Implementation in partial equilibrium*

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Abstract

Consider a society with a finite number of sectors (social issues or commodities). In a partial equilibrium (PE) mechanism a sector authority (SA) aims to elicit agents' preference rankings for outcomes at hand, presuming separability of preferences, while such presumption is false in general and such isolated rankings might be artifacts. Therefore, its participants are required to behave *as if* they had separable preferences. This paper studies what can be Nash implemented if we take such misspecification of PE analysis as a given institutional constraint. The objective is to uncover the kinds of complementarity across sectors that this institutional constraint is able to accommodate. Thus, in our implementation model there are several SAs, agents are constrained to submit their rankings to each SA separately and, moreover, SAs cannot communicate with each other. When a social choice rule (SCR) can be Nash implemented by a product set of PE mechanisms, we say that it can be Nash implemented in PE. We identify necessary conditions for SCRs to be Nash implemented in PE and show that they are also sufficient under a domain condition which identifies the kinds of *admissible complementarities*. Thus, the Nash implementation in PE of SCRs is examined in auction and matching environments.

1. Introduction

The methodology used in the literature of mechanism design in order to understand how to solve a single allocation decision problem whose solution depends on private information held by various agents is that of PE analysis. This methodology isolates outcomes to be allocated as well as people's preferences for those outcomes from the rest of the world, under a ceteris paribus (all else equal) assumption. Because of such isolation, PE mechanism design has provided exact mechanisms and algorithms on how to elicit the private information from agents so as to achieve desirable allocation decisions, and has proved capable of handling a wide variety of issues, not only economic but also political and legal. The prominently successful cases are auction and matching.

Of course, this isolation is legitimate if agents have separable preferences over a product set of outcomes $X = X^1 \times \cdots \times X^\ell$. This is because when an agent has a separable preference, a well-defined marginal preference exists on each component set X^s of the product set, which is independent of the values of other components.

The ceteris paribus assumption, however, cannot be true in general, since people's preferences are generally non-separable. This means that a marginal preference over a component set depends on the values chosen for the other components. For example, which school one would like to be admitted to may depend on where she lives and, moreover, which catchment area she would like to live may depend on which school she could be admitted to. When the school authority assumes that each of its participants has a single preference ranking for schools and requires participants to report their school rankings, it forces its participants to behave *as if* their preferences were separable, while such rankings may be artifacts.

Not least, when we change something in the school admission program, it will have a general equilibrium effect, such as changes in the housing market and how people choose where to live, etc. Likewise, when we change something in an auction rule, it will have a general equilibrium effect on how people consume goods related to the auctioned item and, moreover, will affect bidders' willingness to pay for the item auctioned off, and so on.

Perhaps, a centralized allocation mechanism may be better equipped to deal with issues arising from non-separability of agents' preferences. However, this mechanism is not available or feasible in real life. Given that the goal of implementation theory is to study the relationship between outcomes in a society and the mechanisms under which those outcomes arise, it is important to throw light on how such isolations dictated by the practice of PE mechanisms affect outcomes in society. In this paper, we ask the following questions: What do we lose by ignoring such general equilibrium effects? More specifically, if we take the practice dictated by PE mechanisms as a given institutional constraint, can one describe the requirements on SCRs that are equivalent to Nash implementability by a product set of PE mechanisms? What kind of complementarity, if any, is this practice able to accommodate?

This paper answers the above questions by assuming that there are $\ell \geq 2$ social issues, or sectors, and $n \geq 3$ agents in society. It assumes that every agent in society is involved in all social issues.

Moreover, it supposes that there is a Central Authority (CA) who wishes to Nash implement a SCR, which depends on private information held by various agents. Since the CA cannot design any centralized mechanism and, thus, cannot elicit any private information from agents, he delegates the decision-making authority to independent SAs, such as the school authority, the housing authority, and so on.[1] Thus, the CA cannot control the behaviour of agents. Instead, their interaction is controlled by independent SAs.

Given these delegation arrangements, a SA dealing with the social issue s designs an allocation mechanism or PE mechanism, Γ^s, for the issue at hand. This mechanism asks agents to report only the information pertaining to the issue s as well as assigns outcomes of X^s on the basis of the information elicited from agents. Given a product set of PE mechanisms, one for each issue, denoted by $\Gamma = \Gamma^1 \times \cdots \times \Gamma^\ell$, each agent communicates with each SA separately. Since each SA specifies the PE mechanism in advance, the agents themselves know exactly not only which game induced by Γ^s is being played for the issue s, but also which overall game induced by Γ is being played.

This paper uses Nash equilibrium as the equilibrium concept for solving the game that Γ^s leads to in every environment and for solving the game that Γ leads to. This is because difficulties (to be discussed in section 2) arise when agents' preference are non-separable and agents are forced to behave *as if* they had separable preferences.

For instance, a classic PE mechanism is the so-called Top-Trading-Cycle (TTC) algorithm. Many methods for finding desirable allocations in matching environments are variants of this algorithm. The reason for its success is that the TTC algorithm is strategy-proof; that is, true-telling about her own marginal preferences for houses (or some other indivisible items such as tasks or jobs) is a dominant strategy for each agent. However, when preferences are not separable, a dominant strategy no longer exists. Indeed, there is not even a "true" marginal preference for houses. With non-separable preferences it thus becomes necessary to consider a weaker notion of equilibrium.

Further, to make the analysis consistent with the methodology of PE analysis, the paper makes the following 'behavioral' assumptions:

1. The only concern of the SA is to promote the welfare criterion delegated by the CA.

2. The PE mechanism designed by the SA forces its participants to behave *as if* they had separable preferences.

3. There is no communication between SAs about the information elicited from the agents.

4. Each SA cannot conceive agents' preferences for outcomes of X. Each SA can conceive only marginal preferences that are consistent with allowable separable preferences over X.

5. The CA acts *as if* he had not the ability to distinguish whether a Nash equilibrium outcome of the game induced by Γ comes from by a profile of non-separable preferences or from a profile of separable ones if the marginal preferences over each component set X^s induced by the profiles of agents' preferences are *observationally equivalent with respect to their lower contour sets.*

The first assumption is dictated by the fact that the methodology of PE abstracts entirely from incentive problems of SAs. The second and third one come from the isolation

[1] In line with implementation literature, we use the term SA as an idiom for a social planner who selects the mechanism to implement a SCR and who only cares about the welfare of society.

feature of the methodology. Therefore, on the basis of the second assumption, agents report their school rankings to the school authority, their rankings of houses to the housing authority, and so on, though they may have non-separable preferences.

The fourth assumption comes from the fact that in a multi-item auction setting with private values in which each SA auctions off a single item, the assumption of PE methodology that a buyer has a separable preference for the items being sold implies that her preference is representable by a utility function that is additively separable and linear in income; that is, the buyer has a quasi-linear marginal preference for each item (see Proposition 1 of section 2). However, since marginal preferences induced by non-separable preferences are not necessarily quasi-linear, the SA will notice that there is something wrong with the methodology if he could conceive that a buyer could have non-quasi-linear marginal preferences. The fourth assumption thus rules out this type of situation.

The product set of PE mechanisms Γ induces a game when agents' preferences over the product set of outcomes are $R = (R_1, \cdots, R_n)$. If agents' preferences R are separable, then each preference R_i induces a well-defined independent marginal preference R_i^s over each component set X^s. Moreover, a profile of Nash equilibrium decisions made by SAs when agents' marginal preferences over X^s are $R^s = (R_1^s, \cdots, R_n^s)$ is also a Nash equilibrium outcome of the game induced by Γ at the profile R.

However, this feature no longer holds when the preference profile R consists of non-separable preferences. This is because agent i's ranking induced by R_i for the outcomes of X^s depends on the values fixed for the other components. In cases like this, the CA would be able to detect problems in the equilibrium decisions made by SAs.

Thus, the last assumption rules out this possibility and imposes that a profile $x = \left(x^1, \cdots, x^\ell\right)$ of Nash allocation decisions made by SAs when agents' marginal preferences over each component set X^s are $\hat{R}^s = \left(\hat{R}_1^s, \cdots, \hat{R}_n^s\right)$ is also a Nash equilibrium outcome of the game induced by Γ at R provided that for each agent i and each issue s the lower contour set of $\hat{R}_i^s$ at x^s is identical to the lower contour set of the marginal preference $R_i^s\left(x^{s_C}\right)$ induced by the preference R_i over the component set X^s at x^s when the values of other components of x are fixed at $x^{s_C} = \left(x^1, \cdots, x^{s-1}, x^{s+1}, \cdots, x^\ell\right)$.[2] To make this formulation of observationally equivalence operational we identify a domain-richness condition for the domain of marginal preferences of agents.

In the standard literature of Nash implementation, a SCR is Nash implementable if the authority can design a mechanism whose set of Nash equilibrium outcomes coincides with the outcomes prescribed by the welfare criterion incorporated into the SCR. In our set up, the CA has the same objective as in the standard set up, only now he has to achieve it via a product set of PE mechanisms; that is, via resource allocation mechanisms where the decision-making authority is delegated to SAs. Moreover, SAs have the same objective as in the standard set up, only now their Nash implementation problems pertain only to their respective issues. These objectives are linked by the inability of the CA to distinguish whether a Nash equilibrium outcome is attributable to separable preferences or not. If such a product set of PE mechanism Γ exists, we say the SCR is Nash implementable in PE.

In section 4 we show that a SCR defined on a domain of preferences which can be Nash

[2] We write s_C for the complement of s. Moreover, the profile x^{s_C} is obtained from x by omitting the s-th component.

implemented in PE satisfies a decomposability condition, an indistinguishability condition and a sector-wise Maskin monotonicity condition.

Decomposability requires that the SCR can be decomposed into one-dimensional SCRs, one for each sector, and that the range of the SCR is the product of the ranges of the one-dimensional SCRs if the domain of the SCR consists only of separable preferences.

Sector-wise Maskin monotonicity requires that each one-dimensional SCR needs to satisfy the standard invariance condition due to Maskin (1999).

The indistinguishability condition states that the CA cannot veto a profile of allocations made by SAs for being socially undesirable at one profile of preferences R when each SA decision is based on marginal preferences which are observationally equivalent (with respect to lower contour sets) to the marginal preferences induced by the profile R.

Given these necessary conditions, and under a domain restriction, we characterize Nash implementability in PE with recourse to two conditions reminiscent of the so-called no veto-power condition. It follows from this result that for Nash implementation problems in PE in which there is a private good, Nash implementability in PE of a SCR is nearly equivalent to sector-wise Maskin monotonicity, decomposability and indistinguishability. We also show that the domain restriction, *Property* α, is indispensable for the theorem to hold (see Example 1).

The idea is very simple. Our exercise does not deliver anything new when individuals have separable preferences, because the implementation problem can simply be decomposed into several implementation problems, one for each sector, and each SA solves its problem separately. The interesting case is thus when preferences are not separable. What can we do in this case given the institutional constraint represented by the practice PE mechanisms? Under the domain condition, our answer is, *nothing.* It is not the SAs' business to care about non-separabilities, and the CA cannot do anything about it. School board simply asks households to submit rankings over schools, whether there is indeed such thing or not. Housing agency simply asks households to submit rankings only over houses, whether there is indeed such thing or not. Auction agency simply asks bidders to submit their willingness to pay for an item, whether there is indeed such thing or not, and so on.[3]

To illustrate further, in a perhaps cynical manner, consider that there are two sectors, say L and R, where Sector L allocates left shoes and Sector R allocates right shoes. When an individual reports to the SA for L that she wants some left shoe, we can naturally understand that this is so because she intends and expects to get the right shoe counterpart from SA for R. However, the SA for L here understands only that she likes that left shoe as an individual item, and does not question why she wants it. This is stupid, but this is exactly the nature of PE mechanism design.

[3]We should note that in our discussion we say each individual reports only marginal rankings to each SA just for illustration purpose. Actually, for each of the mechanism we construct, individuals' strategies also include an allocation component and tie-breaking device component. Given that our mechanisms are admittedly abstract and there is no reason to restrict attention to them, one may wonder whether one can obtain more permissive results if each individual is allowed to report information about the entire economy. Unfortunately, the result cannot change because of the institutional constraint of the PE analysis. Indeed, what an individual may report to one SA about the entire economy can be different from what she reports to another SA. Moreover, the SAs do not communicate each other for cross-checking the information collected and there is no additional informational value in doing so.

Being able to remain content with such nature crucially depends on the domain condition, which roughly states that one can never gain by getting worse outcome in each sector. This restricts the kinds of complementarities that are admissible in PE design. One might suspect that this condition is simply a tailor-made one such that the product of sector-wise mechanisms works. However, we show that the condition is indispensable for any PE mechanism to work, not just for the particular mechanism we have constructed. Therefore, in order to accommodate complementarities that lay outside our domain condition there is a need to move away from the practice of PE analysis and to start analysing implementation problems where an incomplete, but yet not negligible, communication is allowed among SAs and where the CA has to make some modelling choice about how SAs communicate.

