Minimum cost connection networks:  
truth-telling and implementation*  

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Abstract  
In the present paper we consider the allocation of costs in connection networks. Agents have connection demands in form of pairs of locations they want to be connected. Connections between locations are costly to build. The problem is to allocate costs of networks satisfying all connection demands. We use three axioms to characterize allocation rules that truthfully implement cost minimizing networks satisfying all connection demands in a game where: (1) a central planner announces an allocation rule and a cost estimation rule; (2) every agent reports her own connection demand as well as all connection costs; and, (3) the central planner selects a cost minimizing network satisfying reported connection demands based on the estimated connection costs and allocates the true connection costs of the selected network.

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1 Introduction

Overview of the paper: In the present paper we consider cost allocation in the connection network (CN) model used in Anshelevich et al. (2008), Chen et al. (2010), Juarez & Kumar (2013) and Moulin (2013). Agents have connection demands in form of pairs of locations they want to be directly or indirectly connected. Connections between locations are costly to establish. Moreover, connections are undirected and free from congestion. Consequently several agents can use the same connection as part of paths satisfying their connection demands. Therefore connections are public goods. In the next part of the introduction we use the German Hansa as an example of network building and cost allocation in practice.

A minimum cost connection network (MCCN) is a network minimizing total cost subject to the constraint that all connection demands have to be satisfied. An allocation rule maps MCCNs, connection demands and connection costs to cost shares for all agents. Depending on the allocation rule there can be a potential conflict between overall welfare aimed at minimizing total cost and individual welfare aimed at minimizing individual cost shares. We characterize the set of allocation rules that truthfully implement MCCNs.

In the minimum cost spanning tree (MCST) model one location is a source and every other location is inhabited by an agent. Every agent wants to be connected to the source making the MCST model a special case of the CN model. Indeed undemanded locations (Steiner nodes) and arbitrary connection demands are allowed in the CN model. In Hougaard & Tvede (2012) we characterized allocation rules that truthfully implement MCCNs in the MCST model in a game:

1. A central planner announces a cost allocation rule and a cost estimation rule.
2. Every agent reports all connection costs.
3. The central planner selects a MCCN based on the estimated connection costs and allocates the true costs of the selected network.

An allocation rule is reductionist provided it depends on irreducible costs, where the irreducible cost of a connection is: the true cost in case the connection is used; and, the lowest cost for which the connection needs not be used in case the connection is unused. An allocation rule is monotonic provided it is monotonic in irreducible costs. In Hougaard & Tvede (2012) we showed for the MCST model that an allocation rule truthfully implements MCCNs if and only if it is reductionist and monotonic.

For the MCST model the set of reductionist and monotonic allocation rules is quite large and includes fixed relative cost shares rules such as the equal split rule, the folk rule discussed in Bogomolnaia & Moulin (2010) and the rest of the family of obligation rules introduced and analyzed in Tijs et al. (2006). An important notion is Stand Alone core
stability: no group of agents should pay more than the minimum cost for a network satisfying all their connection demands. Fixed relative cost shares rules are not Stand Alone core stable. However the family of obligation rules are Stand Alone core stable. Therefore reductionism and monotonicity are compatible with Stand Alone core stability. Hence voluntary participation – even in the strong form of Stand Alone core stability – is compatible with truthful implementation of MCCNs in the MCST model.

Our characterization of allocation rules that truthfully implements MCCNs in the MCST model does not generalize to the CN model. Indeed irreducible costs need not be defined in case of undemanded locations. As a consequence both reductionism and monotonicity need not be defined in case of undemanded locations. Therefore we focus on two crucial properties of reductionism and monotonicity and study the set of allocation rules having these two properties.

An implication of reductionism is that connection costs of unused connections do not influence cost shares. This property is denoted Unobserved Information Independence (UII). An implication of monotonicity is that cost shares are independent of the selected MCCN in case of multiple MCCNs. This property is denoted Network Independence (NI). Finally the property that cost shares are homogeneous of degree one in connection costs is denoted Scale Invariance. For the CN model we characterize allocation rules satisfying UII and NI as well as UII, NI and SI (Theorem 1 and Corollary 1).

In Theorem 1 we show that an allocation rule satisfies UII and NI if and only if cost shares depend on connection demands, total cost and nothing else. In Corollary 1 we show that an allocation rule satisfies UII, NI and SI if and only if relative cost shares depend on connection demands and nothing else. Therefore for given connection demands an allocation rule satisfies UII, NI and SI if and only if relative cost shares are fixed. An important notion is Individual Rationality: no agent should pay more than the minimum cost for a path satisfying her connection demand. Clearly Individual Rationality is much weaker than Stand Alone core stability. Theorem 1 implies reductionism and monotonicity, interpreted as UII and NI, are at odds with Individual Rationality in the CN model.

In order to consider truthful implementation of MCCNs in the CN model we modify the game in Hougaard & Tvede (2012) to include reports on connection demands:

1. A central planner announces a cost allocation rule and a cost estimation rule.
2. Every agent reports her connection demand and all connection costs.
3. The central planner selects a MCCN based on reported demands and the estimated connection costs and allocates the true costs of the selected network.

For the CN model: we show that if an allocation rule truthfully implement MCCNs, then it satisfies UII and NI (Observations 1 and 2); and, we characterize allocation rules satisfying
UII, NI and SI and truthfully implementing MCCNs (Theorem 2 in case connection demands are private information and Corollary 2 in case connection demands are public knowledge). Consequently voluntary participation – even in the weak form of Individual Rationality – is at odds with truthful implemention of MCCNs in the CN model.

Motivating example: The German Hansa is an illustrative example of network building and cost allocation in practice. Below we explain a few aspects concerning the Hansa. A thorough study of the Hansa is found in Dollinger (1970). The Hansa started in the middle of the 12th century as an association of north German merchants, developed into a community of cities in the middle of the 14th century and dissolved in the middle of the 17th century. The Hansa was used for obtaining trading privileges for its merchants as well as protecting and supporting its merchants. Most of the trade involving merchants with Hanseatic trading privileges took place in the area between Novgorod in east and London in west and Cologne in south and Bergen in north. The Hansa is an example of a network with locations being towns whose merchants had Hanseatic privileges and their markets and connections being roads between towns and markets.

