE strategy profiles of (Γ, θ), with s^θ as a typical element.

Definition 1 A sequential mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ implements the SCF $f : \Theta \rightarrow \mathcal{F}$ in SPE if for every $\theta \in \Theta$,

$$f^1[\theta] = g^1(SPE(\Gamma, \theta)), \text{ and}$$

$$f^t\left[\theta | g^{-t}\left(h^t\right)\right] = g^t\left(SPE\left(\Gamma\left(h^t\right), \theta\right)\right), \quad \text{for every } h^t \in H^t \text{ with } t \neq 1.$$

If such a mechanism exists, the SCF f is *sequentially implementable*.

3. Necessary and sufficient conditions

3.1 Sequential decomposability

In this section, we first propose a property, *sequential decomposability*, and show that this is a necessary condition for sequential implementation. While this property is heavy in notation, its idea is simple. This property decomposes the dynamic implementation problem into several "apparently static" implementation problems, one for each period, where every allowable individual i's preference for outcome paths is decomposed into T-periods marginal orderings and where the SCF is decomposed into T-periods *marginal* SCFs.

This necessary condition is derived by using the approach developed by Moore and Repullo (1990) and thus it is stated in terms of the existence of certain sets. These sets are denoted by $\mathcal{Y}^{-t}$, Y^1 and $Y^t(y^{-t})$ and represent respectively the set of feasible past outcome histories up to period $t \neq 1$, the set of period-1 attainable outcomes and the set of period-t attainable outcomes after the past outcome history y^{-t}. Moreover, the condition consists of three parts: the first part characterises the period-T implementation problem, the second one relates to the implementation problem of period $t \neq 1, T$ and the third one relates to the period-1 implementation problem.

Solving backward, for any feasible past outcome history y^{-T}, the *period-T marginal ordering* of individual i in state θ at y^{-T}, that is, at $\theta|y^{-T}$, denoted by $R_i\left[\theta|y^{-T}\right]$, is equal to:

$$y^T R_i\left[\theta|y^{-T}\right] z^T \iff \left(y^{-T}, y^T\right) R_i(\theta)\left(y^{-T}, z^T\right), \quad \text{for every } y^T, z^T \in Y^T\left(y^{-T}\right). \tag{1}$$

We denote by $R\left[\theta|y^{-T}\right]$ the profile of period-T marginal orderings at $\theta|y^{-T}$ and by $\mathcal{D}\left[\Theta|y^{-T}\right]$ the period-T domain of marginal orderings at $\Theta|y^{-T}$, that is:

$$\mathcal{D}\left[\Theta|y^{-T}\right] \equiv \left\{R\left[\theta|y^{-T}\right] | \theta \in \Theta\right\}. \tag{2}$$

Therefore, the first part of the condition can be formulated as follows:

(i) The preference domain $\mathcal{D}\left[\Theta|y^{-T}\right]$ is not empty, and there is a period-T function φ^T : $\mathcal{D}\left[\Theta|y^{-T}\right] \to Y^T\left(y^{-T}\right)$ such that:

$$\varphi^T\left(R\left[\theta|y^{-T}\right]\right) = f^T\left[\theta|y^{-T}\right], \quad \text{for every } \theta \in \Theta. \tag{3}$$

To introduce the second part of the condition, let us suppose that in our way back to period 1 we have reached period $t \neq 1, T$ and that y^{-t} is a feasible past outcome history. Given that in our framework sequential rationality is common knowledge between the players (at every stage of the game) and given that the objective of the planner is to implement a dynamic social choice process prescribed by the SCF f, every player will "look ahead" and a period-t outcome y^t will be evaluated at the past outcome history y^{-t} as well as at the future sure outcome path $f^{+(t+1)}$ prescribed by the SCF in response to the outcome history path (y^{-t}, y^t). On this basis, the *period-t marginal ordering* of individual i in state θ at the past outcome history y^{-t} and at the future sure outcome path prescribed by the social process $f^{+(t+1)}$, that is, at $\theta|y^{-t}, f^{+(t+1)}$, denoted by $R_i\left[\theta|y^{-t}, f^{+(t+1)}\right]$, is equal to:

$$y^t R_i\left[\theta|y^{-t}, f^{+(t+1)}\right] z^t \iff \left(y^{-t}, y^t, f^{+(t+1)}\left[\theta|\left(y^{-t}, y^t\right)\right]\right) R_i\left(\theta\right) \left(y^{-t}, z^t, f^{+(t+1)}\left[\theta|\left(y^{-t}, z^t\right)\right]\right), \tag{4}$$

for every $y^t, z^t \in Y^t\left(y^{-t}\right)$.

Let us denote by $R\left[\theta|y^{-t}, f^{+(t+1)}\right]$ the profile of period-t marginal orderings at $\theta|y^{-t}, f^{+(t+1)}$ for $t \neq 1, T$ and by $\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$ the period-t domain of marginal orderings at $\Theta|y^{-t}, f^{+(t+1)}$, that is:

$$\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right] \equiv \left\{R\left[\theta|y^{-t}, f^{+(t+1)}\right] | \theta \in \Theta\right\}. \tag{5}$$

Therefore, as for the first part of the condition, the second part can be stated as follows:

(ii) The preference domain $\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$ is not empty, and there is a period-t function $\varphi^t : \mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right] \to Y^t\left(y^{-t}\right)$ such that:

$$\varphi^t\left(R\left[\theta|y^{-t}, f^{+(t+1)}\right]\right) = f^t\left[\theta|y^{-t}\right], \quad \text{for every } \theta \in \Theta. \tag{6}$$

Reasoning like that used in the preceding paragraphs, the *period*-1 *marginal ordering* of individual i in state θ at the outcome path prescribed by the social process f^{+2}, that is, at $\theta|f^{+2}$, denoted by $R_i\left[\theta|f^{+2}\right]$, is equal to:

$$y^1 R_i\left[\theta|f^{+2}\right] z^1 \iff \left(y^1, f^{+2}\left[\theta|y^1\right]\right) R_i\left(\theta\right) \left(z^1, f^{+2}\left[\theta|z^1\right]\right), \quad \text{for every } y^1, z^1 \in Y^1. \tag{7}$$

Denoting the profile of period-1 marginal orderings at $\theta|f^{+2}$ by $R\left[\theta|f^{+2}\right]$ and defining the period-1 domain of marginal orderings at $\Theta|f^{+2}$ by:

$$\mathcal{D}\left[\Theta|f^{+2}\right] \equiv \left\{R\left[\theta|f^{+2}\right] | \theta \in \Theta\right\}, \tag{8}$$

the third part of sequential decomposability can be stated as follows:

(iii) The preference domain $\mathcal{D}[\Theta|f^{+2}]$ is not empty, and there is a period-1 function $\varphi^1 : \mathcal{D}[\Theta|f^{+2}] \to Y^1$ such that:

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) = f^1[\theta], \quad \text{for every } \theta \in \Theta. \tag{9}$$

In summary, if the SCF f is sequentially implementable, then the following condition must be satisfied:

Definition 2 The SCF $f : \Theta \to X$ is *sequentially decomposable* if there is a collection of outcome spaces $\{\mathcal{Y}^{-t}\}_{t\in\mathcal{T}\setminus\{1\}}$, there is a period-1 outcome space $Y^1 = \mathcal{Y}^{-2}$ and there is a collection of period-t outcome spaces $\left\{\{Y^t(y^{-t})\}_{y^{-t}\in\mathcal{Y}^{-t}}\right\}_{t\in\mathcal{T}\setminus\{1\}}$ such that $f^1[\Theta] \subseteq Y^1$ and $f^t[\Theta|y^{-t}] \subseteq Y^t(y^{-t})$ for every $t \neq 1$; that for every $t \neq 1$:

$$y^{-t} \in \mathcal{Y}^{-t} \iff y^\tau \in Y^\tau\left(y^{-\tau}\right) \text{ for every } 2 \leq \tau \leq t;$$

that *(i)* is satisfied for every $y^{-T} \in \mathcal{Y}^{-T}$; that *(ii)* is satisfied for every $y^{-t} \in \mathcal{Y}^{-t}$ with $t \neq 1, T$; and that *(iii)* is satisfied.

Our first main result can thus be stated as follows:

Theorem 1 If $I \geq 2$ and the SCF $f : \Theta \to \mathcal{F}$ is sequentially implementable, then it is sequentially decomposable.

Proof. See Appendix. ■

3.2 Sequential Maskin monotonicity

A condition that is central to the Nash implementation thanks to Maskin (1999) is an invariance condition, now widely referred to as Maskin monotonicity. This condition says that if an outcome x is socially optimal at the state θ and this x does not strictly fall in preference for anyone when the state is changed to θ', then x must remain a socially optimal outcome at θ'. An equivalent statement of Maskin monotonicity follows the reasoning that if x is socially optimal at θ but not socially optimal at θ', then the outcome x must have fallen strictly in someone's ordering at the state θ' in order to break the Nash equilibrium via some deviation. Therefore, there must exist some (outcome-)preference reversal if a Nash equilibrium strategy profile at θ is to be broken at θ'. Let us formalize that condition as follows: For any state θ and any individual i and any outcome $x \in X$, the weak lower contour set of $R_i(\theta)$ at x is defined by $L(x, R_i(\theta)) \equiv \{y \in X | x R_i(\theta) y\}$. Therefore:

Definition 3 The SCF $F : \Theta \to X$ is *Maskin monotonic* provided that for all $x \in X$ and all $\bar{\theta}, \theta \in \Theta$, if $L(f(\bar{\theta}), R_i(\bar{\theta})) \subseteq L(f(\bar{\theta}), R_i(\bar{\theta}))$ for every $i \in \mathcal{I}$, then $f(\bar{\theta}) = f(\theta)$.

We basically require an adaptation of Maskin monotonicity to each "apparently static" implementation problem. In other words, sequential Maskin monotonicity requires that every period-t marginal social choice function φ^t that results from the decomposition of the SCF is Maskin monotonic. Therefore, the condition of sequential Maskin monotonicity can be stated as follows:

Definition 4 The sequentially decomposable SCF $f : \Theta \to \mathcal{F}$ is *sequentially Maskin monotonic* provided that: *(i)* the period-T function φ^T over $\mathcal{D}\left[\Theta | y^{-T}\right]$ is Maskin monotonic for every $y^{-T} \in \mathcal{Y}^{-T}$; *(ii)* for every $t \neq 1, T$, the period-t function φ^t over $\mathcal{D}\left[\Theta | y^{-t}, f^{+(t+1)}\right]$ is Maskin monotonic for every $y^{-t} \in \mathcal{Y}^{-t}$; *(iii)* the period-1 function φ^1 over $\mathcal{D}[\Theta | f^{+2}]$ is Maskin monotonic.

Our second main result is that only sequentially Maskin monotonic SCFs are sequentially implementable.

Theorem 2 If $I \geq 2$ and the SCF $f : \Theta \to \mathcal{F}$ is sequentially implementable, then it is sequentially Maskin monotonic.

Proof. See Appendix. ■

3.3 The characterization theorem

In the abstract Arrovian domain, the condition of no veto-power says that if an outcome is at the top of the preferences of all agents but possibly one, then it should be chosen irrespective of the preferences of the remaining agent: that agent cannot veto it. The condition of no veto-power implies two well-known conditions: *unanimity* and *weak no veto-power*. The property of unanimity can be stated as follows for an abstract outcome space X:

DEFINITION 5 The SCF $F : \Theta \rightarrow X$ satisfies *unanimity* provided that for all $\theta \in \Theta$ and all $x \in X$ if $xR_i(\theta)\,y$ for all $i \in \mathcal{I}$ and all $y \in X$, then $x = F(\theta)$. A SCF that satisfies this property is said to be a unanimous SCF.

In other words, it states that if an outcome is at the top of the preferences of all individuals, then that outcome should be selected by the SCF.

As a part of sufficiency, we require an adaptation of the above definition to each period-t implementation problem. In other words, sequential unanimity requires that each of period-t marginal social function φ^t defined over period-t domain of marginal orderings is unanimous. Thus, the condition can be stated as follows:

DEFINITION 6 A sequentially decomposable SCF $f : \Theta \rightarrow \mathcal{F}$ satisfies *sequential unanimity* provided that the following requirements hold: *(i)* the period-T function φ^T over $\mathcal{D}\left[\Theta|y^{-T}\right]$ is unanimous for every $y^{-T} \in \mathcal{Y}^{-T}$; *(ii)* for every $t \neq 1, T$, the period-t function φ^t over $\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$ is unanimous for every $y^{-t} \in \mathcal{Y}^{-t}$; *(iii)* the period-1 function φ^1 over $\mathcal{D}\left[\Theta|f^{+2}\right]$ is unanimous.

Furthermore, the condition of no veto-power implies the condition of weak no veto-power, which states that if an outcome x is socially optimal at the state $\bar{\theta}$ and if the state changes from $\bar{\theta}$ to θ in a way that under the new state an outcome y that was no better than x at $R_i\left(\bar{\theta}\right)$ for some agent i is weakly preferred to all outcomes in the weak lower contour set of $\bar{R}_i(\theta)$ at x according to the ordering $R_i(\theta)$ and this y is maximal for every other agent j in the set X according to $R_j(\theta)$, then this y should be socially optimal at θ. Formally, for an abstract outcome space X:

DEFINITION 7 A SCF $F : \Theta \rightarrow X$ satisfies *weak no veto-power* provided that for every $\bar{\theta}, \theta \in \Theta$ if $y \in L\left(f\left(\bar{\theta}\right), R_i\left(\bar{\theta}\right)\right) \subseteq L\left(y, R_i(\theta)\right)$ for some $i \in \mathcal{I}$ and $X \subseteq L\left(y, R_j(\theta)\right)$ for every $j \in \mathcal{I}\setminus\{i\}$, then $f(\theta) = y$.