Section 5 assesses the implications of our characterization result in matching and auction settings. It shows that some non-dictatorial SCRs defined on preference domains that allow non-separability of preferences are Nash implementable in PE. For instance, in a multi-item auction setting with private values in which buyers have non-separable preferences for items being sold *due to* income effects, one can attain the goal of efficiency as a Nash equilibrium outcome in PE by means of the sector-wise Vickrey (second-price; 1961) auction solution. Roughly speaking, the sector-wise Vickrey auction solution is a solution that assigns each item to the highest bidder and prescribes that this winner pays the amount of the second-highest bid.

The remainder of the paper is organized as follows. Section 2 provides motivating examples in matching and auction environments. Section 3 sets out the theoretical framework and outlines the basic model, while necessary and sufficient conditions are presented in section 4. Section 5 assesses the implications of our characterization result. Section 6 concludes by suggesting directions for future research. Appendix includes proofs not in the main body.

2. Leading examples

To illustrate our points we discuss difficulties that arises from the assumption of separability of preferences in two prominent cases of PE mechanism design: matching and auction.

Matching

Matching theory studies the design and performance of algorithms for transaction between agents. Broadly speaking, it studies who interacts with whom, and how to allocate and exchange transplant organs, dormitory rooms to students, school seats to children, and so on. Many methods for finding desirable allocations are variants of the top trading cycle (TTC) algorithm (Shapley and Scarf, 1974).

Suppose that n agents own an indivisible good (a house) and have strict preferences over the set H of houses. Agent i initially owns house h_i. The TTC algorithm can be described as follows:

Step 1. Each agent points to the owner of her favorite house. Since there are n agents, there is at least one cycle. Give each agent in a cycle the house she points at and remove her from the market with her assigned house. If there is at least one remaining agent, proceed with the next step.

...

Step k. Each remaining agent points to the owner of her favorite house among the remaining ones. Give each agent in a cycle the house she points at and remove her from the market with her assigned house. If there is at least one remaining agent, proceed with the next step.

...

When agents' preferences are separable and their marginal orderings over houses are strict, the TTC assignment is the unique core allocation corresponding to the reported marginal orderings; that is, there is no subset of owners who can make all of its members better off by exchanging the houses they initially own in a different way. Moreover, the TTC algorithm is strategy-proof; that is, true-telling about her own preferences for houses is a dominant strategy for each agent. However, when preferences are not separable a dominant strategy no longer exists. Indeed, there is not even a "true" marginal preference for houses. This is because an agent's marginal preferences for houses will depend on the type of assignment she will get for other objects in the economy. Finally, the simple idea to run the TTC algorithm in order to allocate indivisible goods in the economy when there are forms of preference complementarities may result in an inefficient Nash equilibrium assignment.

Let us show it for the case where there are two agents, that is, $n = 2$. Suppose that each agent i owns an indivisible good of type 1, h_i^1, and an indivisible good of type 2, h_i^2, and has preferences over pairs of goods, that is, over $H^1 \times H^2$, where $H^j = \left\{h_1^j, h_2^j\right\}$ for $j = 1, 2$.

Suppose that agents' preferences from best (left) to worst (right) are:

$$\begin{aligned} \text{for agent A} &: (h_B^1, h_B^2) P(R_A) (h_A^1, h_A^2) P(R_A) (h_A^1, h_B^2) I(R_A) (h_B^1, h_A^2), \\ \text{for agent B} &: (h_A^1, h_A^2) P(R_B) (h_B^1, h_B^2) P(R_B) (h_A^1, h_B^2) I(R_B) (h_B^1, h_A^2), \end{aligned}$$

where $I(R_i)$ denotes the symmetric part of R_i and $P(R_i)$ denotes the asymmetric part of R_i. Goods of type 1 can be viewed as school-seats and goods of type 2 as houses. Suppose that house h_i^2 is in the catchment area of school-seat h_i^1. An interpretation of agent A's preference relation R_A is that agent A strictly prefers a school-seat and a house that are in the same catchment area to a school-seat and a house that are far apart from each other and, moreover, she strictly prefers the pair (h_B^1, h_B^2) to (h_A^1, h_A^2). Finally, the pairs (h_A^1, h_B^2) and (h_B^1, h_A^2) are equally bad for her. A similar interpretation holds for preferences of agent B. Therefore, agents' preferences exhibit a kind of complementarity between school-seats and houses.

The school-seat exchange game form can be summarized as follows:

$$\begin{array}{c|c|c|} & h_A^1 & h_B^1 \\ \hline h_A^1 & (h_A^1, h_B^1) & (h_A^1, h_B^1) \\ \hline h_B^1 & (h_B^1, h_A^1) & (h_A^1, h_B^1) \\ \hline \end{array},$$

where agent A is the row player, and the assignment in each box is the TTC assignment to the action profile to which the box corresponds, with agent A's assignment listed first. With the same convention, the house exchange game form can be summarized as follows:

$$\begin{array}{c|c|c|}
 & h_A^2 & h_B^2 \\
\hline
h_A^2 & (h_A^2, h_B^2) & (h_A^2, h_B^2) \\
\hline
h_B^2 & (h_B^2, h_A^2) & (h_A^2, h_B^2) \\
\hline
\end{array}.$$

The Nash equilibrium outcome of the school-seat game depends on the TTC house-assignment, and vice versa.

Indeed, if the TTC house assignment is that agent i keeps living in the same house h_i^2, then the unique strictly dominant strategy for this agent is to keep her initial school-seat h_i^1. This is because agent i strictly prefers the bundle (h_i^1, h_i^2) to $\left(h_j^1, h_i^2\right)$. On the other hand, if the TTC school-seat assignment is that agent i keeps her school-seat h_i^1, then the unique strictly dominant strategy for agent i is to keep living in her house h_i^2. This is because agent i strictly prefers the bundle (h_i^1, h_i^2) to $\left(h_i^1, h_j^2\right)$. Therefore, if agents coordinate on this type of strategy, the Nash equilibrium outcome of the economy is characterised by no trade. However, the no-trade allocation is not an efficient one. This is so because the move from (h_i^1, h_i^2) to $\left(h_j^1, h_j^2\right)$ is a good deal for both agents. In short, if agents could freely barter exchange items, they would rearrange them so as to arrive at the allocation $((h_B^1, h_B^2), (h_A^1, h_A^2))$, where the first entry is the bundle that agent A gets. We conclude by noting that this allocation is the other (pure strategy) Nash equilibrium outcome induced by the TTC algorithm when each agent always points to the endowment of the other agent.

Auction

There are two sources of non-separability of preferences. One source is represented by the complementarity of items across sectors. In general, willingness to pay for a set of items may not be equal to the sum of willingness to pay for its components. The other source is represented by *income effects*: If there is some change in the transfer payment of one sector, this affects how much one agent is willing to pay\accept for an item of another sector, and vice versa.

The first type of non-separability is typically studied in the literature of multiple-object auctions. It is now known that non-separability across related items creates efficiency issues and strategic interaction issues. For instance, as shown by Avery and Hendershott (2000), when items are complements, running first-price auction for each item separately yields higher expected revenue than auctioning a single bundle. The reason is that a bidder who has a stronger form of preference for complementarity bids more aggressively than other bidders in each auction in order to win all the relevant items, since winning just some of them is valueless for her. Although this is optimal from the seller's viewpoint, it causes a problem of inefficiency, because it increases the probability that a bidder seeking complementarity wins only a part of the items which is valueless by itself alone, hence will resale it.

To the best of our knowledge, no attention has been paid so far to the second source of non-separability of preferences. This is because much of the literature on auctions and, more generally, on social decision problems with income transfers, assumes that participants' utilities are additively separable and linear in income, that is, participants have quasi-linear utilities, meaning that each participant's utility is the value of a decision or item assignment plus-or-minus the value of any income transfer that she receives or makes. In other words, the benefits from a decision assignment or from consuming an item are independent of cash transfers.

In what follows, we first clarify that the assumption of zero income effect is indeed an inevitable consequence of the underlying assumption of separability of preferences. This means that if one wants to integrate the type of income effects described above with mechanism design research, the assumption of separability of preferences needs to be dropped. Second, we show how one theoretical attractive auction mechanism such the *Vickrey auction* (1961) fails to have a dominant strategy equilibrium when income effects are allowed.

The following is our description of preferences over sectors or social issues, where each sector consists of a social decision problem with cash transfers. We adopt this specification throughout the paper. Because our objective is to investigate social decision making in all sectors without taking any particular mechanism as given, we take the physical quantity of "income" as a primitive of the model. This is because we assume that there is a consumption good called "commodity money" and it can be used as a means of payment in all sectors.

Let I denote the set of agents. For the sake of simplicity, let us suppose that there are only two important social issues on the table, denoted by $s = 1, 2$, such as two large public projects. Let D^s denote the set of potential pure social decisions for issue s. Let

$$T = \left\{ t \in [-\bar{t}, \infty)^n : \sum_{i \in I} t_i \leq 0 \right\} \tag{1}$$

denote the set of closed transfers, where the real number $\bar{t} > 0$ denotes some predetermined upper-bound for payments. Let e_i denote the initial endowment of commodity money of agent $i \in I$, which is assumed to be $e_i \geq 2\bar{t}$. A social decision for issue s is thus a pair (d^s, t^s), where the pure decision d^s is an element of D^s and the vector of closed-transfer t^s is an element of T. To economize on notation, let $X^s \equiv D^s \times T$.

Suppose that agent i's preferences R_i for outcomes in $X^1 \times X^2$ can be represented by a utility function $u_i(\cdot; R_i) : X^1 \times X^2 \to \mathbb{R}_+$ of the form

$$u_i\left(x^1, x^2; R_i\right) = U_i(d^1, d^2, t_i^1 + t_i^2 + e_i; R_i), \tag{2}$$

where $U_i(\cdot; R_i) : D^1 \times D^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing in money. This type of utility form encompasses a wide variety of agent's preferences: separable ones and non-separable ones.

In line with Vives (1987) and Hayashi (2013), we show that income effects are ruled out once participants' preferences are assumed to be separable.[4]

[4] Vives (1987) considers an increasing sequence of sets of commodities, and under certain assumptions shows that income effect on each single commodity vanishes as the number of commodity and income tend to infinity at the same rate. Hayashi (2013) considers a continuum of commodity characteristics and shows that when a commodity - described as a subset of the set of commodity characteristics - tends to be arbitrarily small the preference induced over pairs of consumption of the commodity under analysis and income transfer to be allocated to the other commodities converges to a quasi-linear one.

Proposition 1 Suppose that agent $i \in I$'s preferences R_i for outcomes in $X^1 \times X^2$ have a utility representation of the form indicated in (2). Suppose that her willingness to pay\accept is well defined.[5] Then, agent's preferences R_i have a quasi-linear utility form representation if they are separable.

Proof. Let the premises hold. In what follows, we show that the marginal ordering R_i^1 for issue 1 induced by R_i exhibits zero income effects. One can easily see that the fact that U_i is strictly increasing in its third argument assures that more commodity money is better than less according to agent i's marginal ordering R_i^1. Furthermore, the assumption that agent i's willingness to pay\accept is well defined assures that no matter how much better the pure social decision $\hat{d}^1$ is than d^1, according to her marginal ordering R_i^1, some amount of commodity money compensates her for getting d^1 instead of $\hat{d}^1$. Therefore, to see that the marginal ordering R_i^1 induced by R_i has a quasi-linear utility representation in the commodity money, we are left to show that R_i^1 exhibits no income effects. In other words, we need to show that for all d^1 and $\hat{d}^1$ in D^1 and all income transfers t^1, $\tilde{t}^1$, $\hat{t}^1$ and $\bar{t}^1$ in T such that

$$q = \tilde{t}_i^1 - t_i^1 = \bar{t}_i^1 - \hat{t}_i^1, \tag{3}$$

it holds that

$$(d^1, t_i^1 + q) R_i^1 (\hat{d}^1, \hat{t}_i^1 + q) \iff (d^1, t_i^1) R_i^1 (\hat{d}^1, \hat{t}_i^1).$$

Then, consider any two pure social decisions for issue 1, say d^1 and $\hat{d}^1$, and any four income transfers in T^1, say t^1, $\tilde{t}^1$, $\hat{t}^1$ and $\bar{t}^1$, such that (3) holds. The separability requirement implies that for any two outcomes (d^2, t^2) and $(d^2, \hat{t}^2)$ of issue 2 such that agent i's income transfer is q at t^2 and zero at $\hat{t}^2$, it holds that

$$\begin{aligned}
(d^1, t_i^1) R_i^1 (\hat{d}^1, \hat{t}_i^1) \quad &\iff \\
&\iff \quad U_i(d^1, d^2, t_i^1 + t_i^2 + e_i; R_i) \geq U_i(\hat{d}^1, d^2, \hat{t}_i^1 + t_i^2 + e_i; R_i) \\
&\iff \quad U_i(d^1, d^2, \left(t_i^1 + q\right) + \hat{t}_i^2 + e_i; R_i) \geq U_i(\hat{d}^1, d^2, \left(\hat{t}_i^1 + q\right) + \hat{t}_i^2 + e_i; R_i) \\
&\iff \quad (d^1, t_i^1 + q) R_i^1 (\hat{d}^1, \hat{t}_i^1 + q).
\end{aligned}$$

Thus, the marginal ordering R_i^1 satisfies the property of no income effect.