In the middle of the 12th century north German merchants regularly visiting or permanently settling in Gotland formed a community. These merchants wanted to take advantage of commercial opportunities in Russia in Novgorod as well as in Polotsk, Vitebsk and Smolensk on the river Dvina. However the pagan habitants of Finland and the Baltic countries made trade very risky. Around 1200 bishop Albert led a crusade into the Baltic countries. The Gotland community contributed to the crusade by equipping hundreds of crusaders and providing transportation. Lübeck supported the crusade. In 1241 Hamburg and Lübeck agreed to share the cost of keeping the roads between the two towns free from brigands. Both the crusade and the fight against brigandage can be seen as establishing new connections and making these connections safer can be seen as a public good for merchants with Hanseatic privileges as well as everybody else.

The German Hansa lived with the tension between Individual Rationality and truthful implementation. In 1284 the Norwegian king restricted the Hanseatic privileges. The German Hansa responded with a blockade. Bremen did not participate in the blockade probably because being part of the blockade would have favoured merchants from other towns – at least in the short run. Bremen was excluded from the Hansa. The behaviour of Bremen can be interpreted as an attempt to free ride: if the blockade failed, Bremen could continue to trade with Norway; and, if the blockade succeeded, Bremen could benefit from the improved Hanseatic privileges. The Hansa and Denmark were at war between 1367 and 1369 when Denmark asked for peace. The Westphalian towns including Cologne traded mainly with England and the low countries. These towns did not contribute or support the war, but
they were not excluded from the Hansa. The behaviour of the Westphalian towns can be interpreted as a reflection of their commercial interests or equivalently connection demands.

The German Hansa could sanction participants by use of fines, confiscation and exclusion, but as mentioned participation was voluntarily. Based on the different responses to Bremen and the Westphalian towns it appears the Hansa dealt with the incompatibility of voluntary participation and truthful implementation by accepting that cost shares should depend on commercial interests or equivalently connection demands. We guess it worked to relate cost shares and commercial interests because commercial interests of every merchant were known by other merchants.

**Related literature:** It is shown in Megiddo (1978) and Tamir (1991) that the set of Stand Alone core stable allocations can be empty for the CN model in case of undemanded locations. Therefore it is trivial that reductionism and monotonicity are incompatible with Stand Alone core stability. However the implication of Theorem 1 is less trivial: reductionism and monotonicity are incompatible with Individual Rationality.

In Moulin (2013) the folk rule is extended from the MCST model to two subclasses of the CN model for which the set of Stand Alone core stable allocations is nonempty.

Implementation in the CN model has been analyzed in Anshelevich et al. (2008), Chen et al. (2010) and Juarez & Kumar (2013). The same game is used in all three papers:

1. A central planner announces a cost allocation rule.
2. Every agent reports a path between the pair of locations she wants to be connected.
3. The central planner selects the network consisting of all reported paths and allocates costs of the selected network.

In comparison with our game, every agent has less impact on the selection of network, because the paths reported by the other agents are part of the selected network.

In the two former papers properties of a specific allocation rule (the cost of every connection is divided equally between agents whose reported paths include the connection) are studied. It is shown that: Nash equilibria exist; all Nash equilibria can be inefficient; and, there are bounds on the ratios between the cost of the cheapest Nash equilibrium and the cost of a MCCN (price of stability) and the cost of the most expensive Nash equilibrium and the cost of a MCCN (price of anarchy). In the latter paper attention is restricted to the set of allocation rules that depend on costs of reported paths, total cost and nothing else. It is shown that the set of allocation rules implementing MCCNs and the set of fixed relative cost shares rules are identical. In comparison with the present paper a much smaller set of allocation rules is considered. Since attention is restricted to allocation rules that do not depend on connection demands, our Corollary 1 implies that the only allocation rules satisfying UI, NI and SI are fixed relative cost shares rules.
Implementation in the MCST model has been analyzed in Bergantinos & Lorenzo (2004, 2005), Bergantinos & Vidal-Puga (2010) and Hougaard & Tvede (2012). In all four papers existence of Nash equilibria and their properties are considered.

Plan of the paper: In Section 2 we introduce the set up and our axioms; in Section 3 we characterize allocation rules satisfying our axioms; in Section 4 we study implementation of MCCNs; in Section 5 we discuss some of generalizations of our set up; and finally, in Section 6 we end with a few remarks.

2 The CN model

In the present section we introduce the set up and the axioms.

Set up

Let $\mathcal{M} = \{1, \ldots, m\}$ be a set of finitely many agents and $\mathcal{N} = \{1, \ldots, n\}$ a set of finitely many locations. The set of connections between pairs of locations is $\mathcal{N}^2 = \mathcal{N} \times \mathcal{N}$. Every agent $i \in \mathcal{M}$ has a connection demand in the sense that she wants a pair of locations $(a_i, b_i) \in \mathcal{N}^2$ to be directly or indirectly connected. A connection structure $P$ is a collection of individual connection demands $(a_i, b_i)_{i \in \mathcal{M}}$. Let $\mathcal{P}$ be the set of connection structures. A cost structure $C$ describes the costs of connecting all pairs of locations and is defined by a map $c : \mathcal{N}^2 \to \mathbb{R}^+$ with

- $c_{jj} = 0$ for all $j$.
- $c_{jk} > 0$ for all $j$ and $k$ with $j \neq k$.
- $c_{kj} = c_{jk}$ for all $j$ and $k$.

Connections are undirected and free from congestion. A connection problem is a connection structure and a cost structure $(P, C)$. A CN for a connection problem $(P, C)$ is a graph $g$ such that for every agent $i$ there is a path $p_i(g) = a_i \ldots b_i$ between $a_i$ and $b_i$ in $g$ and $g$ is the union of all the paths $g = \bigcup_{i \in \mathcal{M}} p_i(g)$. CNs can contain cycles. Let $v(g, C)$ be the total cost of a CN

$$v(g, C) = \sum_{jk \in g} c_{jk}.$$ 

A MCCN for a connection problem $(P, C)$ is a CN $g$ minimizing total cost subject to the constraint that all connection demands have to be satisfied. Therefore $g$ is a MCCN if and only if $v(g, C) \leq v(h, C)$ for every CN $h$. The set of MCCNs is non-empty and finite because the set of CNs is non-empty and finite. Let $\mathcal{MCCN}(P, C)$ be the set of MCCNs for a given
connection problem \((P, C)\). Clearly every MCCN is either a tree or a forest (a collection of
trees). Indeed if a CN contains a cycle, then removing any connection in the cycle does not
change whether connection demands are satisfied or not.

A cost allocation problem \((g, P, C)\) is a CN \(g\) and a connection problem \((P, C)\) such that
\(g\) is a MCCN for \((P, C)\). In the rest of the paper we focus on cost allocation problems with
undemanded locations. Therefore let \(\mathcal{Z}\) be the set of cost allocation problems \((g, P, C)\)
with \(\mathcal{N} \setminus \cup_i \{a_i, b_i\} \neq \emptyset\). In Section 5 we show how our results can be generalized to all cost
allocation problems with undemanded locations.