As a part of sufficiency, we require the following adaptation of the weak no veto-power condition:

DEFINITION 8 A sequentially decomposable SCF $f : \Theta \rightarrow \mathcal{F}$ satisfies *sequential weak no veto-power* provided that the following requirements hold: *(i)* the period-T function φ^T over $\mathcal{D}\left[\Theta|y^{-T}\right]$ satisfies weak no veto-power for every $y^{-T} \in \mathcal{Y}^{-T}$; *(ii)* for every $t \neq 1, T$, the period-t function φ^t over $\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$ satisfies weak no veto-power for every $y^{-t} \in \mathcal{Y}^{-t}$; *(iii)* the period-1 function φ^1 over $\mathcal{D}\left[\Theta|f^{+2}\right]$ satisfies weak no veto-power.

Our characterization of sequentially implementable SCFs can thus be stated as follows:

THEOREM 3 If $I \geq 3$ and the SCF $f : \Theta \rightarrow \mathcal{X}$ is sequentially decomposable and sequentially Maskin monotonic and if the SCF satisfies sequential weak no veto-power as well as sequential unanimity, then it is sequentially implementable.

PROOF. See Appendix. ■

4. Implications

4.1 Sequential trading

In this section, we investigate sequentially implementable trading rules in a borrowing-lending model with no liquidity constraints, in which agents can transfer wealth between periods by making only one-period-ahead borrowing/lending arrangements. We consider the competitive equilibrium as the natural solution concept for our model, and we investigate whether or not it is sequentially implementable. We assume that markets are complete, and so Arrow-Debreu equilibrium and Radner equilibrium are equivalent for our model (on the equilibrium path).

For the sake of convenience, we assume that there are only *three* consumption periods (CPs), and so *two* trading periods (TPs), and that there is one perfectly divisible commodity in each CP. In TP1 agents transfer consumption between CP1 and CP2, and in TP2 they transfer consumption between CP2 and CP3. Let q^t be the TPt spot price.

Each agent i is endowed with an amount ω_i^t of the commodity in CPt. The total endowment of the commodity in CPt is denoted by ω^t. Agent i's consumption set is $\mathbb{R}_+^3$, and her consumption in CPt is denoted by c_i^t. In state θ, this agent has preference ordering $R_i(\theta)$ over consumption sequences in her consumption set. Endowments are given once and for all, and therefore an *economy* is described by a state θ.

The domain assumption is that at each economy $\theta \in \Theta$ agent i's preference ordering $R_i(\theta)$ is represented by an additively separable utility function

$$U_i(\theta, c_i^1, c_i^2, c_i^3) = v_i^1(\theta, c_i^1) + v_i^2(\theta, c_i^2) + v_i^3(\theta, c_i^3).$$

We focus on the set

$$H = \left\{ z \in \mathbb{R}^I | \sum_{i \in \mathcal{I}} z_i = 0 \right\},$$

which is the set of closed net trades. Thus, the set of closed net trade vectors for TPt can be defined by

$$Z^t = H^t \times H^{t+1}, \qquad \text{for } t = 1, 2.$$

A TPt net trade allocation is thus a vector $z^t = (z^{tt}, z^{tt+1})$ in Z^t, where the ith element z_i^{tt} of z^{tt} denotes agent i's net trade of consumption in CPt, and where the ith element z_i^{tt+1} of z^{tt+1} denotes agent i's net trade of consumption in CP$(t+1)$.

The set of feasible net trade allocations over the two trading periods is denoted by Z and defined by

$$Z = \left\{ (z^1, z^2) \in Z^1 \times Z^2 | \omega_i^1 + z_i^{11} \geq 0,\ \omega_i^2 + z_i^{12} + z_i^{22} \geq 0, \omega_i^3 + z_i^{23} \geq 0,\ \forall i \in \mathcal{I} \right\}.$$

The set of feasible TP1 net trade allocations is given by

$$\bar{Z}^1 = \{z^1 \in Z^1 | (z^1, z^2) \in Z \text{ for some } z^2 \in Z^2\},$$

while the set of TP2 net trade allocation, conditional on z^1, is given by

$$\bar{Z}^2(z^1) = \{z^2 \in Z^2 | (z^1, z^2) \in Z\}, \quad \text{for all } z^1 \in \bar{Z}^1.$$

This description of consumption sets, preferences, and feasible trade allocations over the two trading periods is common with the canonical, Arrow-Debreu general equilibrium model.

In economy $\theta \in \Theta$, agent i's preference ordering $R_i(\theta)$ over consumption sequences induces a preference ordering $\succcurlyeq_i^\theta$ over the set of feasible net trade allocations Z in the natural way: for all $z, \hat{z} \in Z$,

$$z \succcurlyeq_i^\theta \hat{z} \iff U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23}) \geq U_i(\theta, \omega_i^1 + \hat{z}_i^{11}, \omega_i^2 + \hat{z}_i^{12} + \hat{z}_i^{22}, \omega_i^3 + \hat{z}_i^{23}).$$

Though the preference ordering $R_i(\theta)$ exhibits separability over consumption sequences, the derived preference ordering over Z is typically non-separable since consumption in CP2 depends on net trades in both TP1 and TP2.

In contrast to the Arrow-Debreu setting, in sequential trading, trade takes place through time and agents face a sequence of budget sets, one at each TP. Thus, a competitive net trade equilibrium allocation for TP2 can be defined as follows:

Definition 9 For every economy $\theta \in \Theta$ and every $z^1 \in \bar{Z}^1$, the net trade allocation $f^2[\theta|z^1] \in \bar{Z}^2(z^1)$ constitutes a *TP2 competitive net trade allocation*, conditional on z^1, if there is a TP2 spot price $q^2[\theta|z^1]$ such that for every agent i this allocation $f^2[\theta|z^1]$ solves the following problem:

$$\operatorname*{Max}_{z^2 \in \bar{Z}^2(z^1)} \; U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23}), \qquad \text{subject to } z_i^{22} + q^2[\theta|z^1] z_i^{23} \leq 0. \quad (10)$$

Let $R_i^1[\theta, f^2]$ denote agent i's TP1 marginal preference ordering over the set of feasible TP1 net trade allocations and be defined by

$$x^1 R_i^1[\theta, f^2] y^1 \iff U_i(\theta, \omega_i^1 + x_i^{11}, \omega_i^2 + x_i^{12} + f_i^{22}[\theta|x^1], \omega_i^3 + f_i^{23}[\theta|x^1]) \quad (11)$$
$$\geq U_i(\theta, \omega_i^1 + y_i^{11}, \omega_i^2 + y_i^{12} + f_i^{22}[\theta|y^1], \omega_i^3 + f_i^{23}[\theta|y^1]), \qquad \text{for all } x^1, y^1 \in \bar{Z}^1.$$

In contrast to static pure exchange economies where each agent's preferences are defined over her own net trade vectors, in sequential trading, each individual must have preferences over TP1 net trade allocations. This is due to the presence of *intertemporal pecuniary externalities*. Indeed, an outcome of the trading rule in TP2 depends on the net trade allocation assigned in TP1, because trading in TP1 affects the values of endowments in the next trading period. Moreover, the marginal ordering $R_i^1[\theta, f^2]$ may be *non-convex*. In order for it to be a convex preference ordering, it is required that the TP2 function f^2 that maps every economy, conditional on past trades, into a TP2 net trade allocation be a concave function, but this requirement fails for any reasonable trading rule. As is known, although convexity is no more than a sufficient technical condition for things to work, it becomes extremely difficult to establish any reasonable solution once it is violated.

We may proceed in two ways. First, we can still define a concept of competitive equilibrium following the tradition of dynamic general equilibrium theory. Thus, a TP1 competitive net trade equilibrium allocation can be defined as follows:

Definition 10 For every economy $\theta \in \Theta$, a TP1 net trade allocation $f^1[\theta] \in \bar{Z}^1$ constitutes a *TP1 competitive net trade allocation* if there is a TP1 spot price $q^1[\theta]$ such that for every agent i the net trade allocation profile $(f^1[\theta], f^2[\theta|f^1[\theta]])$ solves the following problem:

$$\operatorname*{Max}_{z \in Z} \quad U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23})$$

subject to

$$\begin{aligned} &\text{(i)} \quad & z_i^{11} + q^1[\theta] z_i^{12} &\leq 0 \\ &\text{(ii)} \quad & z_i^{22} + q^2[\theta|z^1] z_i^{23} &\leq 0. \end{aligned}$$

This is consistent with the existing dynamic general equilibrium framework, in the sense that individuals take the price *path* as given. Note that it assumes that each individual perceives that her saving choice does not affect either TP1 spot price $q^1[\theta]$ or TP2 spot price $q^2[\theta|z^1]$. The *path* of consumptions given by this solution is equivalent to Arrow-Debreu and Radner equilibrium. However, this solution is not sequentially implementable. We prove this by means of an example.

Claim 1 Let $I \geq 2$. Then, the Radner solution, defined over Θ, does not satisfy sequential decomposability.

Proof. Suppose that there are three individuals, i, j and k. Assume that agents' intertemporal endowments are as follows:

$$\omega_i = (\omega_i^1, 0, 0),\ \omega_j = (0, \omega_j^2, 0) \text{ and } \omega_k = (0, 0, \omega_k^3),$$

where $\omega_i^1, \omega_j^2, \omega_k^3 > 1$.

Each economy $\theta \in \Theta = (0, 1]$ specifies a preference profile over consumption paths represented by:

$$\begin{aligned} U_i(\theta, c_i^1, c_i^2, c_i^3) &= c_i^1 + \theta \ln c_i^3 \\ U_j(\theta, c_j^1, c_j^2, c_j^3) &= \ln c_j^1 + c_j^2 \\ U_k(\theta, c_k^1, c_k^2, c_k^3) &= \ln c_k^2 + c_k^3. \end{aligned}$$

Then, the TP2 spot price equilibrium is given by:

$$q^2[\theta|x^1] = x_i^{12}, \qquad \text{for all } x^1 \in \bar{Z}^1,$$

and the TP2 competitive net trade allocation is given by:

$$\begin{aligned} f_i^{22}[\theta|x^1] &= -x_i^{12} \\ f_i^{23}[\theta|x^1] &= 1 \\ f_j^{22}[\theta|x^1] &= 0 \\ f_j^{23}[\theta|x^1] &= 0 \\ f_k^{22}[\theta|x^1] &= x_i^{12} \\ f_k^{23}[\theta|x^1] &= -1, \qquad \text{for all } x^1 \in \bar{Z}^1. \end{aligned}$$

The TP1 marginal orderings over $\bar{Z}^1$ induced by TP2 competitive net trade allocations are represented respectively by:

$$\begin{aligned}
U_i^1\left(\theta, x^1|f^2\right) &= \omega_i^1 + x_i^{11} \\
U_j^1\left(\theta, x^1|f^2\right) &= \ln x_j^{11} + \omega_j^2 + x_j^{12} \\
U_k^1\left(\theta, x^1|f^2\right) &= \ln x_i^{12} + \omega_k^3 - 1, \qquad \text{for all } x^1 \in \bar{Z}^1, \text{ for all } \theta \in \Theta.
\end{aligned}$$

For every economy $\theta \in \Theta$, the TP1 equilibrium spot price is:

$$q^1[\theta] = \theta,$$

which results in the following TP2 equilibrium spot price:

$$q^2[\theta|f^1[\theta]] = 1,$$

and in the following competitive equilibrium net trade allocations:

$$\begin{aligned}
f_i^{11}[\theta] &= -\theta \\
f_i^{12}[\theta] &= 1 \\
f_i^{22}[\theta|f^1[\theta]] &= -1 \\
f_i^{23}[\theta|f^1[\theta]] &= 1 \\
f_j^{11}[\theta] &= \theta \\
f_j^{12}[\theta] &= -1 \\
f_j^{22}[\theta|f^1[\theta]] &= 0 \\
f_j^{23}[\theta|f^1[\theta]] &= 0 \\
f_k^{11}[\theta] &= 0 \\
f_k^{12}[\theta] &= 0 \\
f_k^{22}[\theta|f^1[\theta]] &= 1 \\
f_k^{23}[\theta|f^1[\theta]] &= -1.
\end{aligned}$$

We have found that $f^1[\theta] \neq f^1[\theta']$ for all $\theta,\theta' \in \Theta$ with $\theta \neq \theta'$, though TP1 reduced utility profiles are identical across economies in Θ, in violation of part (iii) of sequential decomposability. ■

The second way is to define a concept of intertemporal price equilibrium without commitment, based on the idea of backward-induction, and to restrict attention to economies where there are no intertemporal pecuniary externalities. The latter requirement can be achieved by means of the following restriction.

Condition 1 For all $\theta \in \Theta$, the TP2 spot price $q^2[\theta|x^1]$ is constant in $x^1 \in \bar{Z}^1$.

Note that when the above condition is met, a TP2 competitive net trade vector assigned to individual i depends only on her own past saving/borrowing behavior. For this reason, we write $f_i^{22}[\theta|z_i^{12}]$ and $f_i^{23}[\theta|z_i^{12}]$ for $f_i^{22}[\theta|z^1]$ and $f_i^{23}[\theta|z^1]$ respectively.

Here are examples of domains which satisfy Condition 1. In what follows, let us focus on economies where the quantity ω_i^t is strictly positive for every individual i and every consumption period $t = 1, 2, 3$.

Assumption 1 (Θ^1) Assume that aggregate endowment is constant over time; that is, $\omega^1 = \omega^2 = \omega^3$. Also, assume that the individuals have identical discount factors, while they may exhibit different elasticities of intertemporal substitution. That is, for every economy $\theta \in \Theta^1$ it holds that $\omega^1 = \omega^2 = \omega^3$ and that there is (β^1, β^2) such that every i's preference over consumptions is represented in the form:

$$U_i(\theta, c_i^1, c_i^2, c_i^3) = v_i(\theta, c_i^1) + \beta^1 v_i(\theta, c_i^2) + \beta^1\beta^2 v_i(\theta, c_i^3),$$

where:

- the sub-utility $v_i(\theta, \cdot)$ is twice continuously differentiable, strictly increasing and strictly concave over $\mathbb{R}_{++}$.
- the limit of the first derivative of the sub-utility $v_i(\theta, \cdot)$ is positive infinity as c_i^t approaches 0; that is, $\lim_{c_i^t \to 0} \frac{\partial v_i(\theta, c_i^t)}{\partial c_i^t} = \infty$.