Since the arguments for the other marginal ordering R_i^2 are entirely symmetric, we conclude that agent i's separable ordering R_i has a quasi-linear utility representation in the commodity money. ■

To illustrate a difficulty that arises from the need of integrating income effects in auction design, let us consider the *Vickrey auction*, which is often referred to as the *second-price sealed-bid auction*. According to this auction mechanism, bidders submit sealed bids for the item simultaneously. The bidder who submits the highest bid obtains the item and pays a price equal to the second-highest bid. One important property of the Vickrey auction is that truthful revelation is a (weakly) dominant strategy for each bidder. However, as

[5] In the sense that for any two pure social decisions d^1 and $\hat{d}^1$ of issue 1 there exists an issue-2 outcome (d^2, t^2) and two income transfers for issue 1, say t^1 and $\hat{t}^1$, such that agent i finds $\left(d^1, d^2, t_i^1 + t_i^2 + e_i\right)$ and $\left(\hat{d}^1, d^2, \hat{t}_i^1 + t_i^2 + e_i\right)$ equally good according to U_i.

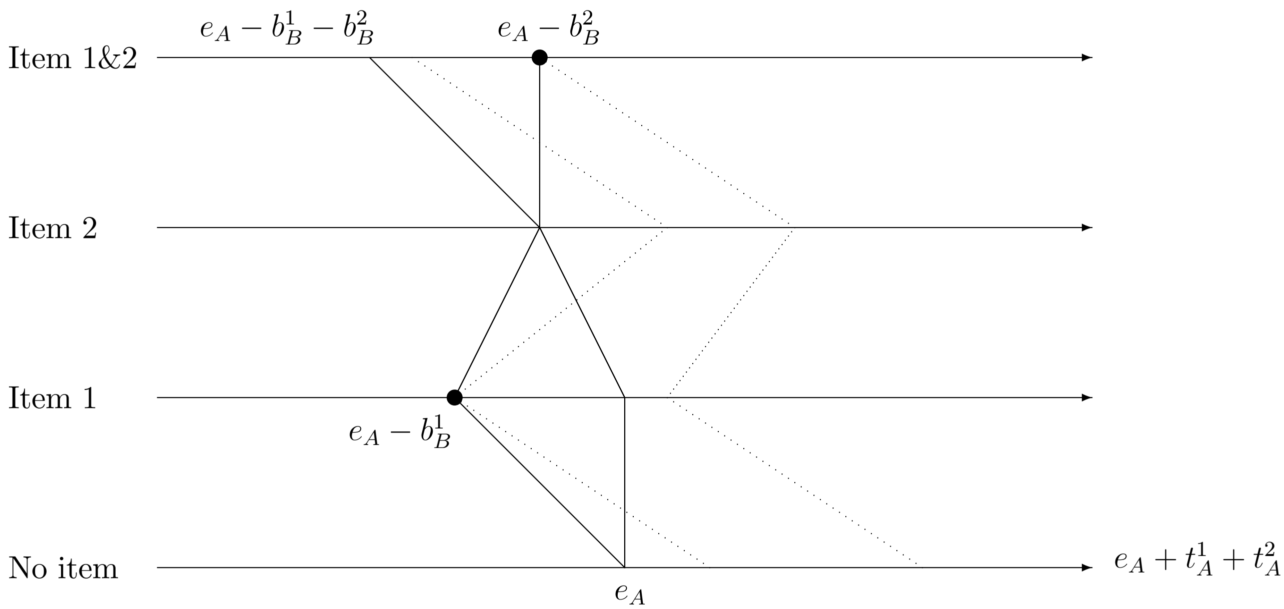

Figure 1:

we show below, if we run the Vickrey auction in each sector and sectors are interlinked, no dominant strategy exists for the bidder who has non-quasi-linear preferences, that is, preferences exhibiting income effects.

There are two bidders, A and B. There are two sectors, each of which consists of allocating a single item with transfers of commodity money. Let us suppose that bidder B has separable preferences. Furthermore, to make the point clearer, we assume the following on bidder A's preferences:

(i) There is no complementarity between the two items, that is, *along each indifference curve* bidder A's willingness to pay for the pair of items is the sum of her willingness to pay for item 1 and that for item 2.

(ii) There is no income effect on item 1, which implies that bidder A's willingness to pay for item 1 is a constant.

(iii) There is income effect on item 2, in the sense that as bidder A has more commodity money her willingness to pay for item 2 increases (i.e., item 2 and money are complements).

Consider running the Vickrey auction in each sector s. Then, in each sector, bidder i pays bidder j's bid, b_j^s, if bidder i is the winner, and nothing otherwise. As a tie-breaking rule, suppose that bidder A obtains the item of sector s if $b_A^s = b_B^s$.

Because bidder A's preference exhibits no complementarity between the items and no income effect on item 1, it is always optimal for bidder A to bid her willingness to pay for it, regardless of her opponent bids and regardless of her own bid for item 2. However, how much bidder A should bid for item 2 and even whether she should win item 2 or not depends on bidder B's bid for item 1. The reason is that her opponent's bid for item 1 determines the money left over for bidder A to bid for item 2, that is, bidder A's valuation of item 2.

Figure 1 gives a graphical representation of consumption spaces and indifference curves for bidder A. When bidder B bids b_B^1 for item 1 and b_B^2 for item 2, bidder A's consumption space is represented by the first solid line (from left to right), and her indifference curve is represented by the first dot line (from left to right). Therefore, when bidder B bids b_B^1 for item 1 and b_B^2 for item 2, bidder A should not win item 2 and win item 1 only. That is, bidder A's bid for item 2 should be below b_B^2.

However, when bidder B bids, let's say, zero for item 1 and b_B^2 for item 2, bidder A's consumption space is represented by the second solid line (from left to right), and her indifference curve is represented by the second dot line (from left to right). Thus, when bidder B bids zero for item 1 and b_B^2 for item 2, bidder A should also win item 2. The reason is that she can get item 1 for free, and for this reason she is willing to pay more for item 2 since this item and money are complements. Thus, bidder A's bid for item 2 should be above (or at least) b_B^2.

3. Preliminaries

We consider a finite set of agents indexed by $i \in I = \{1, \cdots, n\}$ and a finite set of elementary sectors indexed by $s \in S = \{1, \cdots, \ell\}$. The set of outcomes of sector s available to agents is represented by X^s, with x^s as a typical element. X^s is called sector-s outcome space. We assume that the set of outcomes available to agents is the product space

$$X = \prod_{s \in S} X^s.$$

To economize on notation, for any sector s, write s_C for the complement of s in S. Thus, (x^s, x^{s_C}) is an outcome of X, where it is understood that x^{s_C} is an element of the product space $X^{s_C} = \prod_{\bar{s} \in s_C} X^{\bar{s}}$.

In the usual fashion, agent i's preferences over X are given by a complete and transitive binary relation, subsequently an ordering, R_i on X. The corresponding strict and indifference relations are denoted by $P(R_i)$ and $I(R_i)$, respectively. The statement xR_iy means that agent i judges x to be at least as good as y. The statement $xP(R_i)y$ means that agent i judges x better than y. Finally, the statement $xI(R_i)y$ means that agent i judges x and y as equally good.

The condition of separability of preferences that must hold if the isolation of sector-s decision problem from others is legitimate can be formulated as follows. For each x^{s_C}, we define the sector-s marginal ordering, $R_i^s(x^{s_C})$, on X^s by

$$\text{for all } y^s, z^s \in X^s : y^s R_i^s(x^{s_C}) z^s \iff (y^s, x^{s_C}) R_i (z^s, x^{s_C}).$$

We say that the ordering R_i is separable if for all $s \in S$,

$$R_i^s(x^{s_C}) = R_i^s(y^{s_C}) \text{ for all } x^{s_C}, y^{s_C} \in X^{s_C}.$$

In other words, R_i is separable if the agent i's preferences over outcomes of X^s are independent of outcomes chosen from X^{s_C}. Again, to save writing, for any separable ordering R_i, write R_i^s for the sector-s marginal ordering induced by R_i.

We assume that the CA does not know agent i's true preferences. Thus, write $\mathcal{R}(X)$ for the set of orderings on X, $\mathcal{R}^{sep}(X)$ for the set of separable orderings on X, $\mathcal{R}_i$ for the domain of (allowable) orderings on X for agent i, and $\mathcal{R}_i^{sep}$ for the domain of (allowable) separable orderings on X for agent i.

We assume, however, that there is complete information among the agents in I. This implies that the CA knows $\mathcal{R}_i$ for each agent $i \in I$. Then, the CA knows the domain of preferences for the set I, which is the product set of agents' domains, that is,

$$\mathcal{R}_I = \prod_{i \in I} \mathcal{R}_i,$$

with R as a typical profile.

Nash implementation in PE

The goal of the CA is to implement a SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ where $\varphi(R)$ is nonempty for any $R \in \mathcal{R}_I$. We shall refer to $x \in \varphi(R)$ as a φ-optimal outcome at R. The common interpretation is that a SCR represents the social objectives that the society or its representatives want to achieve.

The CA delegates the achievement of the goal(s) to SAs, each of which design a PE mechanism which forces participants to behave *as if* they had separable preferences. Because we endorse the methodology of PE analysis, each SA is assumed to be able to conceive only marginal preferences which are consistent with separable preferences. Formally, for each $s \in S$, let $\mathcal{D}_i^s$ denote the (nonempty) class of allowable sector-s marginal orderings for outcomes of X^s, that is, $\mathcal{D}_i^s \subseteq \{R_i^s | R_i^s \text{ is induced by } R_i \in \mathcal{R}_i^{sep}(X)\}$. The methodology of PE imposes that the domain $\mathcal{D}_i^s$ includes marginal orderings that are induced by elements of $\mathcal{R}_i$ that are separable, that is,

$$\{R_i^s | R_i^s \text{ is induced by } R_i \in \mathcal{R}_i^{sep} \subseteq \mathcal{R}_i\} \subseteq \mathcal{D}_i^s.$$

To see why this is needed, consider the auction environment discussed in the previous section. There, any allowable preference cares for the sum of income transfers across sectors, and cash transfers are not to be evaluated differently: money is money. Put it differently, $\mathcal{R}_i$ need to be a proper subset of $\mathcal{R}(X)$ for this environment - they may coincide in the abstract domain and in the matching domain, though. Under this restriction, any allowable separable preference has to be represented in the form of a sum of quasi-linear functions under the methodology of PE analysis. Then, given that marginal preferences induced by non-separable preferences are not necessarily quasi-linear, the SA will notice (in light of Proposition 1) that there is something wrong with the methodology when he perceives that participants may have non-quasi-linear preferences for outcomes of his sector. In order that PE mechanism design works "successfully," such type of situations have to be avoided. Therefore, we assume that agent i's sector-s domain $\mathcal{D}_i^s$ consists only of marginal preferences which are induced by agent i's allowable separable preferences $\mathcal{R}_i^{sep}$.

Write $\mathcal{D}_I^s$ for the product set of $\mathcal{D}_i^s$'s, with R^s as a typical profile. The goal of sector-s SA is to implement a one-dimensional SCR $\varphi^s : \mathcal{D}_I^s \twoheadrightarrow X^s$ where $\varphi^s(R^s)$ is nonempty for any $R^s \in \mathcal{D}_I^s$. Again, we shall refer to $x^s \in \varphi^s(R^s)$ as a φ^s-optimal sector-s outcome at R^s.

The delegated sector-s SA knows the domain of sector-s preferences $\mathcal{D}_I^s$ and the delegated objective φ^s. However, we assume that this authority is unable to associate any element of $\mathcal{D}_i^s$ with a specific element of agent i's domain $\mathcal{R}_i$. Without repeating the discussion given in section 1, to make the analysis consistent with the methodology of PE analysis we make the following assumptions throughout the paper:

Assumption 1 The only concern of a SA is to promote the goal(s) of the CA.

Assumption 2 The PE mechanism designed by the SA forces its participants to behave *as if* they had separable preferences.

Assumption 3 Absence of communication among SAs.

Assumption 4 Each SA does not know the domain $\mathcal{R}_i$ of (allowable) orderings on X for agent i. Each SA can conceive only marginal preferences that are consistent with allowable separable preferences over X.

Assumption 5 The CA acts *as if* he had not the ability to distinguish whether a Nash equilibrium outcome of the game induced by Γ comes from by a profile of non-separable preferences or from a profile of separable ones if the marginal preferences over each component set X^s induced by the profiles of agents' preferences are *observationally equivalent with respect to their lower contour sets*.

Each SA delegates the choice to agents according to a PE mechanism, which aims to elicit the private information related to sector s from agents. Thus, in pursuing his target(s), sector-s SA designs a PE mechanism $\Gamma^s = \left((M_i^s)_{i\in I}, h^s\right)$, where M_i^s is the strategy space of agent i in sector s and $h^s : M^s \to X^s$, the outcome function, assigns to every strategy profile

$$m^s \in M^s = \prod_{i\in I} M_i^s$$

a unique outcome in X^s.