### Allocation rules and axioms

For a given cost allocation problem \((g, P, C)\) the total cost of the MCCN \(g\) has to be shared
among agents in \(\mathcal{M}\). An allocation rule \(\phi : \mathcal{Z} \rightarrow \mathbb{R}_+^m\) maps a cost allocation problem to an
\(m\)-dimensional vector of positive cost shares,

\[ \phi(g, P, C) = (\phi_1(g, P, C), \ldots, \phi_m(g, P, C)), \]

for which budget-balance \(\sum_{i \in \mathcal{M}} \phi_i(g, P, C) = v(g, C)\) is satisfied.

Most game theoretic solution concepts such as the Shapley value map connection prob-
lems \((P, C)\) rather than cost allocation problems \((g, P, C)\) to cost shares. However maps \(\phi\) from
cost allocation problems to cost shares can be used to define maps \(\Phi\) from connection
problems to cost shares and vice versa. Indeed for a given \(\phi\) let \(\Phi\) be defined by

\[ \Phi(P, C) = \frac{1}{|\mathcal{MCCN}(P, C)|} \sum_{g \in \mathcal{MCCN}(P, C)} \phi(g, P, C) \]

and for a given \(\Phi\) let \(\phi\) be defined by

\[ \phi(g, P, C) = \Phi(P, C) \]

for all \(g \in \mathcal{MCCN}(P, C)\). We use maps \(\phi\) from cost allocation problems rather than connection
problems because the properties of allocation rules we consider thereby become weaker
and clearer.

The first two properties are at the heart of our characterizations.

**Unobserved Information Independence (UII)** For all \((g, P, C), (g, P, D) \in \mathcal{Z}\) with \(c_{jk} = d_{jk}\)
for all \(jk \in g\), \(\phi(g, P, C) = \phi(g, P, D)\).

UII states that for two cost allocation problems with identical MCCNs, connection structures
and observed connection costs, the allocations of costs should be identical. Hence UII
implies the allocation of costs is independent of costs of unobserved connections.
Network Independence (NI) *For all* \((g, P, C), (h, P, C) \in \mathcal{Z}\), \(\phi(g, P, C) = \phi(h, P, C)\).

NI states that for a connection problem with multiple MCCNs, the allocations of costs should be identical for all MCCNs. Hence NI implies the allocation of costs is independent of the chosen MCCN. The last property is standard.

Scale Invariance (SI) *For all* \((g, P, C) \in \mathcal{Z}\) *and* \(\lambda > 0\), \(\phi(g, P, \lambda C) = \lambda \phi(g, P, C)\).

SI states that cost shares are homogeneous of degree one in connection costs.

In case of constructing a network, depending on the perception of fairness, costs of unused connections can be allowed to have more or less influence on the allocation of total cost of the realized network. The position reflected in UII is that since the cost of a realized CN is observed and costs of unused connections are unobserved estimates, there is no obvious reason for letting the allocation of realized cost depend on the costs of unused connections. Moreover for allocation rules that allow costs of unused connections to influence the allocation of total cost there is a conflict between agents over cost estimates of unused connections. This conflict can be an obstacle for implementation of the efficient network as we demonstrate below in Observation 1. All in all, UII can be seen to reflect a positive as well as a normative approach to cost sharing. Allocation rules such as the cost sharing protocol considered in Chen et al. (2010) for which the cost of every connection in the realized graph is split equally between its users satisfy UII. Allocation rules violating UII include most game theoretic solution concepts.

In connection problems with several MCCNs total costs are identical for all MCCNs. Therefore there is no reason for choosing one MCCN over another. The position reflected in NI is that the allocation of costs should not depend on the chosen MCCN. Moreover for allocation rules that allow the choice of MCCN to influence the allocation of costs there is a conflict between agents over which MCCN to choose. This conflict can be an obstacle for implementation of the efficient network as we demonstrate below in Observation 2. All in all, NI can be seen to reflect a positive as well as a normative approach to cost sharing. Most game theoretic solution concepts satisfy NI. Allocation rules violating NI include the cost sharing protocol considered in Chen et al. (2010).

### 3 Characterization results

In the present section we characterize allocation rules satisfying UII and NI as well as allocation rules satisfying UII, NI and SI.
Simple allocation rules

We consider allocation rules for which cost shares depend on the connection structure $P$, the total cost $v(g,C)$ of the MCCN $g$ and nothing else.

Definition 1 A simple allocation rule is an allocation rule $\phi : \mathcal{Z} \rightarrow \mathbb{R}^m_+$ for which there is a map $\Gamma : \mathcal{P} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m_+$ such that for all $(g,P,C) \in \mathcal{Z}$,

$$\phi(g,P,C) = \Gamma(P,v(g,C)).$$

For simple allocation rules, the allocation of costs depends on the connection structure as well as the total cost, but not on any other property of the cost structure.

Theorem 1 An allocation rule $\phi$ satisfies UII and NI if and only if it is simple.

In Hougaard & Tvede (2012) we considered the MCST model where $\mathcal{M} = \{1,\ldots,m\}$, $\mathcal{N} = \{0,\ldots,m\}$ and $(a_i,b_i) = (0,i)$ for all $i$. Moreover we showed that the set of allocation rules satisfying UII and NI is quite large and includes the equal split rule and the folk rule and the rest of the family of obligation rules. All allocation rules in the family of obligation rules are Stand Alone core stable. Theorem 1 implies that adding locations not inhabited by any agent to the MCST model shrinks the set of allocation rules dramatically. Indeed the set of allocation rules satisfying UII and NI contains no Individual Rational allocation rule for the CN model with undemanded locations.

Before we provide a proof for Theorem 1 we discuss the implications of adding SI to UII and NI.

Super simple allocation rules

For simple allocation rules, the allocation of costs can depend on total cost in quite weird ways. Indeed, consider the following example: If total cost is a rational number $v(g,C) \in \mathbb{Q}$, then the total cost is split equally between agents

$$\phi_i(g,P,C) = \frac{v(g,C)}{m}$$

for all $i$ and if the total cost is an irrational number $v(g,C) \in \mathbb{R} \setminus \mathbb{Q}$, then the total cost is allocated between agents such that

$$\phi_i(g,P,C) = \frac{w_i}{\sum_{j \in \mathcal{M}} w_j} v(g,C)$$
where
\[ w_i = \sum_{j \in \{a_i, b_i\}} (|\{i' | j = a_{i'}\}| + |\{i' | j = b_{i'}\}|) \]
for all \( i \). The rule obviously violates SI although each of the two parts of the rule satisfies SI on its domain. The first part of the cost allocation rule is the equal split rule. The second part is a rule where the cost share of agent \( i \) depends on the number of agents who have the locations in the connection demand of agent \( i \) as parts of their connection demands. We consider allocation rules for which relative cost shares depend on connection structures \( P \) and nothing else.