- the limit of the first derivative of the sub-utility $v_i(\theta,\cdot)$ is zero as c_i^t approaches positive infinity; that is, $\lim_{c_i^t\to 0}\frac{\partial v_i(\theta,c_i^t)}{\partial c_i^t}=\infty$.
- the sub-utility $v_i(\theta,\cdot)$ satisfies the requirement that $-\left(\frac{\partial^2 v_i(\theta,c_i^t)}{\partial^2 c_i^t}c_i^t/\frac{\partial v_i(\theta,c_i^t)}{\partial c_i^t}\right)<1$ for all $c_i^t\in\mathbb{R}_{++}$.

For this domain, we obtain that the TP2 competitive spot price, net trade allocations and consumption allocations prescribed for every $\theta\in\Theta^1$ are:

$$\begin{aligned}
q^2\left[\theta|z^1\right] &= \beta^2\\
f_i^{22}\left[\theta|z_i^{12}\right] &= -\frac{\beta^2}{1+\beta^2}\cdot\left(z_i^{12}+\omega_i^2-\omega_i^3\right)\\
f_i^{23}\left[\theta|z_i^{12}\right] &= \frac{1}{1+\beta^2}\cdot\left(z_i^{12}+\omega_i^2-\omega_i^3\right)\\
c_i^{*2}\left[\theta|z^1\right] &= c_i^{*3}\left[\theta|z^1\right]=\frac{z_i^{12}+\omega_i^2+\beta^2\omega_i^3}{1+\beta^2},\quad \forall i\in\mathcal{I}\text{ and }\forall z^1\in\bar{Z}^1.
\end{aligned}$$

Note that period-1 reduced utility on $\bar{Z}^1$ is represented by:

$$U_i\left(\theta,z^1|f^2\right)=v_i\left(\theta,\omega_i^1+z_i^{11}\right)+\beta^1\left(1+\beta^2\right)v_i\left(\theta,\frac{z_i^{12}+\omega_i^2+\beta^2\omega_i^3}{1+\beta^2}\right),\quad \forall i\in\mathcal{I}\text{ and }\forall z^1\in\bar{Z}^1.$$

Assumption 2 (Θ^2) In this domain we drop the assumption of constant aggregate endowment over time, but we assume that individuals have identical CES preferences. That is, for every $\theta\in\Theta^2$ there is a triplet (β^1,β^2,ρ) such that every i's preference ordering over consumptions is represented in the form:

$$U_i(\theta,c_i^1,c_i^2,c_i^3)=\frac{(c_i^1)^{1-\rho}}{1-\rho}+\beta^1\frac{(c_i^2)^{1-\rho}}{1-\rho}+\beta^1\beta^2\frac{(c_i^3)^{1-\rho}}{1-\rho},\qquad\text{with }\rho>0.$$

When agents have identical CES preferences, we obtain that the TP2 competitive equilibrium spot price, net trade allocations and consumption allocations prescribed for every

$\theta \in \Theta^2$ are:

$$\begin{aligned}
q^2\left[\theta|z^1\right] &= \beta^2\left(\frac{\omega^2}{\omega^3}\right)^\rho \\
f_i^{22}\left[\theta|z_i^{12}\right] &= -\frac{z_i^{12}+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\frac{1}{\beta^2}\left(\frac{\omega^2}{\omega^3}\right)^{1-\rho}} \\
f_i^{23}\left[\theta|z_i^{12}\right] &= \frac{\omega^3}{\omega^2}\cdot\frac{z_i^{12}+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\
c_i^{*2}\left[\theta|z^1\right] &= \frac{z_i^{12}+\omega_i^2+\beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^\rho}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\
c_i^{*3}\left[\theta|z^1\right] &= \frac{\omega^3}{\omega^2}\cdot c_i^{*2}\left[\theta|z^1\right], \qquad\qquad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1.
\end{aligned}$$

Next, let us define a TP1 competitive equilibrium when Condition 1 is satisfied.

Definition 11 For every economy θ satisfying Condition 1, a TP1 net trade allocation $\hat{f}^1[\theta] \in \bar{Z}^1$ constitutes a *backward TP1 competitive net trade allocation* if there is a TP1 spot price $q^1[\theta]$ such that for every agent i the net trade allocation $\hat{f}^1[\theta]$ solves the following problem:

$$\operatorname*{Max}_{z^1\in\bar{Z}^1} \quad U_i(\theta,\omega_i^1+z_i^{11},\omega_i^2+z_i^{12}+f_i^{22}\left[\theta|z_i^{12}\right],\omega_i^3+f_i^{22}\left[\theta|z_i^{12}\right]), \quad \text{subject to } z_i^{11}+q^1[\theta]z_i^{12}\leq 0.$$

Using this definition, we obtain that the competitive equilibrium spot prices prescribed for every economy $\theta \in \Theta^1$ are:

$$q^1[\theta] = \beta^1 \text{ and } q^2\left[\theta|\hat{f}^1[\theta]\right] = \beta^2,$$

and so the competitive net trade allocations and the equilibrium consumption allocations are for every individual $i \in \mathcal{I}$ as follows:

$$\begin{aligned}
\hat{f}_i^{11}[\theta] &= -\frac{\beta^1}{1+\beta^1+\beta^1\beta^2}\cdot\left(\omega_i^1\left(1+\beta^2\right)-\omega_i^2-\beta^2\omega_i^3\right)\\
\hat{f}_i^{12}[\theta] &= \frac{1}{1+\beta^1+\beta^1\beta^2}\cdot\left(\omega_i^1\left(1+\beta^2\right)-\omega_i^2-\beta^2\omega_i^3\right)\\
f_i^{22}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= -\frac{\beta^2}{1+\beta^2}\cdot\left(\hat{f}_i^{12}[\theta]+\omega_i^2-\omega_i^3\right)\\
f_i^{23}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= \frac{1}{1+\beta^2}\cdot\left(\hat{f}_i^{12}[\theta]+\omega_i^2-\omega_i^3\right)\\
c_i^{*1}[\theta] &= c_i^{*2}\left[\theta|\hat{f}_i^{12}[\theta]\right]=c_i^{*3}\left[\theta|\hat{f}_i^{12}[\theta]\right]=\frac{\omega_i^1+\beta^1\omega_i^2+\beta^1\beta^2\omega_i^3}{1+\beta^1+\beta^1\beta^2}.
\end{aligned}$$

For economies in Θ^2, we obtain that the equilibrium spot prices prescribed for every $\theta \in \Theta^2$ are:

$$q^1[\theta]=\beta^1\left(\frac{\omega^1}{\omega^2}\right)^{\rho} \text{ and } q^2\left[\theta|\hat{f}^1[\theta]\right]=\beta^2\left(\frac{\omega^2}{\omega^3}\right)^{\rho}.$$

Thus, the competitive net trade allocations are:

$$\begin{aligned}
\hat{f}_i^{11}[\theta] &= -\frac{\omega_i^1\left(\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}+1\right)-\left(\frac{\omega^1}{\omega^2}\right)\left(\omega_i^2+\beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^{\rho}\right)}{1+\frac{1}{\beta^1}\left(\frac{\omega^1}{\omega^2}\right)^{1-\rho}+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}}\\
\hat{f}_i^{12}[\theta] &= \frac{\omega^2}{\omega^1}\cdot\frac{\omega_i^1\left(\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}+1\right)-\left(\frac{\omega^1}{\omega^2}\right)\left(\omega_i^2+\beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^{\rho}\right)}{1+\beta^1\left(\frac{\omega^2}{\omega^1}\right)^{1-\rho}+\beta^1\beta^2\left(\frac{\omega^3}{\omega^1}\right)^{1-\rho}}\\
f_i^{22}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= -\frac{\hat{f}_i^{12}[\theta]+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\frac{1}{\beta^2}\left(\frac{\omega^2}{\omega^3}\right)^{1-\rho}}\\
f_i^{23}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= \frac{\omega^3}{\omega^2}\cdot\frac{\hat{f}_i^{12}[\theta]+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}},
\end{aligned}$$

while the corresponding equilibrium consumption allocations are:

$$\begin{aligned} c_i^{*1}[\theta] &= \frac{\omega_i^1 + \beta^1\omega_i^2\left(\frac{\omega^1}{\omega^2}\right)^\rho + \beta^1\beta^2\omega_i^3\left(\frac{\omega^1}{\omega^3}\right)^\rho}{1+\beta^1\left(\frac{\omega^2}{\omega^1}\right)^{1-\rho} + \beta^1\beta^2\left(\frac{\omega^3}{\omega^1}\right)^{1-\rho}} \\ c_i^{*2}\left[\theta|z^{*1}[\theta]\right] &= \frac{\omega^2}{\omega^1}\cdot c_i^{*1}[\theta] \\ c_i^{*3}\left[\theta|z^{*1}[\theta]\right] &= \frac{\omega^3}{\omega^1}\cdot c_i^{*1}[\theta]. \end{aligned}$$

The *backward competitive solution* of an economy θ is a SCF $\bar{f} = \left(\bar{f}^1[\cdot], \bar{f}^2[\cdot|\cdot]\right)$ associating the period-1 function $\bar{f}^1[\theta]$ with the backward TP1 competitive net trade allocation $\hat{f}^1[\theta]$, that is, $\bar{f}^1[\theta] = \hat{f}^1[\theta] \in \bar{Z}^1$, and the period-2 function $\bar{f}^2[\theta|\cdot]$ with the TP2 competitive net trade allocation for any TP1 net trade allocation in the set $\bar{Z}^1$, that is, $\bar{f}^2[\theta|z^1] = f^2[\theta|z^1]$ for every $z^1 \in \bar{Z}^1$. Thanks to Condition 1, we can now state and prove the following permissive results.

Claim 2 Assume that $I \geq 3$. Suppose that the quantity ω_i^t is strictly positive for every individual i and every consumption period $t = 1, 2, 3$. Then, the backward competitive solution $\bar{f}$ is sequential implementable if it is defined either over Θ^1 or over Θ^2.

Proof. Let the premises hold. To show that $\bar{f}$ is sequential implementable when it is defined either over Θ^1 or over Θ^2, we need to show that this solution is sequentially decomposable and sequential Maskin monotonic. Moreover, we need also to show this solution satisfies sequential unanimity and sequential weak no veto-power.

First, let us show that $\bar{f}$ satisfies sequential decomposability. To this end, let $Y^1 = \mathcal{Y}^{-2} = \bar{Z}^1$ and let $Y^2(z^1) = \bar{Z}^2(z^1)$ for every $z^1 \in \bar{Z}^1$. Then, the sets $Y^1 = \mathcal{Y}^{-2}$ and $Y^2(z^1)$ are not empty sets. Note that for $k = 1, 2$, it holds that $\bar{f}^1\left[\Theta^k\right] \subseteq \bar{Z}^1$ and $\bar{f}^2\left[\Theta^k|z^1\right] \subseteq \bar{Z}^2(z^1)$ for every $z^1 \in \bar{Z}^1$.

Let us define the TP2 marginal ordering of individual i in state θ at $z^1 \in \bar{Z}^1$, denoted by $R_i[\theta|z^1]$, as follows:

$$\begin{aligned} x^2 R_i\left[\theta|z^1\right] y^2 &\iff \\ U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + x_i^{22}, \omega_i^3 + x_i^{23}) &\geq U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + y_i^{22}, \omega_i^3 + y_i^{23}), \end{aligned}$$

for every $x^2, y^2 \in Y^2(z^1)$. We denote by $R[\theta|z^1]$ the profile of TP2 marginal orderings at $\theta|z^1$, by $\mathcal{D}[\Theta^1|z^1]$ the TP2 domain of marginal orderings at $\Theta^1|z^1$ and by $\mathcal{D}[\Theta^2|z^1]$ the TP2 domain of marginal orderings at $\Theta^2|z^1$. For every $k = 1, 2$, let us define the TP2 function $\varphi^2 : \mathcal{D}\left[\Theta^k|z^1\right] \to Y^2(z^1)$ as follows:

$$\varphi^2\left(R\left[\theta|z^1\right]\right) = f^2\left[\theta|z^1\right], \qquad \forall\theta \in \Theta^k.$$

The TP1 marginal ordering of individual i in state θ, denoted by $R_i\left[\theta|\bar{f}^2\right]$, is defined as in (11). Let us denote by $R\left[\theta|\bar{f}^2\right]$ the profile of TP1 marginal orderings at $\theta|\bar{f}^2$, by $\mathcal{D}\left[\Theta^1|\bar{f}^2\right]$ the TP1 domain of marginal orderings at $\Theta^1|\bar{f}^2$, and by $\mathcal{D}\left[\Theta^2|\bar{f}^2\right]$ the TP1 domain of marginal orderings at $\Theta^2|\bar{f}^2$. For every $k = 1, 2$, let us define the TP1 function $\varphi^1 : \mathcal{D}\left[\Theta^k|\bar{f}^2\right] \to Y^1$ as follows:

$$\varphi^1\left(R\left[\theta|\bar{f}^2\right]\right) = \hat{f}^1[\theta], \qquad \forall\theta \in \Theta^k.$$

By the above definitions and by the fact that competitive equilibrium exists in each TP, one can check that the backward competitive solution $\bar{f}$ satisfies sequential decomposability.

To see that $\bar{f}$ also satisfies sequential Maskin monotonicity, it suffices to observe that in each TP the competitive net trade allocation is unique and always an interior allocation, that the TP1 Walrasian solution φ^1 on $\mathcal{D}\left[\Theta^k|\bar{f}^2\right]$ is Maskin monotonic for $k = 1, 2$, and that the TP2 Walrasian solution φ^2 on $\mathcal{D}\left[\Theta^k|z^1\right]$ is also Maskin monotonic for $k = 1, 2$.