A PE mechanism Γ^s together with the profile $R^s \in \mathcal{D}_I^s$ defines a strategic game (Γ^s, R^s) in sector s, in which each agent chooses her strategy and all agents' strategy choices are made simultaneously (that is, when choosing a strategy choice each agent is not informed of the strategy choice chosen by any other agent). A strategy profile $m^s \in M^s$ is a Nash equilibrium (in pure strategies) of (Γ^s, R^s) if for all $i \in I$, it holds that

$$\text{for all } \bar{m}_i^s \in M_i^s : h(m^s)\, R_i^s h\left(\bar{m}_i^s, m_{-i}^s\right).$$

Write $NE(\Gamma^s, R^s)$ for the set of Nash equilibrium profiles of (Γ^s, R^s), and write $h^s(NE(\Gamma^s, R^s))$ for the set of Nash equilibrium outcomes of (Γ^s, R^s).

In delegating the achievement of the goal(s) to SAs, the CA 'loses control' of the mechanism design exercise. In other words, he does not design any mechanism. Moreover, from his point of view, each agent is free to choose strategically from her strategy space $M_i = \prod_{s \in S} M_i^s$ so as to influence the outcomes of PE mechanisms in her favour. Naturally, which outcomes can be obtained by agent i depends on profiles of outcomes that this agent can achieve in each sector s, while keeping her opponents' actions fixed at some strategy profile $m_{-i} \in \prod_{j \in I \setminus \{i\}} \left(\prod_{s \in S} M_j^s \right)$. Therefore, from the point of view of the CA, the mechanism governing communication with agents is a product set of PE mechanisms $\Gamma = \left((M_i)_{i \in N}, h \right)$, where M_i is the strategy space of agent i and $h : M \to X$, the outcome function, assigns to every strategy profile

$$m \in M = \prod_{i \in I} M_i$$

a unique outcome in X such that

$$h(m) = (h^s(m^s))_{s \in S}.$$

A product set of PE mechanisms $\Gamma \equiv (\Gamma^s)_{s \in S}$ and a profile $R \in \mathcal{R}_I$ induce a strategic game (Γ, R). A strategy profile $m \in M$ is a Nash equilibrium (in pure strategies) of (Γ, R) if for all $i \in I$, it holds that

$$\text{for all } \bar{m}_i \in M_i : h(m) R_i h(\bar{m}_i, m_{-i}),$$

where, as usual, m_{-i} is the strategy profile of all agents except i such that $(m_i, m_{-i}) = m$. Write $NE(\Gamma, R)$ for the set of Nash equilibrium profiles of (Γ, R), and write $h(NE(\Gamma, R))$ for the set of Nash equilibrium outcomes of (Γ, R).

Given that in our framework the CA delegates the decision-making authority to SAs, which, in turn, forces agents to behave as if they had separable preferences, the CA should not be able to distinguish whether a Nash equilibrium outcome of a product set of PE mechanisms is coming from a separable profile or from a non-separable preference profile. In other words, they should be observationally equivalent in his eyes. To this end, a formulation of the property of observational equivalence for our Nash implementation problems can be stated as follows: For any sector s, any ordering R_i^s on X^s and any outcome $x^s \in X^s$, the weak lower contour set of R_i^s at x^s is defined by $L(x^s, R_i^s) = \{y^s \in X^s | x^s R_i^s y^s\}$. Therefore:

Definition 1 For each $R \in \mathcal{R}_I$ and $x \in X$, a list of profiles of marginal orderings $(\bar{R}^s)_{s \in S} \in \prod_{s \in S} \mathcal{D}_I^s$ is *equivalent to R at x* if

$$\text{for all } s \in S \text{ and all } i \in I : L(x^s, \bar{R}_i^s) = L(x^s, R_i^s(x^{s_C})).$$

Thus, $(\bar{R}^s)_{s \in S}$ is equivalent to the profile R at x if for any sector s and any agent i, the indifference surface of $R_i^s(x^{s_C})$ through the outcome x^s coincides with the indifference surface of $\bar{R}_i^s$ through the same outcome.

To make this observational equivalence operational we need to assume that agent i's domain of marginal preferences is rich. The following domain-richness condition for $\mathcal{D}_i^s$ assures it.

Definition 2 $\mathcal{D}_i^s$ is rich if for each $R_i \in \mathcal{R}_i$, $x \in X$ and $s \in S$, there exists $\bar{R}_i^s \in \mathcal{D}_i^s$ such that $L\left(x^s, \bar{R}_i^s\right) = L\left(x^s, R_i^s\left(x^{sC}\right)\right)$.

Suppose that the CA wants to Nash implement the SCR φ. Let $\varphi(R)$ represent the set of socially desirable outcomes for the profile R. In the standard literature, the CA provides agents with a mechanism which has the following feature. For every admissible profile of orderings, the set of Nash equilibrium outcomes of the mechanism for that profile is identical to the set of outcomes dictated by the SCR for it. In our set up, the CA has the same objective as in the standard set up, only now he has to achieve it via a product set of PE mechanisms; that is, via a mechanism where the decision-making authority is delegated to SAs (see part (i) of Definition 3 below). Moreover, SAs have the same objective as in the standard set up, only now their Nash implementation problems pertain only to their respective sectors (see part (ii) of Definition 3 below). These objectives are linked by the inability of the CA to distinguish whether a social outcome is attributable to separable preferences or not (see part (iii) of Definition 3 below). This is because, in our framework, the CA delegates the decision-making authority to SAs. SAs, on the other hand, force agents to behave *as if* they had separable preferences in order to Nash implement the delegated target φ^s. As far as such a behavioral equivalence cannot be falsified, the CA is unable to detect problems in the use of PE mechanisms.

Definition 3 The SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ is *Nash implementable in PE* if there exist a product set of PE mechanisms Γ and a sequence $(\varphi^s)_{s\in S}$ of one-dimensional SCRs, where $\varphi^s : \mathcal{D}_I^s \twoheadrightarrow X^s$ for all $s \in S$, such that:
(i) for all $R \in \mathcal{R}_I : \varphi(R) = h(NE(\Gamma, R))$,
(ii) for all $s \in S : \varphi^s(R^s) = h^s(NE(\Gamma^s, R^s))$ for all $R^s \in \mathcal{D}_I^s$,
(iii) for all $R \in \mathcal{R}_I$ and all $x \in X : x \in h(NE(\Gamma, R)) \iff x \in \prod_{s\in S} h^s\left(NE\left(\Gamma^s, \bar{R}^s\right)\right)$ for any $(\bar{R}^s)_{s\in S} \in \prod_{s\in S} \mathcal{D}_I^s$ that is equivalent to R at x.

Let Γ be a product set of PE mechanisms. If a profile R consists of separable orderings, then the Nash equilibrium outcomes of the game (Γ^s, R^s) do not depend on outcomes that agents can obtain from games played in other sectors. Indeed, for cases like this, the Cartesian product of Nash equilibrium outcomes of the games $((\Gamma^s, R^s))_{s\in S}$ constitute the set of Nash equilibrium outcomes of the game (Γ, R); that is, $NE(\Gamma, R) = \prod\limits_{s\in S} NE(\Gamma^s, R^s)$.[6] Thus, the kind of linkages between CA and SAs that is captured by part (iii) of Definition 3 takes place naturally for profiles of separable orderings. Indeed, if the domain $\mathcal{R}_I$ of the SCR φ is represented by the unrestricted domain of profiles of separable preferences, Nash implementation in PE consists only of part (i) of the above definition.[7]

[6] The proof of this can be found in Addendum (Lemma A).

[7] Part (ii) and part (ii) are implied by part (i) of Definition 3. This is because of Lemma A in Addendum as well as of the assumption that the domain of the SCR is the unrestricted domain of separable preferences.

4. Necessary and sufficient conditions

Necessary conditions

In this subsection, we discuss conditions that are necessary for Nash implementation in PE. We end the subsection by showing that no acceptable Pareto optimal SCR defined on the domain of separable orderings can be Nash implemented in PE.

The relevance of implementation theory comes from the fact that it provides a theoretical construct within which to study the way in which a society shall trade off agent preferences to achieve its goals. Unless the SCR is dictatorial, this involves a compromise. The first condition identifies a property of how a SCR must handle the compromise across sectors where agents' preferences are separable.

Definition 4 The SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ is *decomposable* provided that for each $s \in S$, there exists a (nonempty) correspondence $\varphi^s : \mathcal{D}_I^s \twoheadrightarrow X^s$ such that $\varphi(R) = \prod_{s \in S} \varphi^s(R^s)$ for each profile of separable orderings $R \in \mathcal{R}_I$, where $R^s \in \mathcal{D}_I^s$ is the profile of sector-s marginal orderings induced by the profile R.

This says that if a SCR is decomposable, then the sth dimension of the SCR depends only on the profiles of marginal orderings of the sth sector. Differently put, the SCR can be decomposed into the product of one-dimensional SCRs. Furthermore, it implies that the social objectives that a society or its representatives wants to achieve can be decomposed in 'small' social objectives, one for each sector. Therefore, to analyze the way in which the society should trade off agent preferences for the sth sector to achieve its goal, we can ignore consumption trade-offs across sectors and focus only on the profiles of marginal orderings of sth sector.

Theorem 1 The SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ is decomposable if φ is Nash implementable in PE.

Proof. Let the premises hold. Then, by Definition 3, there exist a product set of PE mechanisms Γ and a sequence $(\varphi^s)_{s \in S}$ of one-dimensional SCRs, where $\varphi^s : \mathcal{D}_I^s \twoheadrightarrow X^s$ for all $s \in S$, such that parts (i)-(iii) are satisfied. Furthermore, take any profile of separable orderings $R \in \mathcal{R}_I$. Suppose that $x \in \varphi(R)$. By definition of the domain $\mathcal{D}_i^s$, it follows that the sector-s marginal ordering R_i^s induced by the separable ordering R_i is an element of $\mathcal{D}_i^s$. Part (i) and part (iii) imply that $x \in \prod_{s \in S} h^s(NE(\Gamma^s, R^s))$, and so, by part (ii), we have that $x^s \in \varphi^s(R^s)$ for all $s \in S$, as sought. Conversely, suppose that $x^s \in \varphi^s(R^s)$ for all $s \in S$. Part (ii) implies that $x^s \in h^s(NE(\Gamma^s, R^s))$ for all $s \in S$. Part (iii), combined with part (i), implies that $x \in \varphi(R)$. We conclude that φ is decomposable. ■

In the literature of strategy-proof social choice functions, it has been shown that decomposability is implied by strategy-proofness where agents have separable preferences (as per Barberà et al., 1991; Le Breton and Sen, 1999). A natural question, then, is whether decomposability is implied by Nash implementation.[8] The answer is no (see Example A in Addendum).

A second necessary condition for Nash implementation in PE can be stated as follows:

[8]A SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ is Nash implementable if there exists a mechanism $\gamma \equiv (M, h)$ such that for all $R \in \mathcal{R}_I$, $\varphi(R) = h(NE(\gamma, R))$.

Definition 5 The decomposable SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ satisfies *indistinguishability* if for all $R \in \mathcal{R}_I$ and all $x \in X : x \in \varphi(R) \iff x \in \prod_{s \in S} \varphi^s\left(\bar{R}^s\right)$ for any $(\bar{R}^s)_{s \in S} \in \prod_{s \in S} \mathcal{D}_I^s$ that is equivalent to R at x.

This says that when the overall social objective φ can be decomposed in 'small' social objectives, one for each sector, and when each dimension s of an outcome x represents a socially acceptable compromise for agents, that is, when the sector-s outcome x^s is socially optimal at the profile of sector-s marginal orderings $\bar{R}^s$, then the "bundle of compromises" represented by x needs to be a socially optimal one for the SCR φ at each profile of orderings $\hat{R}$ for which the list of marginal orderings $\left(\bar{R}^s\right)_{s \in S}$ can be considered equivalent to $\hat{R}$ at the outcome x, and vice versa; that is, when the outcome x is φ-optimal at one profile $\hat{R}$, then each x^s needs to be φ^s-optimal at the profile of marginal orderings $\bar{R}^s$ provided that the profile of marginal orderings $\left(\bar{R}^s\right)_{s \in S}$ is equivalent to $\hat{R}$ at x. This condition is related to part (iii) of Definition 3, that is, to the inability of the CA to distinguish whether a social outcome is attributable to separable preferences or not and to his inability to falsify the behavioral equivalence between a list of marginal preferences induced by a profile of separable orderings and a list of marginal orderings induced by a profile of non-separable orderings.

Theorem 2 The decomposable SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ satisfies indistinguishability if φ is Nash implementable in PE.