**Definition 2** A super simple allocation rule is an allocation rule \( \phi : \mathcal{Z} \to \mathbb{R}_+^m \) for which there is a map \( \Lambda : \mathcal{P} \to \mathbb{R}_+^m \) such that for all \( (g, P, C) \in \mathcal{Z} \),
\[ \phi(g, P, C) = \Lambda(P)v(g, C). \]

For super simple allocation rules, the allocation of costs is simple and homogenous of degree one in connection costs. Consequently super simple rules are continuous in cost structures.

**Corollary 1** An allocation rule \( \phi \) satisfies UII, NI and SI if and only if it is super simple.

**Proof:** We leave it to the reader to check that super simple allocation rules satisfy UII, NI and SI. Consequently we focus on the converse claim. According to Theorem 1 if an allocation rule \( \phi : \mathcal{Z} \to \mathbb{R}_+^m \) satisfies UII and NI, then there is a map \( \Gamma : \mathcal{P} \times \mathbb{R}_+^+ \to \mathbb{R}_+^m \) such that for all \( (g, P, C) \in \mathcal{Z} \),
\[ \phi(g, P, C) = \Gamma(P, v(g, C)). \]

If the allocation rule satisfies SI, then \( \phi(h, P, \lambda D) = \lambda \phi(h, P, D) \). Therefore the relative cost shares are independent of the MCCN and the cost structure
\[ \frac{1}{v(g, C)} \phi(g, P, C) = \frac{1}{\lambda v(h, D)} \lambda \phi(h, P, D) = \frac{1}{v(h, D)} \phi(h, P, D). \]

Hence for every connection structure \( P \in \mathcal{P} \) pick an arbitrary cost allocation problem \( (g, P, C) \in \mathcal{Z} \) and let \( \Lambda(P) \) be defined by
\[ \Lambda(P) = \frac{1}{v(g, C)} \phi(g, P, C). \]

Then for all \( (h, P, D) \in \mathcal{Z} \), \( \phi(h, P, D) = \Lambda(P)v(h, D) \). Thus the allocation rule is super simple. \( \square \)
Proof of Theorem 1

We leave it to the reader to check that simple allocation rules satisfy UII and NI. Consequently we focus on the converse claim. The proof consists of a preliminary observation, three lemmas and a closing observation.

First, we make the following preliminary observation. Consider a cost allocation problem \((g, P, C)\) and a finite number of pairs of MCCNs and cost structures \((g^1, C^1), \ldots, (g^N, C^N)\) such that

\[
g, g^1 \in \mathcal{MCN}(P, C^1) \quad \text{and} \quad c^1_{jk} = c_{jk} \quad \text{for all } jk \in g.
\]

\[
g^1, g^2 \in \mathcal{MCN}(P, C^2) \quad \text{and} \quad c^2_{jk} = c^1_{jk} \quad \text{for all } jk \in g^1.
\]

\[
\vdots
\]

\[
g^{N-1}, g^N \in \mathcal{MCN}(P, C^N) \quad \text{and} \quad c^N_{jk} = c^{N-1}_{jk} \quad \text{for all } jk \in g^{N-1}.
\]

Then \(\phi(g^N, P, C^N) = \phi(g, P, C)\) according to UII and NI.

**Lemma 1** Consider a cost allocation problem \((g, P, C)\). Suppose that there is a location \(u\) with \(u \in \mathcal{N}\) and \(u \notin g\). Then for every pair of connections \(rs\) and \(st\) in \(g\) and all cost structures \(C'\) with \(c'_{rs} + c'_{st} = c_{rs} + c_{st}\) and \(c'_{jk} = c_{jk}\) for all other connections, \(\phi(g, P, C) = \phi(g, P, C')\).

**Proof:** Without loss of generality assume that \(c'_{rs} < c_{rs}\) and \(c'_{st} > c_{st}\).

In case removing \(rs\) and \(st\) from \(g\) and adding \(rt\) to \(g\) result in a graph, which is a CN consider the following three steps.

**Step 1:** Define \(g^1\) by \(g^1 = g\) and define \(C^1\) by

\[
c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g, \\ \max\{c_{jk}, v(g, C)\} & \text{for all other connections}. \end{cases}
\]

Then \(g, g^1 \in \mathcal{MCN}(P, C^1)\) and \(c^1_{jk} = c_{jk}\) for all \(jk \in g\). Therefore \(\phi(g^1, P, C^1) = \phi(g, P, C)\) according to UII.

**Step 2:** Define \(g^2\) by removing \(rs\) and \(st\) from \(g^1\) and adding \(rt\). Then \(g^2\) is a connection network. Define \(C^2\) by

\[
c^2_{jk} = \begin{cases} c^1_{rs} + c_{st} & \text{for } jk = rt, \\ c^1_{jk} & \text{for all other connections}. \end{cases}
\]
Then \( g^1, g^2 \in MCCN(P,C^2) \) and \( c_{jk}^2 = c_{jk}^1 \) for all \( jk \in g^1 \). Therefore \( \phi(g^2, P, C^2) = \phi(g^1, P, C^1) \) according to UII and NI.

**Step 3:** Define \( g^3 \) by removing \( rt \) from \( g^2 \) and adding \( rs \) and \( st \) to \( g^2 \). Then \( g^3 \) is a CN. Indeed \( g^3 = g^1 \). Define \( C^3 \) by

\[
c_{jk}^3 = \begin{cases} 
c_{rs} & \text{for } jk = rs \\
c'_{st} & \text{for } jk = st \\
c_{jk}^1 & \text{for all other connections.}
\end{cases}
\]

Then \( g^2, g^3 \in MCCN(P,C^3) \) and \( c_{jk}^3 = c_{jk}^2 \) for all \( jk \in g^1 \). Therefore \( \phi(g^3, P, C^3) = \phi(g^2, P, C^2) \) according to UII and NI. Moreover \( c_{rs}^3 = c_{rs}^1 \) and \( c_{st}^3 = c_{st}^1 \).

In case removing \( rs \) and \( st \) from \( g \) and adding \( rt \) to \( g \) result in a graph, which is not a CN, consider the following four steps.