Finally, to see that the backward competitive solution $\bar{f}$ satisfies sequential unanimity and sequential weak no veto-power it suffices to observe that they are vacuously satisfied since individuals' marginal orderings are strictly monotonic in consumption. ■

When there are indeed pecuniary externalities, we do not have a general answer. Below we provide an answer, although it is far from satisfactory.

Definition 12 (Target rule) Let $\hat{x}^1 \in \bar{Z}^1$ be a *fixed* TP1 net trade allocation. Given an economy θ, the net trade profile in the first period $h^1[\theta] \in \bar{Z}^1$ is defined by

$$h^1[\theta] = \hat{x}^1$$

if

$$
\begin{aligned}
& U_i(\theta, \omega_i^1 + \hat{x}_i^{11}, \omega_i^2 + \hat{x}_i^{12} + f_i^{22}[\theta|\hat{x}^1], \omega_i^3 + f_i^{23}[\theta|\hat{x}^1]) \\
\geq \; & U_i(\theta, \omega_i^1, \omega_i^2 + f_i^{22}[\theta|\mathbf{0}], \omega_i^3 + f_i^{23}[\theta|\mathbf{0}]), \qquad \forall i \in \mathcal{I},
\end{aligned}
$$

otherwise, there is no trade in TP1, that is,

$$h^1[\theta] = \mathbf{0}.$$

The *intertemporal target solution* of an economy θ is a SCF $g = (g^1[\cdot], g^2[\cdot|\cdot])$ associating the period-1 function $g^1[\theta]$ with the net trade allocation $h^1[\theta]$, that is, $g^1[\theta] = h^1[\theta] \in \bar{Z}^1$, and the period-2 function $g^2[\theta|\cdot]$ with the TP2 competitive net trade allocation for any TP1 net trade allocation in the set $\bar{Z}^1$, that is, $g^2[\theta|z^1] = f^2[\theta|z^1]$ for every $z^1 \in \bar{Z}^1$.

Let SCF g be defined over the domain Θ^3, which is such that for every economy $\theta \in \Theta^3$ agent i's preference odering $R_i(\theta)$ is represented by an additively separable utility function

$$U_i(\theta, c_i^1, c_i^2, c_i^3) = v_i^1(\theta, c_i^1) + v_i^2(\theta, c_i^2) + v_i^3(\theta, c_i^3),$$

where:

- Period-t sub-utility $v_i^t(\theta, \cdot)$ is twice continuously differentiable, strictly increasing and strictly concave over $\mathbb{R}_{++}$.
- The limit of the first derivative of period-t sub-utility $v_i^t(\theta, \cdot)$ is positive infinity as c_i^t approaches 0; that is, $\lim_{c_i^t \to 0} \frac{\partial v_i^t(\theta, c_i^t)}{\partial c_i^t} = \infty$.
- The limit of the first derivative of period-t sub-utility $v_i^t(\theta, \cdot)$ is zero as c_i^t approaches positive infinity; that is, $\lim_{c_i^t \to 0} \frac{\partial v_i^t(\theta, c_i^t)}{\partial c_i^t} = \infty$.
- Period-t sub-utility $v_i^t(\theta, \cdot)$ satisfies the requirement that $-\left(\frac{\partial^2 v_i^t(\theta, c_i^t)}{\partial^2 c_i^t} c_i^t / \frac{\partial v_i^t(\theta, c_i^t)}{\partial c_i^t}\right) < 1$ for all $c_i^t \in \mathbb{R}_{++}$.

By setting $Y^1 = \mathcal{Y}^{-2} = \bar{Z}^1$ and $Y^2(z^1) = \bar{Z}^2(z^1)$ for every $z^1 \in \bar{Z}^1$, by defining the

TP2 function $\varphi^2 : \mathcal{D}[\Theta^3|z^1] \to Y^2(z^1)$ by

$$\varphi^2\left(R\left[\theta|z^1\right]\right) = f^2\left[\theta|z^1\right], \qquad \forall\theta \in \Theta^3,$$

and by defining the TP1 function $\varphi^1 : \mathcal{D}[\Theta^3|g^2] \to Y^1$ by

$$\varphi^1\left(R\left[\theta|g^2\right]\right) = h^1[\theta], \qquad \forall\theta \in \Theta^3,$$

one can check that the intertemporal target solution g, defined on Θ^3, satisfies all conditions of Theorem 3, and so it is sequentially implementable when there are three or more individuals.

4.2 Sequential voting

In this section, we consider a bi-dimensional policy space where an *odd* number of individuals vote sequentially on each dimension and where an ordering of the dimensions is exogenously given. We assume that a majority vote is organized around each policy dimension and that the outcome of the first majority vote is known to the voters at the beginning of the second voting stage. This sequential resolution is common in political economy models (see, e.g., Persson and Tabellini, 2000). We are interested in sequentially implementing the simple majority solution, which selects the Condorcet winner in each voting stage.

A policy choice is an ordered pair $(x^1, x^2) \in X^1 \times X^2$, where the policy space of dimension $d = 1, 2$ is an open interval.[3] Each voter i is described by a one-dimensional type θ_i. The type space is the open interval $\left(\underline{\eta}, \bar{\eta}\right)$.

Definition 13 The voter i's utility function $U : \left(\underline{\eta}, \bar{\eta}\right) \times X^1 \times X^2 \to \mathbb{R}$ is a twice-continuously differentiable satisfying:

(a) *Strict concavity*, that is:

$$\frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial^2 x^1} < 0 \quad \text{and} \quad \frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial^2 x^2} < 0, \quad \text{for every } \left(x^1, x^2\right) \in X^1 \times X^2.$$

[3] The choice of a bi-dimensional policy space is motivated by convenience.

(b) *Marginal single-crossing* property, that is:

$$\frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial\theta_i \partial x^1} > 0 \quad \text{and} \quad \frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial\theta_i \partial x^2} > 0, \quad \text{for every } \left(x^1, x^2\right) \in X^1 \times X^2 \text{ and } \theta_i \in \left(\underline{\eta}, \bar{\eta}\right).$$

(c) *Strategic complementarity*, that is:

$$\frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial x^1 \partial x^2} \geq 0, \quad \text{for every } \left(x^1, x^2\right) \in X^1 \times X^2.$$

The marginal single-crossing property simply requires that the marginal utility of both dimensions is increasing in the type of voter. This property can also be found in De Donder et al. (2012).

We now introduce the definition of a Condorcet winner:

Definition 14 Suppose that individuals in $\mathcal{I}$ votes over the set of policies $\mathfrak{P}$. We say that $\mathfrak{p} \in \mathfrak{P}$ is a majority voting outcome, also known as a *Condorcet winner* (CW), if there does not exist any other distinct outcome $\mathfrak{p}' \in \mathfrak{P}$ that is strictly preferred by more than half of voters to the outcome $\mathfrak{p}$.

For any integer $k \geq 2$, the set of states Θ takes the structure of the Cartesian product of allowable independent types for voters, that is, $\Theta \equiv \left(\underline{\eta}, \bar{\eta}\right)^{2k-1}$, with θ as typical element. It simplifies the argument, and causes no loss of generality, to assume that $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{2k-1}$. Therefore, the type θ_k is the median type, denoted by θ_{med}, at state θ.

At the state θ, each voter is assumed to have an ordering preference relation $R_i(\theta)$ over the policy space $X^1 \times X^2$ which is represented by $U(\theta_i, \cdot, \cdot)$.

Solving by backward induction when the state θ is the prevailing state, if $x^1 \in X^1$ is the outcome of the first majority voting, then the stage-2 marginal ordering of voter i on X^2 in state θ at x^1 is denoted by $R_i[\theta|x^1]$ and is represented by $U(\theta_i, x^1, \cdot)$.

The profile of the stage-2 marginal orderings in state θ at x^1 is denoted by $R[\theta|x^1]$. Let $\mathcal{D}[\Theta|x^1]$ be the stage-2 domain of marginal ordering preferences induced by the set Θ as well as by the outcome x^1; that is:

$$\mathcal{D}\left[\Theta|x^1\right] \equiv \left\{R\left[\theta|x^1\right] | \theta \in \Theta\right\}, \quad \text{for every } x^1 \in X^1. \tag{12}$$

If $x^1 \in X^1$ is the outcome of the first majority voting, then the stage-2 majority voting function $f^2 : \mathcal{D}[\Theta|x^1] \to X^2$ is defined as follows:

$$f^2\left[\theta|x^1\right] = CW\left(R\left[\theta|x^1\right]\right),$$

where $CW(R[\theta|x^1])$ denotes the Condorcet winner under the profile $R[\theta|x^1]$. It will be shown below that this outcome is the most-preferred outcome of the median type.

Let us suppose that the stage-2 majority voting function is well-defined for every outcome $x^1 \in X^1$. Then, in stage-1, the utility of a voter i at state θ for the outcome $z^1 \in X^1$ is:

$$U\left(\theta_i, z^1, f^2\left[\theta|z^1\right]\right).$$

Then, the stage-1 marginal ordering of voter i on X^1 in state θ at the majority voting function $f^2[\theta|\cdot]$, denoted by $R_i[\theta|f^2]$, is given by:

$$y^1 R_i\left[\theta|f^2\right] z^1 \iff \left(y^1, f^2\left[\theta|y^1\right]\right) R_i(\theta) \left(z^1, f^2\left[\theta|z^1\right]\right), \quad \text{for every } y^1, z^1 \in X^1.$$

As usual, the profile of the stage-1 marginal orderings in state θ at the majority voting function $f^2[\theta|\cdot]$ is denoted by $R[\theta|f^2]$. Let $\mathcal{D}[\Theta|f^2]$ be the stage-1 domain of marginal ordering preferences induced by the set Θ as well as by the majority voting function f^2; that is:

$$\mathcal{D}\left[\Theta|f^2\right] \equiv \left\{R\left[\theta|f^2\right] | \theta \in \Theta\right\}. \tag{13}$$

Thus, the stage-1 majority voting function $f^1 : \mathcal{D}[\Theta|f^2] \to X^1$ is defined as follows:

$$f^1[\theta] = CW\left(R\left[\theta|f^2\right]\right), \quad \text{for every } \theta \in \Theta,$$

where $CW(R[\theta|f^2])$ denotes the Condorcet winner under the profile $R[\theta|f^2]$.

Definition 15 The SCF $f(\cdot) = (f^1[\cdot], f^2[\cdot|\cdot])$ on Θ is the *majority voting* solution if for every $\theta \in \Theta$:

$$f^1[\theta] = CW\left(R\left[\theta|f^2\right]\right) \text{ and } f^2\left[\theta|x^1\right] = CW\left(R\left[\theta|x^1\right]\right) \text{ for every } x^1 \in X^1.$$

The following lemma shows that the majority voting solution is a single-valued function. The intuition behind it is similar to that of Proposition 4 of De Donder et al. (2012) for the case where there is a continuum of voters. Firstly, the assumption of strict concavity assures the existence and unicity of the Condorcet winner in the second voting stage. This assumption, combined with the assumption of strategic complementarity and with the marginal single-crossing property, assures that the stage-1 marginal ordering of voter i on X^1 in state θ at the majority voting function $f^2[\theta|\cdot]$ is single-crossing. This guarantees the existence and unicity of the Condorcet winner in the first voting stage.

Lemma 1 Suppose that the cardinality of $\mathcal{I}$ is $2k-1$ with $k \geq 2$. Suppose that voter $i \in \mathcal{I}$'s utility function U_i on $\Theta \times X^1 \times X^2$ meets the requirements of Definition 13 and depends only on her own type. Then, the majority voting SCF $f(\cdot) = (f^1[\cdot], f^2[\cdot|\cdot])$ over Θ is a single-valued function on each policy dimension.

Proof. See Appendix. ■

Thanks to the above lemma, we can now state and prove the main result of this section.

Claim 3 Suppose that the cardinality of $\mathcal{I}$ is $2k-1$ with $k \geq 2$. Suppose that voter $i \in \mathcal{I}$'s utility function U_i on $\Theta \times X^1 \times X^2$ meets the requirements of Definition 13 and depends only on her own type. Then, the majority voting solution is sequentially implementable.

Proof. Let the premises hold. By Theorem 3, it suffices to show that the majority voting solution is sequentially decomposable and sequentially Maskin monotonic and, moreover, it satisfies sequential unanimity and sequential weak no veto-power.

Thus, $\mathcal{T} = \{1, 2\}$. Let $Y^1 = X^1$ and $Y^2(x^1) = X^2$ for every $x^1 \in X^1$. Define $\mathcal{D}[\Theta|x^1]$ as in (12) and define $\mathcal{D}[\Theta|f^2]$ as in (13).

For every $x^1 \in X^1$, define the second stage function $\varphi^2 : \mathcal{D}[\Theta|x^1] \to X^2$ by $\varphi^2(R[\theta|x^1]) = CW(R[\theta|x^1])$ for every $R[\theta|x^1] \in \mathcal{D}[\Theta|x^1]$. Moreover, define the first stage function $\varphi^1 : \mathcal{D}[\Theta|f^2] \to X^1$ by $\varphi^1(R[\theta|f^2]) = CW(R[\theta|f^2])$ for every $R[\theta|f^2] \in \mathcal{D}[\Theta|f^2]$. These functions are single-valued by Lemma 1. This shows that the majority voting solution is sequentially decomposable.

By definitions of the preceding paragraph and by the fact that in each period individuals have single crossing preferences, one can see that the majority voting solution is sequentially Maskin monotonic. Since unanimity and weak no veto-power are satisfied, we conclude that the majority voting solution on Θ is sequentially implementable. ■

5. Conclusion

Summary. In a finite-horizon intertemporal setting, the paper has investigated SCFs that are implementable in SPE when society needs to decide and enforce a socially optimal outcome in each period without being able to commit to future ones.