Proof. Let the premises hold. Then, by Definition 3, there exist a product set of PE mechanisms Γ and a sequence $(\varphi^s)_{s \in S}$ of one-dimensional SCRs, where $\varphi^s : \mathcal{D}_I^s \twoheadrightarrow X^s$ for all $s \in S$, such that parts (i)-(iii) are satisfied. Fix any $R \in \mathcal{R}_I$. Suppose that $x \in \varphi(R)$. Part (i) and part (iii) of Definition 3 implies that $x^s \in h^s\left(NE\left(\Gamma^s, \bar{R}^s\right)\right)$ for each $s \in S$, where $\left(\bar{R}^s\right)_{s \in S}$ is equivalent to R at x. Part (ii) implies that $x^s \in \varphi^s\left(\bar{R}^s\right)$ for all $s \in S$, as sought. To prove the statement in the other direction, suppose that $x \in \prod_{s \in S} \varphi^s\left(\bar{R}^s\right)$ for some $(\bar{R}^s)_{s \in S} \in \prod_{s \in S} \mathcal{D}_I^s$ that is equivalent to R at x. Therefore, part (ii) implies that $x^s \in h^s\left(NE\left(\Gamma^s, \bar{R}^s\right)\right)$ for all $s \in S$. Finally, part (iii), combined with part (i), implies that $x \in \varphi(R)$. Thus, the decomposable SCR φ satisfies indistinguishability. ■

A condition that is central to the Nash implementation of SCRs is Maskin monotonicity. This condition says that if an outcome x is φ-optimal at the profile R and this x does not strictly fall in preference for anyone when the profile is changed to R', then x must remain a φ-optimal outcome at R'. We require Maskin monotonicity for each sector s. Let us formalize that condition as follows:

Definition 6 The decomposable SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ is *sector-wise Maskin monotonic* provided that for all $s \in S$, all $x^s \in X^s$ and all $R^s, \bar{R}^s \in \mathcal{D}_I^s$ if $x^s \in \varphi^s(R^s)$ and $L(x^s, R_i^s) \subseteq L(x^s, \bar{R}_i^s)$ for all $i \in I$, then $x^s \in \varphi^s\left(\bar{R}^s\right)$.

Theorem 3 The decomposable SCR $\varphi : \mathcal{R}_I \to X$ is sector-wise Maskin monotonic if φ is Nash implementable in PE.

Proof. The proof can be found in Maskin (1999). ■

A characterization theorem

In implementation theory, it is Maskin's Theorem (Maskin, 1999) that shows that when the CA faces at least three agents, a SCR is implementable in (pure-strategies) Nash equilibrium if it is Maskin monotonic and it satisfies the auxiliary condition of no veto-power.[9]

In the abstract Arrovian domain, the condition of no veto-power says that if an outcome is at the top of the preferences of all agents but possibly one, then it should be chosen irrespective of the preferences of the remaining agent: that agent cannot veto it. The condition of no veto-power implies two conditions. First, it implies the condition of unanimity, which states that if an outcome is at the top of the preferences of all agents, then that outcome should be selected by the SCR. Thus, as a part of sufficiency, we require a variant of unanimity, which states that if all agents agree on which outcome is best for sector s, then this outcome should be chosen by the sth dimension of a decomposable SCR.

Definition 7 A decomposable SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ satisfies *sector-wise unanimity* provided that for all $s \in S$, all $x^s \in X^s$ and all $R^s \in \mathcal{D}_I^s$ if $X^s \subseteq L(x^s, R_i^s)$ for all $i \in I$, then $x^s \in \varphi^s(R^s)$.

Second, the condition of no veto-power implies the condition of weak no veto-power, which states that if an outcome x is φ-optimal at one profile $\bar{R}$ and if the profile change from $\bar{R}$ to R in a way that under the new profile an outcome y that was no better than x at $\bar{R}_i$ for some agent i is weakly preferred to all outcomes in the weak lower contour set of $\bar{R}_i$ at x according to the ordering R_i and this y is maximal for all other agents in the set X, then y should be a φ-optimal outcome at R. As a part of sufficiency, we require the following adaptation of the weak no veto-power condition to our Nash implementation problems.

Definition 8 A decomposable SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ satisfies *sector-wise weak no veto-power* provided that for all $s \in S$, all $x^s \in X^s$ and all $R^s, \bar{R}^s \in \mathcal{D}_I^s$ if $x^s \in \varphi^s(R^s)$, $y^s \in L(x^s, R_i^s) \subseteq L\left(y^s, \bar{R}_i^s\right)$ for some $i \in I$ and $X^s \subseteq L\left(y^s, \bar{R}_j^s\right)$ for all $j \in I \setminus \{i\}$, then $y^s \in \varphi^s\left(\bar{R}^s\right)$.

The main result of the section is also established with the aid of a domain restriction, which we now state below. Examples of domains satisfying Property α are provided in the next section.

Definition 9 The pair $(\mathcal{R}_I, \varphi)$, with $\mathcal{R}_I \subseteq \mathcal{R}_I(X)$, satisfies Property α if for all $R \in \mathcal{R}_I$ and all $x \in \varphi(R)$ there exists a profile of separable orderings $\bar{R} \in \mathcal{R}_I$ such that

$$\text{for all } i \in I : L\left(x, \bar{R}_i\right) \subseteq L(x, R_i).$$

The main result of this subsection can be stated as follows:

Theorem 4 Let $n \geq 3$. Suppose that agent i's domain $\mathcal{D}_i^s$ of sector-s marginal orderings is rich for each sector $s \in S$. The SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ is Nash implementable in PE if φ

[9] Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Lombardi and Yoshihara (2013) refined Maskin's Theorem by providing necessary and sufficient conditions for a SCR to be implementable in (pure strategies) Nash equilibrium. For an introduction to the theory of implementation see Jackson (2001) and Maskin and Sjöström (2002).

satisfies decomposability, indistinguishability, sector-wise Maskin monotonicity, sector-wise unanimity, sector-wise weak no veto-power and Property α.

PROOF. See Appendix. ■

Before discussing the implications of Theorem 4 in the next section, let us first show that the domain restriction represented by Property α is indeed an indispensable requirement for our characterization result.

EXAMPLE 1 *Property α is indispensable for Theorem 4.* Let $n = 3$ and $\ell = 2$. Let $I = \{A, B, C\}$ and let $S = \{1, 2\}$. For sector $s \in S$, let $X^s = \{x^s, y^s\}$ with $x^s \neq y^s$.

Suppose that agent A's domain $\mathcal{R}_A$ consists of the following *strict* orderings:

$$\begin{array}{l}(y^1, y^2)P_A(x^1, x^2)P_A(y^1, x^2)P_A(x^1, y^2)\\(x^1, x^2)\bar{P}_A(x^1, y^2)\bar{P}_A(y^1, x^2)\bar{P}_A(y^1, y^2)\\(y^1, y^2)\tilde{P}_A(y^1, x^2)\tilde{P}_A(x^1, y^2)\tilde{P}_A(x^1, x^2).\end{array}$$

Among the listed orderings, one can check that the only ordering that is not a separable one is P_A. The marginal orderings of P_A are as follows:

$$\begin{array}{lcl}\text{for sector 1} & : & x^1 P_A^1\left(x^2\right) y^1 \text{ and } y^1 P_A^1\left(y^2\right) x^1\\\text{for sector 2} & : & x^2 P_A^2\left(x^1\right) y^2 \text{ and } y^2 P_A^2\left(y^1\right) x^2.\end{array}$$

On the other hand, the marginal orderings of the separable orderings are as follows:

$$\begin{array}{lcl}\text{for sector 1} & : & x^1 \bar{P}_A^1 y^1 \text{ and } y^1 \tilde{P}_A^1 x^1\\\text{for sector 2} & : & x^2 \bar{P}_A^2 y^2 \text{ and } y^2 \tilde{P}_A^2 x^2.\end{array}$$

By definition of the sector-s domain, we have that $\bar{P}_A^s$ and $\tilde{P}_A^s$ are elements of $\mathcal{D}_A^s$. One can check that $\mathcal{D}_A^s$ is rich.

For the sake of simplicity, suppose that $\mathcal{R}_i$ for agent $i \neq A$ consists only of separable orderings. Moreover, suppose that P_i of agent $i \neq A$ is an allowable strict ordering, that is, $P_i \in \mathcal{R}_i$, and it is as follows:

$$\begin{array}{lcl}\text{for agent } B & : & (x^1, x^2)P_B(y^1, x^2)P_B(x^1, y^2)P_B(y^1, y^2)\\\text{for agent } C & : & (y^1, y^2)P_C(y^1, x^2)P_C(x^1, y^2)P_C(x^1, x^2).\end{array}$$

One can check that P_B and P_C are separable orderings on X, and that the marginal orderings of agents B and C are strict and are as follows:

$$\begin{array}{lcl}\text{for sector 1} & : & x^1 P_B^1 y^1 \text{ and } y^1 P_C^1 x^1\\\text{for sector 2} & : & x^2 P_B^2 y^2 \text{ and } y^2 P_C^2 x^2.\end{array}$$

Suppose that the SCR $\varphi : \mathcal{R}_I \twoheadrightarrow X$ satisfies all conditions of Theorem 4 but Property α. Nonetheless, suppose that φ is Nash implementable in PE. Thus, there exists a product set of PE mechanisms Γ such that it Nash implements φ in PE.

The profile $(P_A, P_B, P_C) \equiv R$ is an element of $\mathcal{R}_I$. Suppose that $(x^1, x^2) \in \varphi(R)$. Note that this combination would not be possible if $(\mathcal{R}_I, \varphi)$ satisfied Property α.

Since the SCR φ is decomposable, there exists one-dimensional SCR φ^s on $\mathcal{D}_I^s$ for each $s \in S$. Given that φ is sector-wise Maskin monotonic and, moreover, it satisfies sector-wise unanimity as well as sector-wise weak no veto-power, let

$$\begin{aligned}
\varphi^1\left(\bar{P}_A^1, P_B^1, P_C^1\right) &= x^1 \text{ and } \varphi^2\left(\bar{P}_A^2, P_B^2, P_C^2\right) = x^2 \\
\varphi^1\left(\tilde{P}_A^1, P_B^1, P_C^1\right) &= y^1 \text{ and } \varphi^2\left(\tilde{P}_A^2, P_B^2, P_C^2\right) = y^2.
\end{aligned}$$

Furthermore, since φ satisfies indistinguishability, it also holds that $(y^1, y^2) \in \varphi(R)$.

Since $x \in \varphi(R)$ and, moreover, since Γ Nash implements φ in PE, there exists $m \in M$ such that $h(m) = x$; that is, $h^s(m^s) = x^s$ for each $s \in S$. Since agent C needs not find any profitable unilateral deviation and P_C on X is a separable strict ordering, it holds that $h^s\left(m_{-C}^s, M_C^s\right) = x^s$. Moreover, since agent A also needs not find any profitable unilateral deviation from m, it must be the case that $h^s\left(m_{-A}^s, M_A^s\right) = x^s$ for at least one sector $s \in S$. Fix any of such a sector s. It follows that $m^s \in NE\left(\Gamma^s, \left(\tilde{P}_A^s, P_B^s, P_C^s\right)\right)$ given that $h^s(m^s) = x^s$ is the top ranked outcome for agent B according to P_B^s. Since Γ Nash implements φ in PE, part (ii) of Definition 3 implies that $x^s \in \varphi^s\left(\tilde{P}_A^s, P_B^s, P_C^s\right)$, which contradicts the fact that $\varphi^s\left(\tilde{P}_A^s, P_B^s, P_C^s\right) = y^s$. Thus, Property α is indispensable for Theorem 4.

5. Examples of Nash implementable SCRs in PE

In this section, we present some implications of Theorem 4. More precisely, we consider some interesting domains that are able to accommodate some forms of complementarity.

The example below gives a straightforward domain that satisfies Property α in environments with no monetary transfers. As in the matching leading example discussed above, items of sector 1 can be viewed as school-seats and items of sector 2 as houses. With this in mind, suppose that houses x^2 and y^2 are equally sufficiently close to respective schools x^1 and y^1. Therefore, an interpretation of type of complementarity that the example below accommodates is that agent i strictly prefers the bundles that minimize the distance school-home to other available assignments and she finds the assignments that minimize the distance school-home, that is, (x^1, x^2) and (y^1, y^2), as equally good. Assignments that do not minimize the distance are viewed as equally bad.

Example 2 *Non-separability of preferences in environments with no monetary transfers.* In this example we provide a preference domain that satisfies a stronger variant of Property α, which can be stated as follows: For all $R \in \mathcal{R}_I$ and all $x \in X$, there exists a profile of separable orderings $\bar{R} \in \mathcal{R}_I$ such that

$$\text{for all } i \in I : L\left(x, \bar{R}_i\right) \subseteq L\left(x, R_i\right).$$

Suppose that $S = \{1, 2\}$ and that $X^s = \{x^s, y^s\}$, with $x^s \neq y^s$, for all $s \in S$.

For any agent $i \in I$, define $\mathcal{R}_i$ as follows: $R_i \in \mathcal{R}_i$ if either it is a separable ordering, that is, $R_i \in \mathcal{R}^{sep}(X)$, or for all $x^1, y^1 \in X^1$ and $x^2, y^2 \in X^2$, it holds that

$$(x^1, x^2) I (R_i) (y^1, y^2) P (R_i) (y^1, x^2) R_i (x^1, y^2). \tag{4}$$

One can check that if R_i satisfies (4), then it is not a separable ordering given that the sector-1 marginal ordering $R_i^1(x^2)$ differs from $R_i^1(y^2)$.