**Step 1:** Define \( g^1 \) by \( g^1 = g \) and define \( C^1 \) by

\[
c_{jk}^1 = \begin{cases} 
c_{jk} & \text{for } jk \in g \\
\max\{c_{jk}, v(g, C)\} & \text{for all other connections.}
\end{cases}
\]

Then \( g, g^1 \in MCCN(P,C^1) \) and \( c_{jk}^1 = c_{jk} \) for all \( jk \in g \). Therefore \( \phi(g^1, P, C^1) = \phi(g, P, C) \) according to UII.

**Step 2:** Define \( g^2 \) by removing \( rs \) and \( st \) from \( g^1 \) and adding \( ru, su \) and \( tu \) to \( g^1 \). Then \( g^2 \) is a CN. Define \( C^2 \) by

\[
c_{jk}^2 = \begin{cases} 
\max\{c_{rs}', c_{rs}^1 - c_{st}^1\} & \text{for } jk = ru \\
c_{rs}^1 - \max\{c_{rs}', c_{rs}^1 - c_{st}^1\} & \text{for } jk = su \\
c_{st} & \text{for } jk = tu \\
c_{jk}^1 & \text{for all other connections.}
\end{cases}
\]

Then \( g^1, g^2 \in MCCN(P,C^2) \) and \( c_{jk}^2 = c_{jk}^1 \) for all \( jk \in g^1 \). Therefore \( \phi(g^2, P, C^2) = \phi(g^1, P, C^1) \) according to UII and NI.

**Step 3:** Define \( g^3 \) by removing \( ru, su \) and \( tu \) from \( g^2 \) and adding \( rs \) and \( st \) to \( g^2 \). Then \( g^3 \) is a CN. Indeed \( g^3 = g^1 \). Define \( C^3 \) by

\[
c_{jk}^3 = \begin{cases} 
c_{ru}^2 & \text{for } jk = rs \\
c_{su}^2 + c_{tu}^2 & \text{for } jk = st \\
c_{jk}^2 & \text{for all other connections.}
\end{cases}
\]
Then \( g^2, g^3 \in \mathcal{M}^\mathcal{C}N(P,C^3) \) and \( c^3_{jk} = c^2_{jk} \) for all \( jk \in g^1 \). Therefore \( \phi(g^3,P,C^3) = \phi(g^2,P,C^2) \) according to UII and NI. Moreover \( c^3_{rs} < c^1_{rs} \) and \( c^3_{st} > c^1_{st} \).

**Step 4:** Repeat steps 2 and 3 until the \( c^3_{rs} = c^1_{rs} \) and \( c^3_{st} = c^1_{st} \).

**Lemma 2** Consider a cost allocation problem \((g,P,C)\). For every pair of connections \( rs \) and \( r's' \) in \( g \), where there is no path between \( r \) and \( r' \) in \( g \), and all cost structures \( C' \) with \( c'_rs + c'_{r's'} = c_{rs} + c_{r's'} \) and \( c'_jk = c_{jk} \) for all other connections, \( \phi(g,P,C) = \phi(g,P,C') \).

**Proof:** Without loss of generality assume that \( c'_rs < c_{rs} \) and \( c'_{r's'} > c_{r's'} \).

**Step 1:** Define \( g^1 \) by \( g^1 = g \) and define \( C^1 \) by

\[
c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ \max\{c_{jk}, v(g,C)\} & \text{for all other connections.} \end{cases}
\]

Then \( g, g^1 \in \mathcal{M}^\mathcal{C}N(P,C^1) \) and \( c^1_{jk} = c_{jk} \) for all \( jk \in g \). Therefore \( \phi(g^1,P,C^1) = \phi(g,P,C) \) according to UII.

**Step 2:** Define \( g^2 \) by removing \( rs \) and \( r's' \) from \( g^1 \) and adding \( rr' \), \( r's \) and \( ss' \) to \( g^1 \). Then \( g^2 \) is a CN. Define \( C^2 \) by

\[
c^2_{jk} = \begin{cases} \max\{c'_rs, c^1_{rs}/2\} & \text{for } jk = rr' \\ c^1_{rs} - \max\{c'_rs, c^1_{rs}/2\} & \text{for } jk = r's \\ c_{r's'} & \text{for } jk = ss' \\ c^1_{jk} & \text{for all other connections.} \end{cases}
\]

Then \( g^1, g^2 \in \mathcal{M}^\mathcal{C}N(P,C^2) \) and \( c^2_{jk} = c^1_{jk} \) for all \( jk \in g^1 \). Therefore \( \phi(g^2,P,C^2) = \phi(g^1,P,C^1) \) according to UII and NI.

**Step 3:** Defined \( g^3 \) by removing \( rr' \), \( r's \) and \( ss' \) from \( g^2 \) and adding \( rs \) and \( r's' \) to \( g^2 \), so \( g^3 = g \). Define \( C^3 \) by

\[
c^3_{jk} = \begin{cases} c^2_{rr'} & \text{for } jk = rs \\ c^2_{r's} + c_{ss'} & \text{for } jk = r's' \\ c^2_{jk} & \text{for all other connections.} \end{cases}
\]

Then \( g^2, g^3 \in \mathcal{M}^\mathcal{C}N(P,C^3) \) and \( c^3_{jk} = c^2_{jk} \) for all \( jk \in g^1 \). Therefore \( \phi(g^3,P,C^3) = \phi(g^2,P,C^2) \) according to UII and NI.

**Step 4:** Repeat steps 2 and 3 until the \( c^3_{rs} = c'_rs \) and \( c^3_{r's'} = c'_{r's'} \).

\[\square\]
Lemma 3 Consider two cost allocation problems \((g, P, C)\) and \((h, P, D)\) with \(v(g, C) = v(h, D)\). Then \(\phi(g, P, C) = \phi(h, P, D)\).

Proof: First the cost allocation problem \((g, P, C)\) is transformed into \((g^2, P, C^2)\) with \(\phi(g^2, P, C^2) = \phi(g, P, C)\) such that there is a location \(u\) with \(u \notin g^2\). Consider the following two steps.

Step 1: Define \(g^1\) by \(g^1 = g\) and define \(C^1\) by

\[
c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ v(g, C) & \text{for all other connections}. \end{cases}
\]

Then \(g, g^1 \in \mathcal{M}(P, C^1)\) and \(c^1_{jk} = c_{jk}\) for all \(jk \in g\). Therefore \(\phi(g^1, P, C^1) = \phi(g, P, C)\) according to UII.