A SCF maps each possible state of the environment into a dynamic process of social decisions, which in each period results in a desired allocation of resources for that state on the basis of past decisions. The paper focussed on the sequential implementation of this dynamic process. A SCF is sequentially implementable if there exists a sequence of mechanisms (with observed actions and with simultaneous moves) such that for each possible state of the environment every SPE of games played sequentially by individuals in that state generates the outcome prescribed by the SCF for that state, at every history.

We have identified two necessary conditions for sequential implementability, *sequential decomposability* and *sequential Maskin monotonicity*. The first condition states that a sequential implementable SCF can be decomposed into a sequence of "apparently static" marginal social choice functions, each of which is defined only over marginal preferences induced over outcomes at hand. Each marginal preference is constructed in the manner of backward-induction. This means that a period-t marginal preference over the current component set depends on past decisions as well as on the socially optimal path that the dynamic process will bring about in the future. The second condition states that every such "apparently static" marginal social function needs to satisfies a remarkably strong invariance condition for Nash implementation, now widely referred to as Maskin monotonicity (Maskin, 1999).

We have also shown that under two auxiliary conditions the two necessary conditions are sufficient, as well. The implementing mechanism we have constructed is simple. After each history, we run the canonical Maskin mechanism just to decide the current outcome, in the "apparently static" manner. Participants report messages consisting of a preference profile defined only over current outcomes, an outcome as well as a tie-breaking device.

The last decades have seen impressive advances in the theory of implementation. One conclusion is that the use of refinements in implementation leads to permissive results. This is so because implementation in refinements of Nash equilibrium (Moore and Repullo, 1988; Abreu and Sen, 1990; Palfrey and Srivastava, 1991; Jackson, 1992) allow us to circumvent the limitations imposed by Maskin monotonicity. As Sjöström (1994) pointed out, 'With enough ingenuity the planner can implement "anything"'(p. 503). In contrast to this, we have found that sequential rationality, when used in a context where society is unable to make binding agreements about future outcomes, does not allow the planner to escape the limitations imposed by Maskin monotonicity. Furthermore, we have also shown (in Claim 1) that the condition of sequential decomposability imposes non-trivial restrictions on the class of social dynamic processes that are sequentially implementable.

Indeed, we have applied our analysis to two prominent dynamic problems, voting over time and sequential trading. In the voting application, we have shown that on the domain satisfying the single-crossing property the simple majority solution, which selects the Condorcet winner in each voting stage (after every history), is sequentially implementable.

In a borrowing-lending model with no liquidity constraints, in which individuals trade in spot markets and transfer wealth between any two periods by borrowing and lending, we have noted that intertemporal pecuniary externalities arise because trades in the current period change the spot price of the next period, which, in turn, affects its associated equilibrium allocation. The quantitative implication of this is that every individual's marginal preference ordering concerns not only her own consumption/saving behavior but also the consumption/saving behavior of all other individuals. In this set-up, we have shown that, under such pecuniary externalities, the standard dynamic competitive equilibrium solution is not sequentially implementable. However, we have also identified preference domains – which involve no pecuniary externalities – for which the no-commitment version of the dynamic competitive equilibrium solution is definable and sequentially implementable. It remains an open question how we should deal with intertemporal pecuniary externalities. We hope that this and other topics related to this paper will be investigated in future research.

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Appendix

Proof of Theorem 1

Proof of Theorem 1. Let the premises hold. Thus, there exists a sequential mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ that sequentially implements the SCF f. Therefore, for every $\bar{\theta} \in \Theta$,

$$f^1 \left[\bar{\theta}\right] = g^1 \left(SPE\left(\Gamma, \bar{\theta}\right)\right) \text{ and}$$

$$f^t \left[\bar{\theta}|g^{-t}\left(h^t\right)\right] = g^t \left(SPE\left(\Gamma\left(h^t\right), \bar{\theta}\right)\right) \text{ for every } h^t \in H^t \text{ and every } t \neq 1.$$

Then, there is a strategy profile $s^{\bar{\theta}} \in SPE\left(\Gamma, \bar{\theta}\right)$ of the sequential game $\left(\Gamma, \bar{\theta}\right)$ such that:

$$s^{\bar{\theta}}|h^t \text{ is a Nash equilibrium of } \left(\Gamma\left(h^t\right), \bar{\theta}\right) \text{ for every history } h^t \in H.$$

Moreover, by sequential implementability of f, it also follows that:

$$f^{+t}\left[\bar{\theta}|g^{-t}(h)\right] = g^{+t}\left(s^{\bar{\theta}}|h\right), \quad \text{for every } h \in H^t \text{ with } 2 \leq t \leq T. \tag{14}$$

Fix any period $t \neq 1$. Let us define the set Y^1, the set $\mathcal{Y}^{-t}$ and the set $Y^{-t}\left(g^{-t}(h)\right)$ as follows:

$$Y^1 \equiv \left\{g^1\left(a\left(h^1\right)\right) \in X^t | \text{for some } a\left(h^1\right) \in A\left(h^1\right)\right\}, \tag{15}$$

$$\mathcal{Y}^{-t} \equiv \left\{g^{-t}\left(h\right) \in \mathcal{X}^{-t} | \text{for some } h \in H^t\right\}, \tag{16}$$

and for every $g^{-t}(h) \in \mathcal{Y}^{-t}$:

$$Y^t\left(g^{-t}\left(h\right)\right) \equiv \left\{g^t\left(a\left(h\right)\right) \in X^t\left(g^{-t}\left(h\right)\right) | a\left(h\right) \in A\left(h\right) \text{ for some } h \in H^t\right\}. \tag{17}$$

By their definitions as well as by the assumption that the sequential mechanism Γ implements in SPE the SCF f, one can check that $f^t\left[\Theta | g^{-t}(h)\right] \subseteq Y^t\left(g^{-t}(h)\right)$ and that $f^1[\Theta] \subseteq Y^1$.

Moreover, given that Γ is a sequential mechanism, one can also check that for every period $t \neq 1$:

$$g^{-t}\left(h^t\right) \in \mathcal{Y}^{-t} \iff g^{\tau}\left(a^{\tau}\right) \in Y^{\tau}\left(g^{-\tau}\left(a^1, \cdots, a^{\tau-1}\right)\right) \text{ for every } \tau \text{ such that } 2 \leq \tau \leq t-1,$$

for every $h^t \equiv (a^1, \cdots, a^{t-1}) \in H^t$.

For every $y^{-T} \in \mathcal{Y}^{-T}$, the period-$T$ preference domain $\mathcal{D}\left[\Theta | y^{-T}\right]$ is nonempty, and this follows from its definition in (2) and from the fact that $Y^{-T}\left(y^{-T}\right)$ is not empty. Let the period-T function

$$\varphi^T : \mathcal{D}\left[\Theta | g^{-T}(h)\right] \to Y^T\left(g^{-T}(h)\right)$$

be defined by:

$$\varphi^T\left(R\left[\theta | g^{-T}(h)\right]\right) = g^T(s^{\theta}(h)), \quad \text{for every history } h \in H^T \text{ and state } \theta \in \Theta, \tag{18}$$

where $s^{\theta} \in SPE(\Gamma, \theta)$.

Fix any period $t \neq 1, T$ and any t-head outcome path $y^{-t} \equiv g^{-t}(h) \in \mathcal{Y}^{-t}$ for some $h \in H^t$. Since the set $Y^t\left(g^{-t}(h)\right)$ is not empty and since Γ succesively implements f, one can see that the period-t domain of marginal orderings $\mathcal{D}\left[\Theta | y^{-t}, f^{+(t+1)}\right]$ as defined in (5) is not empty. Similarly, one can see that period-1 domain of marginal orderings $\mathcal{D}\left[(\Theta | f^{+2})\right]$ as defined in (8) is not empty.

For every $t \neq 1, T$, let the period-t function

$$\varphi^t : \mathcal{D}\left[\Theta | g^{-t}(h), f^{+(t+1)}\right] \rightarrow Y^t\left(g^{-t}(h)\right)$$

be defined by:

$$\varphi^t\left(R\left[\theta | g^{-t}(h), f^{+(t+1)}\right]\right) = g^t(s^\theta(h)) \text{ for every } h \in H^t \text{ and every } \theta \in \Theta. \tag{19}$$

Let the period-1 function

$$\varphi^1 : \mathcal{D}\left[\Theta | f^{+2}\right] \rightarrow Y^1$$

be defined by:

$$\varphi^1\left(R\left[\theta | f^{+2}\right]\right) = g^1(s^\theta\left(h^1\right)), \quad \text{for every } \theta \in \Theta. \tag{20}$$

To complete the proof, we need to show that the period-t function φ^t is a function for every $t \in \mathcal{T}$. The following claim establishes it for the case where $t \neq 1, T$. The same arguments, suitably modified, can be used to show that φ^1 and φ^T are functions.

Claim 4 If the SCF f over Θ is sequentially implementable and $R\left[\theta | y^{-t}, f^{+(t+1)}\right] = R\left[\theta' | y^{-t}, f^{+(t+1)}\right]$ for some $y^{-t} \in \mathcal{Y}^{-t}$ with $t \neq 1$ and some $\theta, \theta' \in \Theta$, then $f^t\left[\theta | y^{-t}\right] = f^t\left[\theta' | y^{-t}\right]$.

Proof. Suppose that $y^{-t} = g^{-t}(h)$ for some $h \in H^t$ and that $R\left[\theta | y^{-t}, f^{+(t+1)}\right] = R\left[\theta' | y^{-t}, f^{+(t+1)}\right]$ for some $\theta, \theta' \in \Theta$.

Since $s^\theta \in SPE(\Gamma, \theta)$ and since, moreover, $R\left[\theta | y^{-t}, f^{+(t+1)}\right] = R\left[\theta' | y^{-t}, f^{+(t+1)}\right]$, we have that:

$$s^\theta(h) \in NE\left(\Gamma(h), R\left[\theta | y^{-t}, f^{+(t+1)}\right]\right) \cap NE\left(\Gamma(h), R\left[\theta' | y^{-t}, f^{+(t+1)}\right]\right),$$

and so, for every $i \in \mathcal{I}$ and $a_i(h) \in A_i(h)$, it holds that:

$$s^\theta(h)\, R_i\left[\theta' | y^{-t}, f^{+(t+1)}\right]\left(a_i(h), s^\theta_{-i}(h)\right).$$

From the definition of $R_i\left[\theta' | y^{-t}, f^{+(t+1)}\right]$ and from (14), it follows that for every $i \in \mathcal{I}$

and $a_i(h) \in A_i(h)$ it holds that:

$$\left(g^{-t}(h), g^{t}\left(s^{\theta}(h)\right), g^{+(t+1)}\left(s^{\theta'} \mid \left(h, s^{\theta}(h)\right)\right)\right) R_i(\theta') \qquad (21)$$
$$\left(g^{-t}(h), g^{t}\left(a_i(h), s^{\theta}_{-i}(h)\right), g^{+(t+1)}\left(s^{\theta'} \mid \left(h, \left(a_i(h), s^{\theta}_{-i}(h)\right)\right)\right)\right).$$

Let $s_i|h \equiv (s_i^{\tau})_{\tau \geq t}$ denote the individual i's strategy according to which this i plays $s_i^t = s_i^{\theta}(h)$ after the history h and she conforms to the strategy $s_i^{\theta'}$ thereafter; that is, $s_i^{\bar{t}} = \left(s_i^{\theta'}\right)^{\bar{t}}$ for every $\bar{t} > t$. Note that $s|h'$ is a Nash equilibrium of $(\Gamma(h'), \theta')$ for every history $h' \in H|(h, a(h))$ since $s^{\theta'}$ is a strategy profile in $SPE(\Gamma, \theta')$. Thus, to have that the strategy profile $s|h$ is a SPE strategy profile of $(\Gamma(h), \theta')$, we need to show that $s|h$ is a Nash equilibrium of $(\Gamma(h), \theta')$.

Since the action profile $s(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$, it follows that (21) holds for every $i \in I$ and every $a_i(h) \in A_i(h)$. Thus, no individual i can gain by deviating from the action $s_i(h)$ and thereafter conforming to s_i. Since the one deviation property (see, e.g., Osborne and Rubinstein, 1994; Lemma 98.2) holds for a finite-horizon multi-period game with observed actions and simultaneous moves, it follows that the strategy profile $s|h$ is a SPE of $(\Gamma(h), \theta')$. This means that $g^t(SPE(\Gamma(h), \theta)) = g^t(SPE(\Gamma(h), \theta'))$. Since the sequential mechanism Γ implements the SCF f in SPE, we have that $f^t[\theta'|g^{-t}(h)] = f^t[\theta|g^{-t}(h)]$. ■

The statement follows by the above arguments. ■

Proof of Theorem 2

Proof of Theorem 2. Let the premises hold. Thus, there exists a sequential mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ that sequentially implements the SCF f. Therefore, for every $\bar{\theta} \in \Theta$,

$$f^1\left[\bar{\theta}\right] = g^1\left(SPE\left(\Gamma, \bar{\theta}\right)\right) \text{ and}$$

$$f^t\left[\bar{\theta}|g^{-t}\left(h^t\right)\right] = g^t\left(SPE\left(\Gamma\left(h^t\right), \bar{\theta}\right)\right) \text{ for every } h^t \in H^t \text{ and every } t \neq 1.$$

Consider any state $\bar{\theta}$. Then, there is a strategy profile $s^{\bar{\theta}} \in SPE\left(\Gamma, \bar{\theta}\right)$ of the sequential game $\left(\Gamma, \bar{\theta}\right)$ such that:

$$s^{\bar{\theta}}|h^t \text{ is a Nash equilibrium of } \left(\Gamma\left(h^t\right), \bar{\theta}\right) \text{ for every history } h^t \in H.$$

Moreover, by sequential implementability of f, it also follows that:

$$f^{+t}\left[\bar{\theta}|g^{-t}\left(h\right)\right] = g^{+t}\left(s^{\bar{\theta}}|h\right), \quad \text{for every } h \in H^t \text{ with } 2 \leq t \leq T.$$

Since the SCF f is sequentially decomposable, define the set Y^1, the set $\mathcal{Y}^{-t}$ and the set $Y^t\left(g^{-t}\left(h^t\right)\right)$ as in (15), (16) and (17) of the proof of Theorem 1, respectively.