In order to check that $\prod_{i \in I} \mathcal{R}_i$ satisfies the stronger variant of Property α stated above, let the following separable orderings be elements of $\mathcal{R}_i$:

$$\begin{aligned}
\text{given } (x^1, x^2) &: (x^1, x^2) P\left(\bar{R}_i\right) (y^1, x^2) P\left(\bar{R}_i\right) (x^1, y^2) P\left(\bar{R}_i\right) (y^1, y^2) \\
\text{given } (y^1, y^2) &: (y^1, y^2) P\left(\tilde{R}_i\right) (x^1, y^2) P\left(\tilde{R}_i\right) (y^1, x^2) P\left(\tilde{R}_i\right) (x^1, x^2) \\
\text{given } (x^1, y^2) &: (y^1, x^2) P\left(\hat{R}_i\right) (y^1, y^2) I\left(\hat{R}_i\right) (x^1, x^2) P\left(\hat{R}_i\right) (x^1, y^2) \\
\text{given } (y^1, x^2) &: (x^1, y^2) P\left(R'_i\right) (x^1, x^2) I\left(R'_i\right) (y^1, y^2) P\left(R'_i\right) (y^1, x^2).
\end{aligned}$$

One can now easily check that the stronger variant of Property α mentioned above is satisfied.

The next result also shows that in auction\public decisions environments with monetary transfers, Property α accommodates non-separability of preferences due to income effects.

Proposition 2 Let $S = \{1, 2\}$. For each $s \in S$, let $X^s = D^s \times T$, where T is the set of closed transfers defined in (1). Assume that agent i's preferences belonging to $\mathcal{R}_i$ are represented in the form given in (2). Suppose that her willingness to pay\accept is well defined.[10] For each agent $i \in I$, suppose that $R_i \in \mathcal{R}_i$ satisfies the following property: For all $d^1, \bar{d}^1 \in D^1$, $d^2, \bar{d}^2 \in D^2$ and $t^1, t^2 \in T$, if

$$\begin{aligned}
U_i(d^1, d^2, t_i^1 + t_i^2 + e_i; R_i) &= U_i(\bar{d}^1, d^2, t_i^1 + \Delta t_i^1 + t_i^2 + e_i; R_i) \\
&= U_i(d^1, \bar{d}^2, t_i^1 + t_i^2 + \Delta t_i^2 + e_i; R_i),
\end{aligned} \tag{5}$$

then

$$U_i(\bar{d}^1, \bar{d}^2, t_i^1 + \Delta t_i^1 + t_i^2 + \Delta t_i^2 + e_i; R_i) = U_i(d^1, d^2, t_i^1 + t_i^2 + e_i; R_i). \tag{6}$$

Let $\varphi : \prod_{i \in I} \mathcal{R}_i \twoheadrightarrow X^1 \times X^2$ be a SCR. Then, $\left(\prod_{i \in I} \mathcal{R}_i, \varphi\right)$ satisfies Property α.

Proof. Let the premises hold. Take any $(x^1, x^2) \in \varphi(R)$. Since $R \in \mathcal{R}_I$ and, moreover, since each R_i of agent i has a utility representation of the form described in (2), $L((x^1, x^2), R_i)$ is equivalent to $L((d^1, d^2, t_i^1 + t_i^2 + e_i), U_i)$. Since for each agent i the income effect is fixed at one given level and, moreover, since R_i satisfies the above property, there

[10] To assure that agent i's willingness to pay/accept is well defined, we also assume that U_i satisfies the following property: For all $d^1, \bar{d}^1 \in D^1$, all $d^2, \bar{d}^2 \in D^2$, all $t^1, t^2 \in T$, there exist $\bar{t}^1, \bar{t}^2 \in T$ such that

$$U_i\left(d^1, d^2, t_i^1 + t_i^2 + e_i; R_i\right) = U_i\left(\bar{d}^1, \bar{d}^2, \bar{t}_i^1 + \bar{t}_i^2 + e_i; R_i\right).$$

exists a separable preference $R_i' \in \mathcal{R}_i$ for agent i such that the indifference surface of U_i passing through the bundle $(d^1, d^2, t_i^1 + t_i^2 + e_i)$ coincides exactly with the indifference surface of R_i' through that bundle.[11] ■

Given that there are domains that satisfy Property α, in the next two subsections we provide two examples of SCRs that are Nash implementable in PE. In line with the leading examples discussed above, we show that the sector-wise (weak) core solution and the sector-wise Vickrey-Clarke-Groves solution are Nash implementable in PE.

Sector-wise (weak) core solution

A sector-s coalitional game is a four-tuple $(I, X^s, R^s, \upsilon^s)$, where:

- I is a finite set of agents, with $n \geq 3$.
- X^s is a non-empty set of outcomes available from sector s.
- R^s is a profile of orderings for agents on X^s.
- υ^s is a sector-s characteristic function $\upsilon^s : 2^N \backslash \{\varnothing\} \to 2^{X^s}$, which assigns for each nonempty coalition T a subset of outcomes.

Definition 10 For any sector-s coalitional game $(I, X^s, R^s, \upsilon^s)$, an outcome $x^s \in X^s$ is blocked by a coalition T if there is $y \in \upsilon^s(T)$ such that $(y, x) \in P(R_i^s)$ for each $i \in T$.

We consider a situation in which the SA knows what is feasible for each coalition, that is, the characteristic function υ^s, but he does not know agents' preferences. This situation is modeled by a four-tuple $(I, X^s, \mathcal{D}_I^s, \upsilon^s)$, which we refer to as a sector-s coalitional game environment.

The sector-s core solution, denoted by φ_{Core}^s, is a correspondence on $\mathcal{D}_I^s$ such that for each profile R^s,

$$\varphi_{Core}^s(R^s) \equiv \{x^s \in \upsilon^s(I) \,|\, x^s \text{ is not blocked by any coalition } T\}.$$

Definition 11 The SCR $\varphi_{S\text{-}Core} : \mathcal{R}_I \twoheadrightarrow X$ is the *sector-wise core solution* provided that for all $R \in \mathcal{R}_I$ and all $x \in X$:

$$x \in \varphi_{S\text{-}Core}(R) \iff x \in \prod_{s \in S} \varphi_{Core}^s\left(\bar{R}^s\right)$$

for an arbitrary list of profiles $(\bar{R}^s)_{s\in S} \in \prod_{s\in S} \mathcal{D}_I^s$ that is equivalent to R at x.

[11] It is also possible to show (see Lemma B and Proposition 2A in Addendum) that the property that $R_i \in \mathcal{R}_i$ is required to satisfy is indeed equivalent to the following property: For all $R_i \in \mathcal{R}_i$ and all $x \in X$, there exists a separable ordering $\bar{R}_i \in \mathcal{R}_I$ such that for all $s \in S : L^s(x, R_i) \subseteq L^s(x, \bar{R}_i)$, and such that $L\left(x, \bar{R}_i\right) \subseteq L(x, R_i)$, where $L^s(x, R_i) = \{(y^s, x^{sC}) \in X | x R_i (y^s, x^{sC})\}$.

In Example 2 we have provided an example of preference domain in environments with no income transfers that satisfies Property α. In light of it, there are interesting domains consisting of non-separable preferences for which the sector-wise core solution is Nash implementable in PE. Formally:

THEOREM 5 Take any $\mathcal{R}_I$ such that the pair $(\mathcal{R}_I, \varphi_{S\text{-}Core})$ satisfies Property α. Suppose that agent i's domain $\mathcal{D}_i^s$ is rich for each sector $s \in S$ and agent $i \in I$. Suppose that there are at least two sectors, $\ell \geq 2$. Let $(I, X^s, \mathcal{D}_I^s, v^s)$ be any coalitional game environment for sector $s \in S$. The sector-wise core solution $\varphi_{S\text{-}Core} : \mathcal{R}_I \twoheadrightarrow X$ is Nash implementable in PE.

PROOF. Let the premises hold. By construction, $\varphi_{S\text{-}Core}$ satisfies decomposability and indistinguishability. Moreover, it is well-known that sector-s core solution is unanimous and Maskin monotonic. Thus, $\varphi_{S\text{-}Core}$ satisfies sector-wise Maskin monotonicity and sector-wise unanimity. We are left to show that $\varphi_{S\text{-}Core}$ satisfies sector-wise weak no veto-power. This is shown below for an arbitrary $s \in S$.

Take any $R^s, \hat{R}^s \in \mathcal{D}_I^s$ and suppose that $x^s \in \varphi_{Core}^s(R^s)$, that $y^s \in L(x^s, R_i^s) \subseteq L\left(y^s, \hat{R}_i^s\right)$ for some $i \in I$, and that $X^s \subseteq L\left(y^s, \hat{R}_j^s\right)$ for any other agent $j \in I \backslash \{i\}$. We show that $y^s \in \varphi_{Core}^s\left(\hat{R}^s\right)$. Assume, to the contrary, that $y^s \notin \varphi_{Core}^s\left(\hat{R}^s\right)$.

Then, there exists a coalition T and an outcome $z^s \in v^s(T)$ such that $z^s P\left(\hat{R}_k^s\right) y^s$ for all $k \in T$. Since the outcome y^s is maximal for each agent $j \neq i$, it must be the case that $T = \{i\}$. Given that $z^s P\left(\hat{R}_i^s\right) y^s$, it follows that $z^s \notin L\left(y^s, \hat{R}_i^s\right)$, and so $z^s \notin L(x^s, R_i^s)$. Therefore, $z^s P\left(\hat{R}_i^s\right) x^s$, which contradicts the supposition that x^s is a sector-s core allocation for the coalitional game (I, X^s, R^s, v^s). Thus, φ_{Core}^s satisfies weak no veto-power. We conclude that $\varphi_{S\text{-}Core}$ satisfies sector-wise weak no veto-power.

Theorem 4 implies that $\varphi_{S\text{-}Core}$ is Nash implementable in PE. ■

Sector-wise VCG solution

Let us consider the auction\public decision environment with income transfers described above. It is well-known that the Vickrey auction is a special case of the Vickrey-Clarke-Groves (VCG) mechanism, which we now proceed to define below.[12] Recall that by definition of $\mathcal{D}_i^s$ and by Proposition 1, it holds that each $R_i^s \in \mathcal{D}_i^s$ has a quasi-linear utility representation

$$u_i^s(x^s, R_i^s) = v^s(d^s, R_i^s) + (t_i^s + e_i),$$

where $v^s(d^s, R_i^s)$ denotes the benefit that agent i of type R_i^s receives from a decision $d^s \in D^s$, and t_i^s a payment to agents.

A sector-s VCG game environment is a five-tuple $(I, X^s, \mathcal{D}_I^s, d^s, \tau^s)$, where:

- I is a finite set of agents, with $n \geq 3$.
- $X^s \equiv D^s \times T$ is a non-empty set of outcomes available from sector-s, where D^s denotes the set of pure decisions and T is the set of closed transfers defined in (1).

[12] The Clarke-Groves mechanism is introduced in Clarke (1971) and Groves (1973).

- $\mathcal{D}_I^s$ is the domain of agents' quasi-linear preferences for outcomes in X^s.
- $d^s : \mathcal{D}_I^s \to D^s$ is a decision rule that prescribes the pure decision that is efficient contingent on preferences $\hat{R}^s$ reported by agents, that is,

$$d^s\left(\hat{R}^s\right) \in \arg\max_{\bar{d}^s \in D^s}\left(\sum_{j\in I} v_j^s\left(\bar{d}^s, R_j^s\right)\right).$$

- $\tau^s : \mathcal{D}_I^s \to T$ is a transfer rule that stipulates a payment to agents if $\tau_i^s\left(\hat{R}^s\right) > 0$ (and from agents if $\tau_i^s\left(\hat{R}^s\right) < 0$) contingent on preferences $\hat{R}^s$ reported by agents, as follows:

$$\tau_i^s\left(\hat{R}^s\right) = h_i\left(\hat{R}_{-i}^s\right) - \sum_{j\in I\setminus\{i\}} v_j^s\left(d^s\left(\hat{R}^s\right), R_j^s\right), \tag{7}$$

  where $h_i(\cdot)$ is an arbitrary function that is independent of agent i's report.

The outcome $(d^s, t^s) \in X^s$ is a sector-s VCG outcome of the VCG game $(I, X^s, R^s, d^s, \tau^s)$ if $d^s = d(R^s)$ and $t^s = \tau^s(R^s)$. The sector-s VCG solution, denoted by φ_{VCG}^s, is a correspondence on $\mathcal{D}_I^s$ such that for each profile R^s,

$$\varphi_{VCG}^s(R^s) \equiv \{x^s \in X^s | x^s = (d^s(R^s), \tau^s(R^s))\}.$$

Definition 12 The SCR $\varphi_{S\text{-}VCG} : \mathcal{R}_I \twoheadrightarrow X$ is the *sector-wise VCG solution* if for each $R \in \mathcal{R}_I$,

$$x \in \varphi_{S\text{-}VCG}(R) \iff x \in \prod_{s\in S} \varphi_{VCG}^s(\bar{R}^s)$$

for an arbitrary list of profiles $(\bar{R}^s)_{s\in S} \in \prod_{s\in S} \mathcal{D}_I^s$ that is equivalent to R at x.