Step 2: Define \(g^2\) by removing \(ru\) and \(su\) from \(g^1\) and adding \(rs\) to \(g^1\) as well as replacing all other connections to \(u\) in \(g^1\) with connections to \(r\). Then \(g^2\) is a CN. Define \(C^2\) by

\[
c^2_{jk} = \begin{cases} c^1_{ru} + c^1_{su} & \text{for } jk = rs \\ c^1_{ju} & \text{for all } j \text{ with } ju \in g \text{ and } k = r \\ c^1_{jk} & \text{for all other connections}. \end{cases}
\]

Then \(g^1, g^2 \in \mathcal{M}(P, C^2)\) and \(c^2_{jk} = c^1_{jk}\) for all \(jk \in g^1\). Therefore \(\phi(g^2, P, C^2) = \phi(g^1, P, C^1)\) according to UII and NI.

Let \(M\) be a subset of \(\mathcal{M}\) such that

- If \(h\) is a CN for \((a_i, b_i)_{i \in M}\), then \(h\) is a CN for \(P\).
- For all \(i \in M\) there exists a CN \(h\) for \((a_i, b_i)_{i \in M \setminus \{i\}}\) such that \(h\) is not a CN for \(P\).

Second \((g^2, P, C^2)\) is transformed into \((\tilde{g}, P, \tilde{C})\) with \(\phi(\tilde{g}, P, \tilde{C}) = \phi(g^2, P, C^2)\) such that \(\tilde{g} = \bigcup_{i \in M} a_i, b_i\) and \(\tilde{C}\) is arbitrary with \(\sum_{i \in M} \tilde{C}_{a_i,b_i} = v(g, P, C)\) and \(\tilde{c}_{jk} = v(g, P, C)\) for all other connections. Consider the following three steps.

Step 3: Pick \(i \in M\) with \(a_i b_i \notin g^2\). Then there exists a connection \(j'k'\) in the path \(p_i(g^2)\) between \(a_i\) and \(b_i\) such that \(j'k' \notin \{(a_r, b_r)_{r \in M}\}\). Apply Lemma 1 to move connection costs in the path \(p_i(g^2)\) such that \(c_{j'k'} \geq \sum_{j \in p_i(g^2) \setminus j'k'} c_{jk}\). Define \(h\) by removing \(j'k'\) from \(g^2\) and adding \(a_i b_i\) to \(g^2\). Define \(g^3\) by removing all connections in \(h \setminus \bigcup_{i \in M} P_i(h)\) from \(h\). Define \(C^3\) by

\[
c^3_{jk} = \begin{cases} c^2_{j'k'} + \sum_{j''k'' \in h \setminus \bigcup_{i \in M} P_i(h)} c^3_{j''k''} & \text{for } jk = a_i b_i \\ c^2_{jk} & \text{for all other connections}. \end{cases}
\]
Then $g^2, g^3 \in \mathcal{MCCN}(P, C^3)$ and $c^3_{jk} = c^2_{jk}$ for all $jk \in g^2$. Therefore $\phi(g^3, P, C^3) = \phi(g^2, P, C^2)$ according to UII and NI.

**Step 4:** Repeat step 3 until $a_i b_i \in g^3$ for all $i \in M$ so $g^3 = \cup_{i \in M} \{a_i, b_i\}$.

**Step 5:** Apply Lemmas 1 and 2 to move cost to an arbitrary $\tilde{C}$ with $\sum_{i \in M} \tilde{c}_{ai, bi} = v(g, P, C)$ and $\tilde{c}_{jk} = v(g, P, C)$ for all other connections. Then $\tilde{g} \in \mathcal{MCCN}(P, \tilde{C})$ for $\tilde{g} = g^3$ and $\phi(\tilde{g}, P, \tilde{C}) = \phi(g^3, P, C^3)$.

All in all $\phi(g, P, C) = \phi(\tilde{g}, P, \tilde{C}) = \phi(h, P, D)$. □

We conclude the proof of Theorem 1 with the following observation: according to Lemma 3 the allocation of costs depends on the connection structure $P$ and the total cost of the MCCN $v(g, C)$ and no other feature of the cost allocation problem $(g, P, C)$. Therefore the allocation rule is simple.

## 4 Implementation

In the present section we consider a network formation game with $m$ agents and a central planner. The agents have private information about their connection demands and know all connection costs. The planner is ignorant. The agents need to ensure that their individual connection demands are satisfied and want to minimize their individual cost shares. The planner wants minimize total cost subject to the constraint that all connection demands has to be satisfied.

### The game

The network formation game has three stages:

1. A central planner announces a cost allocation rule and a cost estimation rule.
2. Every agent reports her connection demand and all connection costs.
3. The central planner selects a MCCN based on reported demands and the estimated connection costs and allocates the true costs of the selected network.

The reports of the agents are used by the planner to estimate the connection and cost structures. Agents can misreport both their individual connection demands $(a_i, b_i)$ and the cost structure $C$. Misreporting influences the estimates of connection demands and costs. Estimates of connection demands and costs influence the selection of MCCNs. However, true costs rather than estimated costs of a selected MCCN are observed. Therefore true costs rather than estimated costs of a selected MCCN are allocated among agents. Hence misreporting has an indirect influence rather than direct influence on cost shares.
Formally, the rules of the game consist of an allocation rule $\phi : \mathcal{Z} \to \mathbb{R}^m$ and a connection cost estimation rule $\tau : \mathbb{R}^m_+ \to \mathbb{R}_+$. Let $\sigma = (\sigma_1, \ldots, \sigma_m)$, where $\sigma_i = (\sigma_{i,j}^k)_{1 \leq j < k \leq n}$ for all $i$ and $\sigma_{i,j}^k > 0$ for all $i$ and $j$, be a collection of individual cost reports. The estimated cost of a connection is supposed to be between the minimum and maximum reported cost

$$\tau(\sigma_{i,j}^k, \ldots, \sigma_{i,m}^k) \in [\min_i\{\sigma_{i,j}^k\}, \max_i\{\sigma_{i,j}^k\}]$$

and upward unbounded

$$\lim_{\sigma_{i,j}^k \to \infty} \tau(\sigma_{i,j}^k, \ldots, \sigma_{i,m}^k) = \infty$$

for every $i$. Upward unboundedness implies that every agent is able to influence the cost estimate of the planner.