Fix any $g^{-T}\left(h\right) \in \mathcal{Y}^{-T}$ with $h \in H^T$ and suppose that for every $i \in \mathcal{I}$ and every $a\left(h\right) \in A\left(h\right)$, it holds that:

$$\begin{aligned} &\varphi^T\left(R\left[\theta|g^{-T}\left(h\right)\right]\right) R_i\left[\theta|g^{-T}\left(h\right)\right] g^T\left(a\left(h\right)\right) \implies \\ &\qquad \varphi^T\left(R\left[\theta|g^{-T}\left(h\right)\right]\right) R_i\left[\theta'|g^{-T}\left(h\right)\right] g^T\left(a\left(h\right)\right), \end{aligned} \tag{22}$$

for some $R\left[\theta|g^{-T}\left(h\right)\right]$ and $R\left[\theta'|g^{-T}\left(h\right)\right]$ in $\mathcal{D}\left[\Theta|g^{-T}\left(h\right)\right]$.

Since the sequential mechanism Γ implements the SCF f in SPE, we have that:

$$\varphi^T\left(R\left[\theta|g^{-T}\left(h\right)\right]\right) = g^T\left(s^\theta\left(h\right)\right) = f^T\left[\theta|g^{-T}\left(h\right)\right],$$

and that action profile $s^\theta\left(h\right)$ is a Nash equilibrium of $\left(\Gamma\left(h\right), R\left[\theta|g^{-T}\left(h\right)\right]\right)$.

From the definitions of $R_i\left[\theta|g^{-T}\left(h\right)\right]$ and $R_i\left[\theta'|g^{-T}\left(h\right)\right]$ given in (1), we have that:

$$\begin{aligned} &g^T\left(s^\theta\left(h\right)\right) R_i\left[\theta|g^{-T}\left(h\right)\right] g^T\left(a\left(h\right)\right) \iff \\ &\qquad \left(g^{-T}\left(h\right), g^T\left(s^\theta\left(h\right)\right)\right) R_i\left(\theta\right) \left(g^{-T}\left(h\right), g^T\left(a\left(h\right)\right)\right), \end{aligned} \tag{23}$$

and that:

$$\begin{aligned} &g^T\left(s^\theta\left(h\right)\right) R_i\left[\theta'|g^{-T}\left(h\right)\right] g^T\left(a\left(h\right)\right) \iff \\ &\qquad \left(g^{-T}\left(h\right), g^T\left(s^\theta\left(h\right)\right)\right) R_i\left(\theta'\right) \left(g^{-T}\left(h\right), g^T\left(a\left(h\right)\right)\right). \end{aligned} \tag{24}$$

If there exist $i \in \mathcal{I}$ and $a_i(h) \in A_i(h)$ such that:

$$g^T\left(a_i(h), s_{-i}^{\theta}(h)\right) P_i\left[\theta'|g^{-T}(h)\right] g^T(s^{\theta}(h)),$$

it follows from (22)-(24) that:

$$g^T\left(a_i(h), s_{-i}^{\theta}(\theta)(h)\right) P_i\left[\theta|g^{-T}(h)\right] g^T(s^{\theta}(h)),$$

which contradicts the fact that the action profile $s^{\theta}(h)$ is a Nash equilibrium of $\left(\Gamma\left(h^T\right), R\left[\theta|g^{-T}(h)\right]\right)$. Thus, this action profile $s^{\theta}(h)$ is also a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-T}(h)\right]\right)$. Also, note that this profile $s^{\theta}(h)$ is also a Nash equilibrium of $(\Gamma(h), \theta')$.

Since the period-T SCF f^T is a function and since the action profile $s^{\theta}(h)$ is a Nash equilibrium of $(\Gamma(h), \theta')$, it needs to be the case that $g^T\left(s^{\theta}(h)\right) = f^T\left(\theta'|g^{-T}(h)\right)$. It follows from the fact that the SCF f is sequentially decomposable that $g^T\left(s^{\theta}(h)\right) = \varphi^T\left(R\left[\theta'|g^{-T}(h)\right]\right)$, as was to be proved.

Fix any $t \neq 1, T$ and consider any $g^{-t}(h) \in \mathcal{Y}^{-t}$ with $h \in H^t$. Furthermore, consider any profile $R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]$ and any profile $R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]$ in $\mathcal{D}\left[\Theta|g^{-t}(h), f^{+(t+1)}\right]$. Suppose that for every $i \in \mathcal{I}$ and every $a(h) \in A(h)$:

$$\begin{aligned} &\varphi^t\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right) R_i\left[\theta|g^{-t}(h), f^{+(t+1)}\right] g^t(a(h)) \implies \\ &\qquad \varphi^t\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right) R_i\left[\theta'|g^{-t}(h), f^{+(t+1)}\right] g^t(a(h)). \end{aligned} \tag{25}$$

Since the sequential mechanism Γ implements the SCF f in SPE, we have that:

$$\varphi^t\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right) = f^t\left[\theta|g^{-t}(h)\right] = g^t\left(s^{\theta}(h)\right).$$

Moreover, from the definitions of $R_i\left[\theta|g^{-t}(h), f^{+(t+1)}\right]$ and $R_i\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]$ given in (4) and from the fact that Γ sequentially implements the SCF f, one can see that the action profile $s^{\theta}(h)$ is a Nash equilibrium of $\left(\Gamma, R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right)$, that:

$$\begin{aligned} &g^t\left(s^{\theta}(h)\right) R_i\left[\theta|g^{-t}(h), f^{+(t+1)}\right] g^t(a(h)) \iff \\ &\quad \left(g^{-t}(h), g^{+t}\left(s^{\theta}|h\right)\right) R_i(\theta)\left(g^{-t}(h), g^t(a(h)), g^{+(t+1)}\left(s^{\theta}|(h, a(h))\right)\right), \end{aligned} \tag{26}$$

and that:

$$g^t\left(s^\theta(h)\right) R_i\left[\theta'|g^{-t}(h), f^{+(t+1)}\right] g^t(a(h)) \iff$$
$$\left(g^{-t}(h), g^t\left(s^\theta(h)\right), g^{+(t+1)}\left(s^{\theta'}|\left(h, s^\theta(h)\right)\right)\right) R_i(\theta) \left(g^{-t}(h), g^t(a(h)), g^{+(t+1)}\left(s^{\theta'}|(h, a(h))\right)\right). \tag{27}$$

If there exist $i \in \mathcal{I}$ and $a_i(h) \in A_i(h)$ such that:

$$g^t\left(a_i(h), s^\theta_{-i}(h)\right) P_i\left[\theta'|g^{-t}(h), g^{+(t+1)}\left(s^{\theta'}|(h,\cdot)\right)\right] g^t(s^\theta(h)),$$

it follows from (25)-(27) that:

$$g^t\left(a_i(h), s^\theta_{-i}(h)\right) P_i\left[\theta|g^{-t}(h), g^{+(t+1)}\left(s^{\theta}|(h,\cdot)\right)\right] g^t(s^\theta(h)),$$

which contradicts the fact that $s^\theta(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right)$. Thus, the action profile $s^\theta(h)$ is also a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$.

Let $s_i|h \equiv (s_i^\tau)_{\tau \geq t}$ denote the individual i's strategy according to which this i plays $s_i^t = s_i^\theta(h)$ after the history h and she conforms to the strategy $s_i^{\theta'}$ thereafter; that is, $s_i^{\bar{t}} = \left(s_i^{\theta'}\right)^{\bar{t}}$ for every $\bar{t} > t$. Note that $s|h'$ is a Nash equilibrium of $(\Gamma(h'), \theta')$ for every history $h' \in H|(h, a(h))$ since $s^{\theta'}$ is a strategy profile in $SPE(\Gamma, \theta')$. Thus, to have that the strategy profile $s|h$ is a SPE strategy profile of $(\Gamma(h), \theta')$, we need to show that $s|h$ is a Nash equilibrium of $(\Gamma(h), \theta')$.

Since the action profile $s(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$, it follows from (27) that for every $i \in I$ and every $a_i(h) \in A_i(h)$:

$$\left(g^{-t}(h), g^{+t}(s)\right) R_i(\theta') \left(g^{-t}(h), g^t(a_i(h), s_{-i}(h)), g^{+(t+1)}(s|(h, (a_i(h), s_{-i}(h))))\right).$$

Thus, no individual i can gain by deviating from the action profile $s(h)$ and thereafter conforming to s_i, and so the strategy profile $s|h$ is a SPE of $(\Gamma(h), \theta')$.

Since the sequential mechanism Γ implements the SCF f in SPE, we have that $f^t[\theta'|g^{-t}(h)] = g^t(SPE(\Gamma(h), \theta'))$. Moreover, given that the period-t social function f^t is a function and that the strategy profile $s|h$ is a SPE of $(\Gamma(h), \theta')$, it also follows that $f^t[\theta'|g^{-t}(h)] = g^t(s(h))$. Since f is sequentially decomposable, we have $g^t(s(h)) = \varphi^t\left(R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$, as was to be shown.

Consider some $R[\theta|f^{+2}]$ and some $R[\theta'|f^{+2}]$ in $\mathcal{D}[\Theta|f^{+2}]$. Suppose that for every $i \in \mathcal{I}$ and every $a(h^1) \in A(h^1)$:

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) R_i\left[\theta|f^{+2}\right] g^1\left(a\left(h^1\right)\right) \implies \varphi^1\left(R\left[\theta|f^{+2}\right]\right) R_i\left[\theta'|f^{+2}\right] g^1\left(a\left(h^1\right)\right). \tag{28}$$

Since f is sequentially decomposable, we have that

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) = f^1[\theta] = g^1\left(s^\theta\left(h^1\right)\right).$$

Moreover, it also follows from the definitions of $R_i[\theta|f^{+2}]$ and $R_i[\theta'|f^{+2}]$ given in (7) and from the fact that Γ sequentially implements the SCF f that the action profile $s^\theta(h^1)$ is a Nash equilibrium of $(\Gamma, R[\theta|f^{+2}])$, that:

$$\begin{aligned}&\varphi^1(R[\theta|f^{+2}])\, R_i[\theta|f^{+2}]\, g^1(a(h^1)) \iff \\ &\qquad \left(g^1\left(s^\theta\left(h^1\right)\right), g^{+2}\left(s^\theta|s^\theta\left(h^1\right)\right)\right) R_i(\theta)\left(g^1\left(a\left(h^1\right)\right), g^{+2}\left(s^\theta|a\left(h^1\right)\right)\right),\end{aligned} \tag{29}$$

and that:

$$\begin{aligned}&\varphi^1(R[\theta|f^{+2}])\, R_i[\theta'|f^{+2}]\, g^1(a(h^1)) \iff \\ &\qquad \left(g^1\left(s^\theta\left(h^1\right)\right), g^{+2}\left(s^{\theta'}|s^\theta\left(h^1\right)\right)\right) R_i(\theta')\left(g^1\left(a\left(h^1\right)\right), g^{+2}\left(s^{\theta'}|a\left(h^1\right)\right)\right).\end{aligned} \tag{30}$$

Suppose that

$$g^1\left(a_i\left(h^1\right), s^\theta_{-i}\left(h^1\right)\right) P_i\left(\theta'|f^{+2}\right) g^1\left(s^\theta\left(h^1\right)\right)$$

for some $i \in I$ and some $a_i(h^1) \in A_i(h^1)$. Thus, it follows from (28)-(30) that:

$$\begin{aligned}&g^1\left(a_i(h^1), s^\theta_{-i}(h^1)\right) P_i(\theta|f^{+2})\, g^1\left(s^\theta(h^1)\right) \iff \\ &\quad \left(g^1\left(a_i(h^1), s^\theta_{-i}(h^1)\right), g^{+2}\left(s^\theta|\left(a_i(h^1), s^\theta_{-i}(h^1)\right)\right)\right) P_i(\theta)\left(g^1\left(s^\theta(h^1)\right), g^{+2}\left(s^\theta|s^\theta(h^1)\right)\right),\end{aligned}$$

which contradicts the fact that action profile $s^\theta(h^1)$ is a Nash equilibrium of $(\Gamma, R[\theta|f^{+2}])$. Therefore, the profile $s^\theta(h^1)$ is also a Nash equilibrium of $(\Gamma, R[\theta'|f^{+2}])$.

As we did previously, let $s_i \equiv (s_i^\tau)_{\tau\geq 1}$ denote the individual i's strategy according to which this i plays $s_i^1 \equiv s_i^\theta(h^1)$ at the start of the game and thereafter she conforms to the strategy $s_i^{\theta'}$; that is, $s_i^t \equiv \left(s_i^{\theta'}\right)^t$ for every $t \geq 2$.

Note that $s|h'$ is a Nash equilibrium of $(\Gamma(h'), \theta')$ for every nontrivial history $h \in H$ since $s^{\theta'}$ is a strategy profile in $SPE(\Gamma, \theta')$. Thus, to have that the strategy profile s is a SPE of (Γ, θ'), we need to show that s is also a Nash equilibrium of (Γ, θ').

Since the action profile $s(h^1)$ is a Nash equilibrium of $(\Gamma(h^1), R[\theta'|f^{+2}])$, it follows from (27) that for every $i \in I$ and every $a_i(h^1) \in A_i(h^1)$:

$$(g(s)) R_i(\theta') \left(g^1\left(a_i\left(h^1\right), s_{-i}\left(h^1\right)\right), g^{+2}\left(s|\left(a_i\left(h^1\right), s_{-i}\left(h^1\right)\right)\right)\right).$$

Thus, no individual i can gain by deviating from $s_i(h^1)$ and thereafter conforming to s_i, and so the strategy profile s is a SPE of $(\Gamma(h), \theta')$.