In Proposition 2 we show that Property α accommodates non-separability of preferences due to income effects. In light of it, we show that the sector-wise VCG solution is Nash implementable in PE.

Theorem 6 Take any $\mathcal{R}_I$ such that the pair $(\mathcal{R}_I, \varphi_{S\text{-}VCG})$ satisfies Property α. Suppose that agent i's domain $\mathcal{D}_i^s$ is rich for each sector $s \in S$ and agent $i \in I$. Suppose that there are at least two sectors, $\ell \geq 2$. Let $(I, X^s, \mathcal{D}_I^s, d^s, \tau^s)$ be any VCG game environment for sector $s \in S$. The sector-wise VCG solution $\varphi_{S\text{-}VCG} : \mathcal{R}_I \twoheadrightarrow X$ is Nash implementable in PE.

Proof. Let the premises hold. By construction, $\varphi_{S\text{-}VCG}$ satisfies decomposability and indistinguishability. Moreover, it is well-known that sector-s VCG solution is Maskin monotonic and a unanimous SCR. Moreover, sector-s VCG solution satisfies weak no veto-power vacuously. Thus, $\varphi_{S\text{-}VCG}$ satisfies sector-wise Maskin monotonicity, and sector-wise weak no veto-power and sector-wise unanimity. Theorem 4 implies that $\varphi_{S\text{-}VCG}$ is Nash implementable in PE. ■

6. Concluding remarks

A product set of PE mechanisms is a mechanism in which its participants are constrained to submit their rankings to sector authorities separately and, moreover, sector authorities cannot communicate with each other, due to misspecification by the CA that preferences are separable or due to technical/institutional constraints. Therefore, a key property of a single PE mechanism is that participants are required to behave *as if* they had separable preferences.

We identify a set of necessary conditions for the implementation of SCRs via a product set of PE mechanisms, that is, for the implementation in PE. Furthermore, under mild auxiliary conditions, reminiscent of Maskin's Theorem (1999), we have also shown that they are sufficient for the implementation in PE.

We conclude by discussing future research directions. The first thing to come next will be to quantify how much we lose by the type of misspecification considered in this paper. Theoretical, empirical and experimental studies will be helpful there.

It is also worth investigating what can be implemented when an incomplete yet not negligible communication is allowed among SAs, while the central designer has to make some modeling choice about how SAs communicate.

Another direction will be to study how we can improve the mechanism in a sector while keeping fixed the mechanisms in other sectors and, given such change, how we can improve the mechanism in another sector while keeping fixed those in other sectors, and so on. There is no obvious way do it because under general equilibrium effects it is not obvious whether or not a change regarded as an "improvement" from the point of view of PE mechanism design is indeed an improvement. That research direction will answer the question of how we should change the PE mechanism in an improving manner.

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Appendix

Proof of Theorem 4

Let the premises hold. The proof is based on the construction of a product set of PE mechanisms $\Gamma = (\Gamma^s)_{s\in S}$, where sector-$s$ PE mechanism, $\Gamma^s = (M^s, h^s)$, is a canonical mechanism.

Sector $s \in S$ PE mechanism:
Agent i's message space is defined by[13]

$$M_i^s = \mathcal{D}_I^s \times X^s \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers. Thus, agent i's strategy consists of an outcome in X^s, a profile of orderings and a nonnegative integer. Thus, a typical strategy played by agent i is denoted by $m_i^s = \left((R^s)^i, (x^s)^i, z^i\right)$. The message space of agents is the product space

$$M^s = \prod_{i\in I} M_i^s,$$

with m^s as a typical strategy profile. The outcome function h^s is defined with the following three rules:

Rule 1: If $m_i^s = \left((R^s)^i, (x^s)^i, 0\right) = \left(\bar{R}^s, x^s, 0\right)$ for each agent $i \in I$ and $x^s \in \varphi^s\left(\bar{R}^s\right)$, then $h^s(m^s) = x^s$.

[13]Note that $\mathcal{D}_i^s$ is nonempty for each agent $i \in I$ since it is rich.

Rule 2: If $n-1$ agents play $m_j^s = \left(\bar{R}^s, x^s, 0\right)$ with $x^s = \varphi^s\left(\bar{R}^s\right)$, but agent i plays $m_i^s = \left(\left(R^s\right)^i, \left(x^s\right)^i, z^i\right) \neq \left(\bar{R}^s, x^s, 0\right)$, then we can have two cases:

1. If $x^s \bar{R}_i^s \left(x^s\right)^i$, then $h^s\left(m^s\right) = \left(x^s\right)^i$.

2. If $\left(x^s\right)^i P\left(\bar{R}_i^s\right) x^s$, then $h^s\left(m^s\right) = x^s$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with lowest index among them.) This agent is declared the winner of the game and the alternative implemented is the one she selects.

Since φ is decomposable, there exists a sequence $\left(\varphi^s\right)_{s\in S}$ of one-dimensional SCRs, where $\varphi^s : \mathcal{D}_I^s \twoheadrightarrow X^s$ for each $s \in S$. Also, note that the proof of part (ii) of Definition 3 follows very closely the proof of Repullo (1987; pp. 40-41). To complete the proof, we show that part (i) and part (iii) of Definition 3 are satisfied, as well. Thus, let us first show part (i); that is, for all $R \in \mathcal{R}_I$, $\varphi\left(R\right) = h\left(NE(\Gamma, R)\right)$. Fix any $R \in \mathcal{R}_I$.

We first show that $h\left(NE(\Gamma, R)\right) \subseteq \varphi\left(R\right)$. Take any $x \in h\left(NE(\Gamma, R)\right)$. Then, there exists $m \in NE(\Gamma, R)$ such that $h\left(m\right) = \left(h^s\left(m^s\right)\right)_{s\in S} = x$. To economize on notation, for any sector $s \in S$ and any strategy profile $\bar{m} \in \prod\limits_{s\in S} M^s$, write $h^{sC}\left(\bar{m}^{sC}\right)$ for the profile of outcomes $\left(\left(h^{\hat{s}}\left(\bar{m}^{\hat{s}}\right)\right)_{\hat{s}\in S\setminus\{s\}}\right)$, so that $h\left(\bar{m}\right) = \left(h^s\left(\bar{m}^s\right), h^{sC}\left(\bar{m}^{sC}\right)\right)$. Then, given that $m \in NE(\Gamma, R)$, for each agent $i \in I$ it holds that

$$\text{for each } s \in S : h^s\left(m^s\right) R_i^s\left(h^{sC}\left(m^{sC}\right)\right) h^s\left(\hat{m}_i^s, m_{-i}^s\right) \text{ for each } \hat{m}_i^s \in M_i^s.$$

Given that D_i^s is rich, it follows from Definition 2 that there exists an ordering $\bar{R}_i^s \in D_i^s$ such that $L\left(h^s\left(m^s\right), \bar{R}_i^s\right) = L\left(h^s\left(m^s\right), R_i^s\left(h^{sC}\left(m^{sC}\right)\right)\right)$ for each $i \in I$ and $s \in S$. Then, $h^s\left(m^s\right) \in NE\left(\Gamma^s, \bar{R}^s\right)$ for each $s \in S$ and so part (ii) of Definition 3 assures that $h^s\left(m^s\right) \in \varphi^s\left(\bar{R}^s\right)$ for each $s \in S$. Finally, since $(\bar{R}^s)_{s\in S} \in \prod_{s\in S} \mathcal{D}_I^s$ is equivalent to R at x and since, moreover, φ satisfies indistinguishability, we have that $h\left(m\right) \in \varphi\left(R\right)$.

For the converse, suppose that $x \in \varphi\left(R\right)$. Given that the $\left(\mathcal{R}_I, \varphi\right)$ satisfies Property α, it follows that there exists a profile of separable orderings $\bar{R} \in \mathcal{R}_I$ such that

$$\text{for all } i \in I : L\left(x, \bar{R}_i\right) \subseteq L\left(x, R_i\right). \tag{8}$$

Furthermore, given that $\bar{R} \in \mathcal{R}_I^{sep}\left(X\right)$, decomposability implies $\varphi\left(\bar{R}\right) = \prod_{s\in S} \varphi^s(\bar{R}^s)$, where $R^s \in \mathcal{D}_I^s$ is the profile of sector-s marginal orderings induced by $\bar{R}$. Part (ii) of Definition 3 implies that $\varphi\left(\bar{R}\right) = \prod_{s\in S} h^s\left(NE\left(\Gamma^s, \bar{R}^s\right)\right)$. Moreover, given that $NE\left(\Gamma, \bar{R}\right) = \prod_{s\in S} h^s\left(NE\left(\Gamma^s, \bar{R}^s\right)\right)$, we have that $\varphi\left(\bar{R}\right) = NE\left(\Gamma, \bar{R}\right)$. Thus, there exists $m \in \prod\limits_{i\in I} M_i \equiv \prod\limits_{i\in I}\left(\prod\limits_{s\in S} M_i^s\right)$ such that $h\left(m\right) = x$. Since $h\left(m\right)$ is a Nash equilibrium outcome of $\left(\Gamma, \bar{R}\right)$, it holds that

$$\text{for all } i \in I : \left\{h\left(m_i', m_{-i}\right) \in X | m_i' \in M_i\right\} \subseteq L\left(x, \bar{R}_i\right).$$

Finally, given that (8) holds, it follows that

$$\text{for all } i \in I : \{h(m_i', m_{-i}) \in X | m_i' \in M_i\} \subseteq L(x, R_i).$$

We have that $h(m) = x \in h(NE(\Gamma, R))$. Thus we have established part (i) of Definition 3; that is, $\varphi(R) = NE(\Gamma, R)$ for all $R \in \mathcal{R}_I$.

Part (iii) of Definition 3 follows from the fact that φ satisfies indistinguishability as well as from parts (i)-(ii) of Definition 3. ■

Addendum (not for publication)

Lemma A

Let Γ be a product set of PE mechanisms. For all $R \in \mathcal{R}_I^{sep}$,

$$NE(\Gamma, R) = \prod_{s \in S} NE(\Gamma^s, R^s),$$

where for all $i \in I$ and all $s \in S$, R_i^s is the sector-s marginal ordering induced by R_i.

PROOF. Let Γ be a product set of PE mechanisms. Take any $R \in \mathcal{R}_I^{sep}$. For any $i \in I$ and any $s \in S$, write R_i^s for the sector-s marginal ordering induced by R_i. Consider any $m \in NE(\Gamma, R)$. Thus, it follows that

$$h(m) R_i h(\bar{m}_i, m_{-i}) \text{ for all } \bar{m}_i \in M_i.$$

Fix any $s \in S$ and any $i \in I$. Since $R_i \in \mathcal{R}_i^{sep}$, it holds that

$$\text{for all } \bar{m}_i^s \in M_i^s : h^s(m^s) R_i^s h^s\left(\bar{m}_i^s, m_{-i}^s\right).$$

Since it holds for any $i \in I$, we have that $m^s \in NE(\Gamma^s, R^s)$. Finally, given that the choice of s was arbitrary, we have that $m \in \prod_{s \in S} NE(\Gamma^s, R^s)$.

Consider any $m \in \prod_{s \in S} NE(\Gamma^s, R^s)$. Thus,

$$\text{for all } s \in S \text{ and all } i \in I : h^s(m^s) R_i^s h^s\left(\bar{m}_i^s, m_{-i}^s\right) \text{ for all } \bar{m}_i^s \in M_i^s.$$

Assume, to the contrary, that $m \notin NE(\Gamma, R)$. Then, for at least one $i_o \in I$ and one $\bar{m}_{i_o} \in M_{i_0}$, it holds that $h(\bar{m}_{i_o}, m_{-i_o}) P(R_{i_o}) h(m)$.