Let $\omega$, where $\omega = (\omega_1, \ldots, \omega_m)$ and $\omega_i = (\alpha_i, \beta_i) \in \mathcal{N}^2$ for all $i$, be a collection of individual reports on connection demands. For the collection of individual connection demand reports $\omega$, the estimated connection structure is the reports $P^e(\omega) = \omega$. For a collection of individual cost reports $\sigma$ the estimated cost structure is $C^e(\sigma)$, where $c_{i,j}^{e,k} = \tau(\sigma_{i,j}^k, \ldots, \sigma_{i,m}^k)$ for all $j,k$. The planner allocates observed costs. Therefore for every network $g$ in $\mathcal{MCCN}(P^e(\omega), C^e(\sigma))$ the allocation of observed costs is $\phi(g, P^e(\omega), C)$. The planner randomly selects a network $g$ in $\mathcal{MCCN}(P^e(\omega), C^e(\sigma))$. Hence for fixed collections of individual reports $(\omega, \sigma)$ the expected allocation of costs is

$$\Phi(P^e(\omega), C^e(\sigma), C) = \frac{1}{|\mathcal{MCCN}(P^e(\omega), C^e(\sigma))|} \sum_{g \in \mathcal{MCCN}(P^e(\omega), C^e(\sigma))} \phi(g, P^e(\omega), C).$$

Agents can choose their reports strategically.

**Equilibrium**

The notion of equilibrium is Nash equilibrium.

**Definition 3** A Nash equilibrium is a collection of individual reports $(\bar{\omega}, \bar{\sigma})$ such that for every agent $i$ and all reports $(\omega_i, \sigma_i)$,

$$\Phi_i(P^e(\omega_i, \bar{\omega}_{-i}), C^e(\sigma_i, \bar{\sigma}_{-i}), C) \geq \Phi_i(P^e(\bar{\omega}), C^e(\bar{\sigma}), C).$$

**No truth-telling without UII and NI**

In the two observations below we show that if truth-telling is a Nash equilibrium, then the allocation rule satisfies both UII and NI provided the estimation rule is well behaved.
Misreporting connection costs of unused connections influences the cost estimates of unused connections. Moreover if an allocation rule does not satisfy UII, cost estimates of unused connections influence cost shares. Therefore agents can manipulate their cost shares by misreporting provided UII is not satisfied. Furthermore misreporting need not be revealed.

**Observation 1** Suppose an allocation rule $\phi$ is continuous in cost structures and an estimation rule $\tau$ is continuous and unbounded in cost reports: for every $i$,

$$ \lim_{\sigma_i^{jk} \to 0} \tau(\sigma_1^{jk}, \ldots, \sigma_m^{jk}) = 0 \quad \text{and} \quad \lim_{\sigma_i^{jk} \to \infty} \tau(\sigma_1^{jk}, \ldots, \sigma_m^{jk}) = \infty. $$

If truth-telling is a Nash equilibrium, then $\phi$ satisfies UII.

**Proof:** Suppose a cost allocation rule $\phi$ does not satisfy UII. Then there is a pair cost allocation problems $(g, P, C)$ and $(g, P, D)$ with $c_{jk} = d_{jk}$ for all $jk \in g$ and an agent $i$ such that $\phi_i(g, P, C) \neq \phi_i(g, P, D)$. For all $\epsilon > 0$ define two other cost structures $C^1$ and $D^1$ by

$$ c_{jk}^1 = \begin{cases} c_{jk} & \text{for } jk \in g \\ c_{jk} + \epsilon & \text{for all other connections} \end{cases} $$

and

$$ d_{jk}^1 = \begin{cases} d_{jk} & \text{for } jk \in g \\ d_{jk} + \epsilon & \text{for all other connections}. \end{cases} $$

Then $g$ is the unique MCCN for connection problems $(P, C^1)$ and $(P, D^1)$ for all $\epsilon$. For $\epsilon$ sufficiently small $\phi_i(g, P, C^1) \neq \phi_i(g, P, D^1)$ because $\phi$ is continuous in cost structures. Without loss of generality suppose $\phi_i(g, P, C^1) > \phi_i(g, P, D^1)$. For the connection problem $(P, C^1)$ if all agents except agent $i$ are telling the truth, then the strategy $(\omega_i, \sigma_i)$, where $\omega_i = (a_i, b_i)$ and $\sigma_i$ is such that $\tau(\sigma_i^{jk}, (c_{jk})_{j \neq i}) = d_{jk}^1$ lowers the cost share of agent $i$. There exists $\sigma^i$ such that $\tau(\sigma_i^{jk}, (c_{jk})_{j \neq i}) = d_{jk}^1$ because $\tau$ is continuous and unbounded. Therefore truth-telling is not a Nash equilibrium. \hfill $\square$

Misreporting connection costs influences cost estimates. Moreover cost estimates influence the set of estimated MCCNs. If an allocation rule does not satisfy NI, then cost shares depends on the selected MCCN. Therefore agents can manipulate their cost shares by misreporting provided NI is not satisfied.

**Observation 2** Suppose an allocation rule $\phi$ is continuous in cost structures and an estimation rule $\tau$ is continuous and upward unbounded in cost reports: for every $i$,

$$ \lim_{\sigma_i^{jk} \to \infty} \tau(\sigma_1^{jk}, \ldots, \sigma_m^{jk}) = \infty $$

If truth-telling is a Nash equilibrium, then $\phi$ satisfies NI.
Proof: Suppose a cost allocation rule $\phi$ does not satisfy NI. Then there is a connection problem $(P, C)$ and an agent $i$ such that
\[
\min_{g \in \mathcal{M} \in \mathcal{N}(P, C)} \phi_i(g, P, C) < \max_{h \in \mathcal{M} \in \mathcal{N}(P, C)} \phi_i(h, P, C)
\]
for some $i$. Therefore suppose $\phi_i(g, P, C) \leq \phi_i(h, P, C)$ for $g$ and $i$ and all $h \in \mathcal{M} \in \mathcal{N}(P, C)$ and $\phi_i(g, P, C) < \phi_i(h', P, C)$ for $g$ and $i$ and some $h' \in \mathcal{M} \in \mathcal{N}(P, C)$. For $\epsilon > 0$ define another cost structure $C^1$ by
\[
c_{jk}^1 = \begin{cases} c_{jk} & \text{for } jk \in g \\ c_{jk} + \epsilon & \text{for all other connections} \end{cases}
\]
Then $g$ is the unique MCCN for connection problem $(P, C^1)$ for all $\epsilon$. For $\epsilon$ sufficiently small
\[
\phi_i(g, P, C^1) < \frac{1}{|\mathcal{M} \in \mathcal{N}(P, C)|} \sum_{h \in \mathcal{M} \in \mathcal{N}(P, C)} \phi_i(h, P, C)
\]
because $\phi$ is continuous in cost structures. For the connection problem $(P, C)$ if all agents except agent $i$ are telling the truth, then the strategy $(\omega, \sigma_i)$, where $\omega = (a_i, b_i)$ and $\sigma_i$ is such that $\tau(\sigma_i^{jk}, (c_{jk})_{\tau \neq i}) = c_{jk}^1$, lowers the expected cost share of agent $i$. There exists $\sigma_i$ such that $\tau(\sigma_i^{jk}, (c_{jk})_{\tau \neq i}) = c_{jk}^1$ because $\tau$ is continuous and upward unbounded. Hence truth-telling is not an equilibrium for $(P, C)$. \hfill \Box

Truth-telling

Both UII and NI are necessary for truthful reporting as shown in Observations 1 and 2 and SI is standard. Consequently we focus on super simple allocation rules.