Since the sequential mechanism Γ implements the SCF f in SPE, we have that $f^1[\theta'] = g^1(SPE(\Gamma, \theta'))$. Moreover, given that the period-1 social function f^1 is a function and that the strategy profile s is a SPE of (Γ, θ'), it also follows that $f^1[\theta'] = g^1(s(h^1))$. Since f is sequentially decomposable, we have $g^1(s(h^1)) = \varphi^1(R[\theta'|f^{+2}])$, as was to be shown. ■

Proof of Theorem 3

Proof of Theorem 3. The proof is based on the construction of a sequential mechanism Γ, where each period-t mechanism is a canonical mechanism.

Period-1 mechanism:

Individual i's period-1 action space is defined by:

$$A_i\left(H^1\right) \equiv \mathcal{D}\left[\Theta|f^{+2}\right] \times Y^1 \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers and H^1 is the null set. Thus, a period-1 action of individual i consists of an element of the set Y^1, an element of the period-1 domain of marginal preferences induced by the set Θ at the socially optimal 2-tail outcome paths f^{+2}, and a nonnegative integer. A typical period-1 action played by individual i is denoted by

$a_i\left(h^1\right) \equiv \left(\left(R\left[\bar{\theta}|f^{+2}\right]\right)^i, \left(x^1\right)^i, \left(z\right)^i\right).$

Period-1 action space of individuals is the product space:

$$A\left(H^1\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(H^1\right),$$

with $a\left(h^1\right)$ as a typical period-1 action profile.

The period-t outcome function g^1 is defined by the following three rules:

Rule 1: If $a_i\left(h^1\right) \equiv \left(R\left[\bar{\theta}|f^{+2}\right], x^1, 0\right)$ for every $i \in \mathcal{I}$ and $x^1 = \varphi^1\left(R\left[\bar{\theta}|f^{+2}\right]\right)$, then $g^1\left(a\left(h\right)\right) = x^1$.

Rule 2: If $n-1$ individuals play $a_j\left(h^1\right) \equiv \left(R\left[\bar{\theta}|f^{+2}\right], x^1, 0\right)$ with $x^1 = \varphi^1\left(R\left[\bar{\theta}|f^{+2}\right]\right)$ but individual i plays $a_i\left(h^1\right) \equiv \left(\left(R\left[\bar{\theta}|f^{+2}\right]\right)^i, \left(x^1\right)^i, \left(z\right)^i\right) \neq a_j\left(h^1\right)$, then we can have two cases:

1. If $x^1 R_i\left[\bar{\theta}|f^{+2}\right]\left(x^1\right)^i$, then $g^1\left(a\left(h^1\right)\right) = \left(x^1\right)^i$.

2. If $\left(x^1\right)^i P_i\left[\bar{\theta}|f^{+2}\right] x^1$, then $g^1\left(a\left(h^1\right)\right) = x^1$.

Rule 3: Otherwise, an integer game is played: identify the individual who plays the highest integer (if there is a tie at the top, pick the individual with the lowest index among them.) This individual is declared the winner of the game, and the alternative implemented is the one she selects.

Period-t mechanism with $t \neq 1, T$:

Individual i's period-t action space after history $h \in H^t$ such that $g^{-t}\left(h\right) \in \mathcal{Y}^{-t}$ is defined by:

$$A_i\left(h\right) \equiv \mathcal{D}\left[\Theta|g^{-t}\left(h\right), f^{+(t+1)}\right] \times Y^t\left(g^{-t}\left(h\right)\right) \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers. Thus, a period-t action of individual i after history $h \in H^t$ consists of an element of the set $Y^t\left(g^{-t}\left(h\right)\right)$, an element of the period-t domain of marginal preferences induced by the set Θ at the t-head outcome path $g^{-t}\left(h\right)$ and at the socially optimal $t+1$-tail outcome paths $f^{+(t+1)}$, and a nonnegative integer. A typical period-t action played by individual i after history $h \in H^t$ is denoted by $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)^i, (x^t)^i, (z)^i\right)$.

Period-t action space of individuals after history $h \in H^t$ is the product space:

$$A(h) \equiv \prod_{i \in \mathcal{I}} A_i(h),$$

with $a(h)$ as a typical period-t action profile after history $h \in H^t$.

The period-t outcome function g^t is defined by the following three rules for every $h \in H^t$ such that $g^{-t}(h) \in \mathcal{Y}^{-t}$:

Rule 1: If $a_i(h) \equiv \left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right], x^t, 0\right)$ for every $i \in \mathcal{I}$ and $x^t = \varphi^t\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)$, then $g^t(a(h)) = x^t$.

Rule 2: If $n-1$ individuals play $a_j(h) \equiv \left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right], x^t, 0\right)$ with

$$x^t = \varphi^t\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)$$

but individual i plays $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)^i, (x^t)^i, (z)^i\right) \neq a_j(h)$, then we can have two cases:

1. If $x^t R_i\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right](x^t)^i$, then $g^t(a(h)) = (x^t)^i$.

2. If $(x^t)^i P_i\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right] x^t$, then $g^t(a(h)) = x^t$.

Rule 3: Otherwise, an integer game is played: identify the individual who plays the highest integer (if there is a tie at the top, pick the individual with the lowest index among them.) This individual is declared the winner of the game, and the alternative implemented is the one she selects.

Period-T mechanism:

Individual i's period-T action space after history $h \in H^T$ such that $g^{-T}(h) \in \mathcal{Y}^{-T}$ is defined by:

$$A_i(h) \equiv \mathcal{D}\left[\Theta|g^{-T}(h)\right] \times Y^T\left(g^{-T}(h)\right) \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers. Thus, a period-T action of individual i after history $h \in H^T$ consists of an element of the set $Y^T\left(g^{-T}(h)\right)$, an element of the period-T domain of marginal preferences induced by the set Θ and the T-head outcome path $g^{-T}(h)$, and a nonnegative integer. A typical period-T action played by individual i after history $h \in H^T$ is denoted by $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)^i, \left(x^T\right)^i, (z)^i\right)$.

Period-T action space of individuals after history $h \in H^T$ is the product space:

$$A(h) \equiv \prod_{i \in \mathcal{I}} A_i(h),$$

with $a(h)$ as a typical period-T action profile after history $h \in H^T$.

The period-T outcome function g^T is defined by the following three rules for every $h \in H^T$ such that $g^{-T}(h) \in \mathcal{Y}^{-T}$:

Rule 1: If $a_i(h) \equiv \left(R\left[\bar{\theta}|g^{-T}(h)\right], x^T, 0\right)$ for every $i \in \mathcal{I}$ and $x^T = \varphi^T\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)$, then $g^T(a(h)) = x^T$.

Rule 2: If $n-1$ individuals play $a_j(h) \equiv \left(R\left[\bar{\theta}|g^{-T}(h)\right], x^T, 0\right)$ with $x^T = \varphi^T\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)$ but individual i plays $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)^i, \left(x^T\right)^i, (z)^i\right) \neq a_j(h)$, then we can have two cases:

1. If $x^T R_i\left[\bar{\theta}|g^{-T}(h)\right]\left(x^T\right)^i$, then $g^T(a(h)) = \left(x^T\right)^i$.
2. If $\left(x^T\right)^i P_i\left[\bar{\theta}|g^{-T}(h)\right] x^T$, then $g^T(a(h)) = x^T$.

Rule 3: Otherwise, an integer game is played: identify the individual who plays the highest integer (if there is a tie at the top, pick the individual with the lowest index among them.) This individual is declared the winner of the game, and the alternative implemented is the one she selects.

Let

$$H \equiv \bigcup_{t \in \mathcal{T}} H^t$$

be the set of all possible histories, let $A_i \equiv \bigcup_{h \in H} A_i(h)$ be the set of all actions for individual $i \in \mathcal{I}$, let $A(H)$ be the set of all profiles of actions available to individuals, defined by

$$A(H) \equiv \bigcup_{h \in H} A(h),$$

and let $g \equiv (g^1, \cdots, g^T)$ be the sequence of outcome functions, one for each period $t \in \mathcal{T}$. Note that g satisfies the following properties: a) the outcome function g^1 assigns to period-1 action profile $a(h^1) \in A(h^1)$ a unique outcome in Y^1, and b) for every period $t \neq 1$ and every nontrivial history $h^t \in H^t$, the outcome function g^t assigns to each period-t action profile $a(h^t) \in A(h^t)$ a unique outcome in $Y^t(g^{-t}(h^t))$. Thus, by construction, $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ is a sequential mechanism.

We now prove that (a) for every $\theta \in \Theta$, there exists a SPE strategy $s^\theta \in S$ of (Γ, θ) such that $g^1(s^\theta(h^1)) = f^1[\theta]$, $f^t[\theta|g^{-t}(h^t)] = g^t(s^\theta(h^t))$ for every nontrivial $h^t \in H^t$, and (b) for every $\theta \in \Theta$ and for every $s^\theta \in SPE(\Gamma, \theta)$, $g^1(s^\theta(h^1)) = f^1[\theta]$ and $f^t[\theta|g^{-t}(h^t)] = g^t(s^\theta(h^t))$ for every nontrivial $h^t \in H^t$. Thus, fix any state $\theta \in \Theta$.

Let us first prove (a). Since the SCF is sequentially decomposable, we have that $f^1[\theta] = \varphi^1(R[\theta|f^{+2}])$, that $f^t[\theta|g^{-t}(h)] = \varphi^t(R[\theta|g^{-t}(h), f^{+(t+1)}])$ for every nontrivial $h \in H^t$ and every $t \neq 1, T$ and that $f^T[\theta|g^{-T}(h)] = \varphi^T(R[\theta|g^{-T}(h)])$ for every $h \in H^T$.

Let us define individual $i \in \mathcal{I}$'s strategy $s_i^\theta : H \to A_i$ by:

$$s_i^\theta(h^1) = \left(R\left[\theta|f^{+2}\right], \varphi^1\left(R\left[\theta|f^{+2}\right]\right), 0\right),$$

$$s_i^\theta(h) = \left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right], \varphi^t\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right), 0\right), \quad \text{for every } h \in H^t \text{ with } t \neq 1, T,$$

$$s_i^\theta(h) = \left(R\left[\theta|g^{-T}(h)\right], \varphi^T\left(R\left[\theta|g^{-t}(h)\right]\right), 0\right), \quad \text{for every } h \in H^T.$$

For every period t and history $h^t \in H^t$, to show that $s^\theta|h^t \equiv (s_1^\theta|h^t, \cdots, s_I^\theta|h^t)$ is a SPE of $(\Gamma(h^t), \theta)$ it suffices to show that no individual i can gain by deviating from $s_i^\theta|h^t$ in a single period $\tau \geq t$ and conforming to $s_i^\theta|h^t$ thereafter. To this end, first note that for every history $h \in H$, the strategy profile $s^\theta(h)$ falls into *Rule 1*. Thus, by construction and the fact that the SCF is sequentially decomposable, one can check that $g^1(s^\theta(h^1)) = f^1[\theta]$,

$f^t [\theta|g^{-t}(h^t)] = g^t \left(s^\theta (h^t)\right)$ for every $h^t \in H^t$ and every $t \neq 1$.

Fix any period t and any history $h^t \in H^t$. Suppose that individual i deviates from $s_i^\theta|h^\tau$ with $h^\tau \in H|h^t$ by changing only the action $s_i^\theta(h^\tau)$ into $a_i(h^\tau) \in A_i(h^\tau)$. Given that no unilateral deviation from $s^\theta(h^\tau)$ can induce *Rule 3*, the outcome is thus determined by *Rule 2*. But then, under this rule the outcome would only change to be the period-τ outcome announced by this i in her deviation if this outcome is not better than the outcome $g^\tau \left(s^\theta (h^\tau)\right)$ according to the period-τ marginal ordering $R_i [\theta|f^{+2}]$ if $\tau = 1$, to the period-τ marginal ordering $R_i \left[\theta|g^{-\tau}(h^\tau), f^{+(t+1)}\right]$ if $\tau \neq 1, T$, and to the period-τ marginal ordering $R_i [\theta|g^{-\tau}(h^\tau)]$ if $\tau = T$. By noting that $R_i [\theta|f^{+2}]$ is the true period-1 marginal ordering of individual i in state θ at the socially optimal 2-tail outcome paths $f^{+2}[\theta|\cdot]$ if $\tau = 1$, that $R_i \left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]$ is the true period-τ marginal ordering of individual i in state θ at the head-path $g^{-\tau}(h^\tau)$ and the socially optimal τ-tail outcome paths $f^{+(\tau+1)}[\theta|\cdot]$ if $\tau \neq 1, T$ and that $R_i [\theta|g^{-\tau}(h^\tau)]$ is the true period-τ marginal ordering of individual i in state θ at the head-path $g^{-\tau}(h^\tau)$ if $\tau = T$, individual i will not benefit from such a deviation. Since the choice of individual i as well as of the history $h^\tau \in H^\tau|h^t$ are arbitrary, we conclude that the strategy profile $s^\theta|h^t$ is a SPE of $(\Gamma(h^t), \theta)$. Hence, the proposed strategy profile $s^\theta|h$ is a SPE of $(\Gamma(h), \theta)$ for every history $h \in H$, whose outcomes are such that $g^1 \left(s^\theta (h^1)\right) = f^1[\theta]$, $f^t [\theta|g^{-t}(h^t)] = g^t \left(s^\theta (h^t)\right)$ for every $h^t \in H^t$ and every $t \neq 1$. This proves our goal (a) stated above. The rest of the proof shows that our goal (b) holds, too.