Since for sector 1, it holds that

$$h^1\left(m^1\right) R_{i_o}^1 h^1\left(\bar{m}_{i_o}^1, m_{-i_o}^1\right),$$

it follows from $R_{i_o} \in \mathcal{R}_{i_o}^{sep}$ that

$$h(m) R_{i_o} \left(h^1\left(\bar{m}_{i_o}^1, m_{-i_o}^1\right), \left(h^s(m^s)\right)_{s \in S \setminus \{1\}}\right).$$

Reasoning like that used in the preceding lines shows that for any $s \in S \setminus \{1, \ell\}$, it holds that

$$\left(\left(h^p\left(\bar{m}_{i_o}^p, m_{-i_o}^p\right)\right)_{p=1,\cdots,s-1}, \left(h^q(m^q)\right)_{q=s,\cdots,\ell}\right) R_{i_o} \left(\left(h^p\left(\bar{m}_{i_o}^p, m_{-i_o}^p\right)\right)_{p=1,\cdots,s}, \left(h^q(m^q)\right)_{q=s+1,\cdots,\ell}\right).$$

Likewise, for sector ℓ, it holds that

$$\left(\left(h^p\left(\bar{m}_{i_o}^p, m_{-i_o}^p\right)\right)_{p=1,\cdots,\ell-1}, h^\ell\left(m^\ell\right)\right) R_{i_o} h(\bar{m}_{i_o}, m_{-i_o}).$$

Since R_i is transitive, it follows that

$$h(m)\, R_{i_o} h(\bar{m}_{i_o}, m_{-i_o}),$$

in violation of $h(\bar{m}_{i_o}, m_{-i_o})\, P(R_{i_o})\, h(m)$. Thus, $m \in NE(\Gamma, R)$. ■

Example A

There are two types of agents, say type A and type B, two sectors, say sector 1 and sector 2, and two distinct items per sector, say x^s and y^s. Consider a profile R where the separable strict orderings of types are

$$\begin{array}{rcl} \text{for type } A & : & (x^1, x^2) P(R_A)(x^1, y^2) P(R_A)(y^1, x^2) P(R_A)(y^1, y^2) \\ \text{for type } B & : & (y^1, y^2) P(R_B)(x^1, y^2) P(R_B)(y^1, x^2) P(R_B)(x^1, x^2). \end{array}$$

Furthermore, consider a profile $\bar{R}$ where the separable strict orderings of types are

$$\begin{array}{rcl} \text{for type } A & : & (x^1, x^2) P\left(\bar{R}_A\right)(y^1, x^2) P\left(\bar{R}_A\right)(x^1, y^2) P\left(\bar{R}_A\right)(y^1, y^2) \\ \text{for type } B & : & (y^1, y^2) P\left(\bar{R}_B\right)(y^1, x^2) P\left(\bar{R}_B\right)(x^1, y^2) P\left(\bar{R}_B\right)(x^1, x^2). \end{array}$$

One can check that R and $\bar{R}$ induce the following marginal strict orderings:

$$\begin{array}{rcl} \text{for type } A \text{, sector 1} & : & x^1 P\left(R_A^1\right) y^1 \\ \text{for type } A \text{, sector 2} & : & x^2 P\left(R_A^2\right) y^2 \\ \text{for type } B \text{, sector 1} & : & y^1 P\left(R_B^1\right) x^1 \\ \text{for type } B \text{, sector 2} & : & y^2 P\left(R_B^2\right) x^2. \end{array}$$

Suppose that there are three agents, where agents 1 and 2 are of type A and agent 3 is of type B. Furthermore, suppose that the profiles R and $\bar{R}$ are the only allowable profiles of separable orderings.

Consider the SCR $\varphi : \{R, \bar{R}\} \twoheadrightarrow X$ such that

$$\varphi(R) = \left\{(x^1, y^2), \left(x^1, x^2\right)\right\} \neq \varphi(\bar{R}) = \left\{\left(y^1, x^2\right), \left(x^1, x^2\right)\right\}. \tag{9}$$

This SCR is Maskin monotonic and satisfies the condition of no veto-power.[14] Therefore, the SCR φ is Nash implementable, according to Maskin's Theorem (Maskin, 1999).

Suppose that the SCR φ is decomposable. By construction, one has that the set of marginal orderings of sector 1 and sector 2 induced by R and $\bar{R}$ are

$$\begin{array}{rcl} \text{for type } A & : & \mathcal{D}_A^1 = \left\{R_A^1\right\} \text{ and } \mathcal{D}_A^2 = \left\{R_A^2\right\} \\ \text{for type } B & : & \mathcal{D}_B^1 = \left\{R_B^1\right\} \text{ and } \mathcal{D}_B^2 = \left\{R_B^2\right\}. \end{array}$$

[14] No veto-power says that if an outcome x is at the top of the preferences of all but possibly one of the agents, then x should be selected by the SCR φ.

Decomposability implies that

$$\varphi(R) = \varphi^1(R_A^1, R_A^1, R_B^1) \times \varphi^2(R_A^2, R_A^2, R_B^2) = \varphi(\bar{R}),$$

in violation of (9). Thus, the SCR φ is not decomposable.

Proposition 2A

In what follows, we first present two domain restrictions, namely Property β and Property β^*, and show in Lemma B that Property β implies Property β^*, and that these two domain restrictions are indeed equivalent if X^s is a finite set for each sector $s \in S$ and the domain of allowable orderings for agent i, $\mathcal{R}_i$, includes the set of separable orderings on X. Finally, we show that in auction settings Property β accommodates non-separability of agents' preferences due to income effects. We show this since Property β is easier to check than Property α.

Definition 13 The domain $\mathcal{R}_i \subseteq \mathcal{R}(X)$ satisfies Property β if, for all $R_i \in \mathcal{R}_i$ and all $x \in X$, there exists a separable ordering $\bar{R}_i \in \mathcal{R}_i$ such that

$$\text{for all } s \in S : L^s(x, R_i) \subseteq L^s(x, \bar{R}_i), \tag{10}$$

$$\text{and } L(x, \bar{R}_i) \subseteq L(x, R_i), \tag{11}$$

where $L^s(x, R_i) = \{(y^s, x^{s_C}) \in X | xR_i(y^s, x^{s_C})\}$.

A domain condition which is implied by Property β can be defined as follows:

Definition 14 The domain $\mathcal{R}_i \subseteq \mathcal{R}(X)$ satisfies Property β^* if for all $R_i \in \mathcal{R}_i$ and all $x, y \in X$ it holds that

$$xR_i(y^s, x^{s_C}) \text{ for all } s \in S \implies xR_iy.$$

Property β^* is easier to check than Property β. The next result shows that Property β^* is implied by Property β and that the two properties are equivalent if for each $s \in S$, X^s is a finite set, and the domain $\mathcal{R}_i$ includes the set of separable orderings on X.

Lemma B If $\mathcal{R}_i \subseteq \mathcal{R}(X)$ satisfies Property β, then $\mathcal{R}_i$ satisfies Property β^*. The converse is true provided that X^s is finite for all $s \in S$ and that $\mathcal{R}^{sep}(X) \subseteq \mathcal{R}_i$.

Proof. Consider any $R_i \in \mathcal{R}_i$ and $x, y \in X$ such that $xR_i(y^s, x^{s_C})$ for all $s \in S$. Suppose that $\mathcal{R}_i$ satisfies Property β. Then, there exists a separable ordering $\bar{R}_i \in \mathcal{R}_i^{sep}$ such that (10) and (11) hold. Since, by hypothesis, $xR_i(y^s, x^{s_C})$, it follows from (10) that $x\bar{R}_i(y^s, x^{s_C})$, and so the sector-s marginal ordering is such that $x^s\bar{R}_i^s y^s$. Given that $\bar{R}_i$ is a separable ordering, we have that

$$x\bar{R}_i(y^1, x^{1_C}),$$

that

$$\text{for all } s \in S \backslash \{1, \ell\} : \left((y^q)_{q=1,\cdots,s-1}, (x^q)_{q=s,\cdots,\ell}\right) \bar{R}_i \left((y^q)_{q=1,\cdots,s}, (x^q)_{q=s+1,\cdots,\ell}\right)$$

and that

$$\left(y^{\ell_C}, x^{\ell}\right) \bar{R}_i y.$$

Since $\bar{R}_i$ is transitive, it follows that $x\bar{R}_i y$. Given that (11) holds, we have that xR_iy. Thus, $\mathcal{R}_i$ satisfies Property β^*.

To show the converse, suppose X^s is finite for all $s \in S$ and that $\mathcal{R}^{sep}(X) \subseteq \mathcal{R}_i$. Moreover, suppose that $\mathcal{R}_i$ satisfies Property β^*. Assume, to the contrary, that Property β is violated. Fix any $R \in \mathcal{R}_i$ and $x \in X$.

For each $s \in S$, fix a representation of the sector-s marginal ordering $R_i^s(x^{s_C})$, which is denoted by v_i^s. Then, for any $\lambda > 0$, let $\bar{R}_i^\lambda$ be a separable ordering represented in the form

$$\bar{u}_i^\lambda(y) = \sum_{s\in S} \exp \lambda(v_i^s(y^s) - v_i^s(x^s)).$$

For λ sufficiently large it holds that

$$x\bar{R}_i^\lambda y \Longrightarrow xR_i(y^s, x^{s_C}) \text{ for all } s \in S.$$

This is because if $x\bar{R}_i^\lambda y$ but $(y^s, x^{s_C})P(R_i)\, x$ for some $s \in S$, then for λ sufficiently large the term $\exp \lambda(v_i^s(y^s) - v_i^s(x^s))$ becomes arbitrarily large, which leads to $yP\left(\bar{R}_i^\lambda\right)x$.

Fix any $s \in S$. Suppose that $xR_i\,(y^s, x^{s_C})$ for some $y^s \in X^s$. Then, $v_i^s\,(x^s) \geq v_i^s\,(y^s)$ given that $x^s R_i^s\,(x^{s_C})\, y^s$. We need to rule out the case that $(y^s, x^{s_C})\, P\left(\bar{R}_i^\lambda\right) x$ to conclude that $x\bar{R}_i^\lambda\,(y^s, x^{s_C})$. Thus, suppose that $(y^s, x^{s_C})\, P\left(\bar{R}_i^\lambda\right) x$. By definition of $\bar{u}_i^\lambda$, it must hold that $\bar{u}_i^\lambda\,(y^s, x^{s_C}) > \bar{u}_i^\lambda\,(x)$ or, equivalently, it must be the case that

$$\exp \lambda(v_i^s(y^s) - v_i^s(x^s)) > 1,$$

which is false given that $v_i^s\,(x^s) \geq v_i^s\,(y^s)$ and $\lambda > 0$.

Suppose that there exists $y \in X$ such that $x\bar{R}_i^\lambda y$ but $yP\,(R_i)\,x$. Since $x\bar{R}_i^\lambda y$, then for λ sufficiently large it holds that $xR_i(y^s, x^{s_C})$ for all $s \in S$. Property β^* implies that xR_iy, which is a contradiction. Thus, $\mathcal{R}_i$ satisfies Property β^*. ■

Proposition 2A Let $S = \{1, 2\}$. For each $s \in S$, let $X^s = D^s \times T$, where T is the set of closed transfers defined in (1). Assume that agent i's preferences belonging to $\mathcal{R}_i$ are represented in the form given in (2). Suppose that her willingness to pay\accept is well defined.[15] Property β is equivalent to the following property: for all $d^1, \bar{d}^1 \in D^1$, $d^2, \bar{d}^2 \in D^2$ and $t^1, t^2 \in T$, if

$$\begin{aligned} U_i(d^1, d^2, t_i^1 + t_i^2 + e_i) &= U_i(\bar{d}^1, d^2, t_i^1 + \Delta t_i^1 + t_i^2 + e_i) \\ &= U_i(d^1, \bar{d}^2, t_i^1 + t_i^2 + \Delta t_i^2 + e_i), \end{aligned} \tag{12}$$

[15]To assure that agent i's willingness to pay\accept is well defined, we also assume that U_i satisfies the following property: For all $d^1, \bar{d}^1 \in D^1$, all $d^2, \bar{d}^2 \in D^2$, all $t^1, t^2 \in T$, there exist $\bar{t}^1, \bar{t}^2 \in T$ such that

$$U_i\left(d^1, d^2, t_i^1 + t_i^2 + e_i\right) = U_i\left(\bar{d}^1, \bar{d}^2, \bar{t}_i^1 + \bar{t}_i^2 + e_i\right).$$

then

$$U_i(\bar{d}^1, \bar{d}^2, t_i^1 + \Delta t_i^1 + t_i^2 + \Delta t_i^2 + e_i) = U_i(d^1, d^2, t_i^1 + t_i^2 + e_i). \tag{13}$$

PROOF. To show that the above property is implied by Property β, pick any $d^1, \bar{d}^1 \in D^1$, $d^2, \bar{d}^2 \in D^2$ and $t^1, t^2 \in T$. Take any Δt_i^1 and Δt_i^2 such that the equalities in (12) hold. We need to show (13). Since agent i's willingness to pay\accept is well defined, by assumption, there exists $\overline{\Delta t_i^2}$ such that

$$U_i(\bar{d}^1, \bar{d}^2, t_i^1 + \Delta t_i^1 + t_i^2 + \overline{\Delta t_i^2} + e_i) = U_i(\bar{d}^1, d^2, t_i^1 + \Delta t_i^1 + t_i^2 + e_i), \tag{14}$$

and so, from (12), it follows that

$$U_i(\bar{d}^1, \bar{d}^2, t_i^1 + \Delta t_i^1 + t_i^2 + \overline{\Delta t_i^2} + e_i) = U_i(d^1, \bar{d}^2, t_i^1 + t_i^2 + \Delta t_i^2 + e_i). \tag{15}$$

Then, by Lemma B, we can apply Property β^* to the equalities (14) and (15) so as to obtain that

$$U_i(\bar{d}^1, \bar{d}^2, t_i^1 + \Delta t_i^1 + t_i^2 + \overline{\Delta t_i^2} + e_i) = U_i(\bar{d}^1, \bar{d}^2, t_i^1 + \Delta t_i^1 + t_i^2 + \Delta t_i^2 + e_i). \tag{16}$$

This implies $\overline{\Delta t_i^2} = \Delta t_i^2$. Therefore, combining (5) and (14) with (16), we obtain (13). Thus, $\mathcal{R}_i$ satisfies the above property if it satisfies Property β.

The converse is true, because the indifference surface passing through $(d^1, d^2, t_i^1 + t_i^2 + e_i)$ coincides exactly with the indifference surface of the corresponding separable preference. ■