To see this, assume that the strategy profile s is a SPE of (Γ, θ). Moreover, fix any history $h \in H$. Thus, the strategy profile $s|h$ is a SPE of $(\Gamma(h), \theta)$. Assume, to the contrary, that there is a period $t \in \mathcal{T}$ as well as a history $h^t \in H|h$ such that either $f^t [\theta|g^{-t}(h^t)] \neq g^t(s(h^t))$ if $t \neq 1$ or $f^1[\theta] \neq g^1(s(h^1))$ if $t = 1$. Among all such histories, let $h^\tau \in H|h$ be one of the longest histories. Thus, it must be the case that $f^\tau [\theta|g^{-\tau}(h^\tau)] \neq g^\tau(s(h^\tau))$ and, moreover, that $f^{\hat{\tau}} \left[\theta|g^{-\hat{\tau}}\left(h^{\hat{\tau}}\right)\right] = g^{\hat{\tau}}\left(s\left(h^{\hat{\tau}}\right)\right)$ for every $h^{\hat{\tau}} \in H|(h^\tau, s^\tau(h^\tau))$ if $\tau \neq T$. Note that for the case where $\tau \neq T$ the sequential decomposability of the SCF f implies that:

$$g^{\hat{\tau}}\left(s\left(h^{\hat{\tau}}\right)\right) = \varphi^{\hat{\tau}}\left(R\left[\theta|g^{-\hat{\tau}}\left(h^{\hat{\tau}}\right), f^{+(\hat{\tau}+1)}\right]\right) \text{ for every } h^{\hat{\tau}} \in H|(h^\tau, s^\tau(h^\tau)) \text{ with } \hat{\tau} \neq T, \text{ and that:}$$

$$g^T\left(s\left(h^T\right)\right) = \varphi^T\left(R\left[\theta|g^{-T}\left(h^T\right)\right]\right) \text{ for every } h^T \in H|(h^\tau, s^\tau(h^\tau)).$$

Also, note that the true profile of period-τ marginal orderings at true state θ is:

$$R\left[\theta|f^{+(\tau+1)}\right] \text{ if } \tau = 1,$$

$$R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right] \text{ if } \tau \neq 1, T,$$

$$R\left[\theta|g^{-\tau}\left(h^{\tau}\right)\right] \text{ if } \tau = T.$$

Let us suppose that $\tau \neq 1, T$. Then, the action profile $s\left(h^{\tau}\right)$ is a Nash equilibrium of $\left(\Gamma, R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)$.

Suppose that $s\left(h^{\tau}\right)$ falls into *Rule 1* of period-τ mechanism. Thus, $g^{\tau}\left(s\left(h^{\tau}\right)\right) = \varphi^{\tau}\left(R\left[\bar{\theta}|g^{-\tau}\left(h\right), f^{+(\tau+1)}\right]\right)$, and this outcome is an element of $Y^{\tau}\left(g^{-\tau}\left(h^{\tau}\right)\right)$. Since f is sequentially decomposable, an immediate contradiction is obtained if $g^{\tau}\left(s\left(h^{\tau}\right)\right) = \varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h\right), f^{+(\tau+1)}\right]\right)$. Therefore, let us suppose that $g^{\tau}\left(s\left(h^{\tau}\right)\right) \neq \varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h\right), f^{+(\tau+1)}\right]\right)$.

Since f is sequentially Maskin monotonic and since

$$\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right) \neq \varphi^{\tau}\left(R\left[\bar{\theta}|g^{-\tau}\left(h\right), f^{+(\tau+1)}\right]\right),$$

there exists an individual i and a period-τ outcome $y^{\tau} \in Y^{\tau}\left(g^{-\tau}\left(h^{\tau}\right)\right)$ such that

$$\varphi^{\tau}\left(R\left[\bar{\theta}|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right) R_i y^{\tau}$$

and

$$y^{\tau} P_i\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right] \varphi^{\tau}\left(R\left[\bar{\theta}|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right).$$

By changing $s_i\left(h^{\tau}\right)$ into $a_i\left(h^{\tau}\right) = \left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right], y^{\tau}, 1\right)$, individual i can induce *Rule 2* and obtain $g^{\tau}\left(a_i\left(h^{\tau}\right), s_{-i}\left(h^{t}\right)\right) = y^{\tau}$, thereby contradicting the fact that the action profile $s\left(h^{\tau}\right)$ is a Nash equilibrium of $\left(\Gamma, R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)$.

Suppose that $s\left(h^{\tau}\right)$ falls into *Rule 2* of period-τ mechanism. Thus, for every individual $j \neq i$, the period-τ outcome determined by this rule is maximal for this j in $Y^{\tau}\left(g^{-\tau}\left(h^{\tau}\right)\right)$ according to her period-τ marginal ordering $R_j\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]$. Moreover, given that the action profile $s\left(h^{\tau}\right)$ is a Nash equilibrium of $\left(\Gamma, R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)$, for individual i it holds that the outcome $g^{\tau}\left(s\left(h^{t}\right)\right)$ is such that $g^{\tau}\left(s\left(h^{t}\right)\right)$ is an element of the weak lower contour set of $R_i\left[\bar{\theta}|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]$ at $\varphi^{\tau}\left(R\left[\bar{\theta}|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)$ and that

$$g^{\tau}\left(s\left(h^{t}\right)\right) R_i\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right] x^{\tau}$$

for every x^{τ} in the weak lower contour set of $R_i\left[\bar{\theta}|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]$ at $\varphi^{\tau}\left(R\left[\bar{\theta}|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)$. Since the SCF f satisfies the sequentially weak no veto-power, this implies that

$$g^{\tau}\left(s\left(h^{\tau}\right)\right)=\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right).$$

The sequential decomposability of f implies that $\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)=f^{\tau}\left[\theta|g^{-\tau}\left(h^{\tau}\right)\right]$, which is a contradiction.

Suppose that $s\left(h^{\tau}\right)$ falls into *Rule 3* of period-τ mechanism. Thus, for every individual j, the period-τ outcome determined by this rule is maximal for this j in $Y^{\tau}\left(g^{-\tau}\left(h^{\tau}\right)\right)$ according to her period-τ marginal ordering $R_j\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]$. Since the SCF f satisfies the sequentially unanimity, we have that $g^{\tau}\left(s\left(h^{\tau}\right)\right)=\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)$. The sequential decomposability of f implies that $\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right), f^{+(\tau+1)}\right]\right)=f^{\tau}\left[\theta|g^{-\tau}\left(h^{\tau}\right)\right]$, which is a contradiction.

We conclude the proof by mentioning that, suitably modified, the above proof provided for the case where $\tau \neq 1, T$ applies to the case where $\tau=1$ as well as to the case where $\tau=T$. ■

Proof of Lemma 1

PROOF OF LEMMA 1. Let the premises hold. Fix any $x^1 \in X^1$ and any $\theta \in \Theta$. Let $x^2\left[\eta|x^1\right]$ be the solution to:

$$\frac{\partial U(\eta, x^1, x^2)}{\partial x^2}=0.$$

By the implicit function theorem, we have that:

$$\frac{\partial x^2\left[\eta|x^1\right]}{\partial \eta}=-\frac{\frac{\partial^2 U\left(\eta, x^1, x^2\left[\eta|x^1\right]\right)}{\partial^2 x^2}}{\frac{\partial^2 U\left(\eta, x^1, x^2\left[\eta|x^1\right]\right)}{\partial \eta \partial x^2}}>0.$$

Therefore, the peak for the median type $\eta = \theta_{med}$ is always the peak in the second voting stage for each $x^1 \in X^1$. Write $x^2\left[\theta_{med}|x^1\right]$ for the peak of the median type in the second voting stage conditional on x^1.

Since it holds that:

$$\frac{\partial U\left(\theta_{med}, x^1, x^2\left[\theta_{med}|x^1\right]\right)}{\partial x^2} = 0,$$

from the implicit function theorem we obtain that:

$$\frac{\partial x^2\left[\theta_{med}|x^1\right]}{\partial x^1} = -\frac{\frac{\partial^2 U\left(\theta_{med}, x^1, x^2\left[\theta_{med}|x^1\right]\right)}{\partial x^1 \partial x^2}}{\frac{\partial^2 U\left(\theta_{med}, x^1, x^2\left[\theta_{med}|x^1\right]\right)}{\partial^2 x^2}} \geq 0.$$

Let us show that $x^2\left[\theta_{med}|x^1\right]$ is the Condorcet winner under $R\left[\theta|x^1\right]$ for every $x^1 \in X^1$. For every allowable type $\eta \in \left(\underline{\eta}, \bar{\eta}\right)$ and policy (x^1, x^2), let:

$$\Phi(\eta, x^1, x^2) = U\left(\eta, x^1, x^2\left[\theta_{med}|x^1\right]\right) - U(\eta, x^1, x^2).$$

Then, for every $x^2 < x^2\left[\theta_{med}, x^1\right]$, we have that:

$$\Phi(\theta_{med}, x^1, x^2) = \int_{x^2}^{x^2\left[\theta_{med}|x^1\right]} \frac{\partial U(\theta_{med}, x^1, z^2)}{\partial z^2} dz^2.$$

Furthermore, for every $\eta > \theta_{med}$, it holds that:

$$\Phi(\eta, x^1, x^2) - \Phi(\theta_{med}, x^1, x^2) = \int_{\theta_k}^{\eta} \int_{x^2}^{x^2\left[\theta_{med}|x^1\right]} \frac{\partial^2 U(\alpha, x^1, z^2)}{\partial \alpha \partial z^2} dz^2 d\alpha > 0.$$

Since

$$\Phi(\theta_{med}, x^1, x^2) = U(\theta_{med}, x^1, x^2\left[\theta_{med}|x^1\right]) - U(\theta_{med}, x^1, x^2) \geq 0,$$

it follows that:

$$\Phi(\eta, x^1, x^2) > 0,$$

which, in turn, guarantees that:

$$U(\eta, x^1, x^2\left[\theta_{med}|x^1\right]) > U(\eta, x^1, x^2).$$

Therefore, for every voter $j = k+1, \cdots, 2k-1$, it holds that:

$$U(\theta_j, x^1, x^2\left[\theta_{med}|x^1\right]) > U(\theta_j, x^1, x^2).$$

Likewise, for every $x^2 > x^2[\theta_{med}|x^1]$, one can show that for every voter $j = 1, \cdots, k-1$ it holds that:

$$U(\theta_j, x^1, x^2\left[\theta_{med}|x^1\right]) > U(\theta_j, x^1, x^2).$$

Therefore, $x^2[\theta_{med}|x^1]$ is a Condorcet winner under $R[\theta|x^1]$, that is, $CW(R[\theta|x^1]) = x^2[\theta_{med}|x^1]$, and so the majority voting function $f^2[\cdot|\cdot]$ is a single-valued function for every $\theta \in \Theta$ and every $x^1 \in X^1$.

Let $x[\theta_{med}] = (x^1[\theta_{med}], x^2[\theta_{med}])$ be the global peak for the median type θ_{med}. Next, we show that $x^1[\theta_{med}]$ is the Condorcet winner under $R[\theta|f^2]$.

Solving backward, given that the majority voting function $f^2[\theta|x^1] = x^2[\theta_{med}|x^1]$ for every $x^1 \in X^1$, we have that the reduced utility of type η is:

$$V(\eta, x^1) = U(\eta, x^1, x^2\left[\theta_{med}|x^1\right]).$$

Then, we have that:

$$\frac{\partial V(\eta, x^1)}{\partial x^1} = \frac{\partial U(\eta, x^1, x^2[\theta_{med}|x^1])}{\partial x^1} + \frac{\partial U(\eta, x^1, x^2[\theta_{med}|x^1])}{\partial x^2}\frac{\partial x^2[\theta_{med}|x^1]}{\partial x^1},$$

and so, by Definition 13, it follows that:

$$\frac{\partial^2 V(\eta, x^1)}{\partial \eta \partial x^1} = \frac{\partial^2 U(\eta, x^1, x^2[\theta_{med}|x^1])}{\partial \eta \partial x^1} + \frac{\partial^2 U(\eta, x^1, x^2[\theta_{med}|x^1])}{\partial \eta \partial x^2}\frac{\partial x^2[\theta_{med}|x^1]}{\partial x^1} > 0.$$

Then, for every $x^1 \in X^1$, let:

$$\Pi(\eta, x^1) = V(\eta, x^1[\theta_{med}]) - V(\eta, x^1)$$

Next, take any $x^1 < x^1[\theta_{med}]$. Then, it holds that:

$$\Pi(\theta_{med}, x^1) = \int_{x^1}^{x^1[\theta_{med}]} \frac{\partial V(\theta_{med}, z^1)}{\partial z^1} dz^1.$$

Moreover, for every $\eta > \theta_{med}$, it also holds that:

$$\Pi(\eta, x^1) - \Pi(\theta_{med}, x^1) = \int_{\theta_{med}}^{\eta} \int_{x^1}^{x^1[\theta_{med}]} \frac{\partial^2 V(\alpha, z^1)}{\partial \alpha \partial z^1} dz^1 d\alpha > 0.$$

Since

$$\Pi(\theta_{med}, x^1) = V(\theta_{med}, x^1[\theta_{med}]) - V(\theta_{med}, x^1) \geq 0,$$

we have that:

$$\Pi(\eta, x^1) > 0,$$

which, in turn, guarantees that:

$$V(\eta, x^1[\theta_{med}]) > V(\eta, x^1).$$

Therefore, for every voter $j = k+1, \cdots, 2k-1$, we have that:

$$V(\theta_j, x^1[\theta_{med}]) > V(\theta_j, x^1).$$

Likewise, for every $x^1 > x^1[\theta_{med}]$ one can also show that:

$$V(\theta_j, x^1[\theta_{med}]) > V(\theta_j, x^1), \quad \text{for every voter } j = 1, \cdots, k-1.$$

We conclude that $x^1[\theta_{med}]$ is a Condorcet winner under $R[\theta|f^2]$, that is, $CW(R[\theta|f^2]) = x^1[\theta_{med}]$, and so the majority voting function $f^1[\cdot]$ is a single-valued function for every $\theta \in \Theta$.

■