Definition 4 An allocation rule $\phi$ is implementable provided truth-telling is a Nash equilibrium for every connection problem $(P, C) \in \mathcal{P}$.

Let $P_{-i} = (a_{i'}, b_{i'})_{i' \neq i}$ be a collection of connection demands for all agents except agent $i$. Then a super simple allocation rule is implementable if and only if for all $i$ and $P_{-i}$, the relative cost share of agent $i$ is independent of her strategy $(\alpha_i, \beta_i)$.

Theorem 2 Suppose an allocation rule $\phi$ is super simple. Then truth-telling is a Nash equilibrium if and only if $\Lambda_i((a_i, b_i), P_{-i}) = \Lambda_i((\alpha_i, \beta_i), P_{-i})$ for all $i$, $P_{-i}$, $(a_i, b_i)$ and $(\alpha_i, \beta_i)$.

Proof: Suppose $\Lambda_i((a_i, b_i), P_{-i}) < \Lambda_i((\alpha_i, \beta_i), P_{-i})$ for some $P_{-i}$ and $(a_i, b_i)$ and $(\alpha_i, \beta_i)$. Consider a cost structure $C$ such that for some $i'$ not a path between $a_{i'}$ and $b_{i'}$ going through all locations is the unique MCCN. Suppose the true connection structure is $P$. Then agent $i$
can lower her cost share by changing her strategy from \((a_i, b_i)\) to \((\alpha_i, \beta_i)\). Therefore truth-telling is not a Nash equilibrium.

Suppose truth-telling is not a Nash equilibrium for some connection problem \((P, C)\). Then there are \(i\) with \((\alpha_i, \beta_i) \neq (a_i, b_i)\) and \(h\) such that

\[
\Lambda_i((\alpha_i, \beta_i), P_{-i}) v(h, C) < \Lambda_i(((a_i, b_i), P_{-i}) v(g, C).
\]

Hence \(\Lambda_i((\alpha_i, \beta_i), P_{-i}) < \Lambda_i(((a_i, b_i), P_{-i})\), because \(v(h, C) \geq v(g, C)\).

Theorem 2 implies that the price of stability is one provided the allocation rule satisfies the assumption in the theorem. However a simple example shows that the the price of anarchy is unbounded. Consider a situation with at least two agents \(i \in \{1, 2\}\) and three locations \(j \in \{1, 2, 3\}\). Suppose the connection demands are \((a_i, b_i) = (1, 2)\) for both \(i\) and the cost structure is \(c_{12} = 1, c_{13} = 2(1 + \varepsilon)\) and \(c_{23} = \varepsilon\) where \(\varepsilon > 0\). Then the unique MCCN is \(g = 12\) with \(v(g, C) = 1\). Let the allocation rule be the equal split rule. Suppose both agents misreport their connection demands \((\alpha_i, \beta_i) = (1, 3)\) for both \(i\) and report the true cost structure. Based on the reports the network \(h = 12, 23\) is selected with \(v(h, C) = 1 + \varepsilon\). Moreover it is not possible for any of the two agents to bring down her cost share by changing her report. Therefore the price of anarchy is \(1 + \varepsilon\) where \(\varepsilon\) is arbitrary.

Suppose connection demands are known, so every agent reports only connection costs rather than both her connection demand and connection costs. Then for arbitrary rules of the game the set of MCCNs for the estimated cost structure is a subset of the set of MCCNs for the true cost structure. Hence the price of anarchy drops to one.

**Corollary 2** Suppose connection demands are known. If \(\sigma\) is a Nash equilibrium, then \(\mathcal{M}C\mathcal{E}(P, C^{eq}(\sigma)) \subset \mathcal{M}C\mathcal{E}(P, C)\).

**Proof:** The proof follows the proof of Theorem 2 in Hougaard & Tvede (2012).

\[\square\]

## 5 Generalizations

Lemma 2 implies our characterizations of simple and super simple allocation rules extend to cost allocation problems \((g, P, C)\) with \(\mathcal{N} \cup \{a_i, b_i\} = \emptyset\) provided \(g\) is a forest with multiple trees.

Our characterizations of simple and super simple allocation rules can be extended to all cost allocation problems from cost allocation problems with undemanded locations by adding an axiom linking the allocations of costs of problems in \(\mathcal{Z}\) and not in \(\mathcal{Z}\). Suppose the set of locations can be an arbitrary finite subset of \(\mathbb{N}\). Let \(\mathcal{N}(C) \subset \mathbb{N}\) be the set of locations for a given cost structure \(C\).
Definition 5 A cost structure $C$ is **embedded** in a sequence of cost structures $(C^\mu)_{\mu \in \mathbb{N}}$, provided:

- $\mathcal{N}(C) \subset \mathcal{N}(C^\mu)$ for all $\mu$.
- $c_{jk}^\mu = c_{jk}^\nu$ for all $\mu$ and $jk \in \mathcal{N}(C)^2$.
- $\mathcal{N}(C^\mu) = \mathcal{N}(C^\nu)$ for all $\mu$ and $\nu$.

Then the additional axiom is as follows.

**Limit Independence (LI)** For all $(g, P, C)$ and $(g, P, C^\mu)_{\mu \in \mathbb{N}}$, where $C$ is embedded in $(C^\mu)_{\mu \in \mathbb{N}}$ and $\lim_{\mu \to \infty} c_{jk}^\mu = \infty$ for all $jk \notin \mathcal{N}(C)^2$, $\lim_{\mu \to \infty} \phi(g, P, C^\mu) = \phi(g, P, C)$.

Intuitively, LI states that adding locations whose connection costs tend to infinity should not change the allocation of costs in the limit. Axioms UII and LI enable us to compare allocations of costs of a cost allocation problem $(g, P, C) \not\in \mathcal{X}$ and a sequence of cost allocation problems $(g, P, C^\mu)_{\mu \in \mathbb{N}} \in \mathcal{X}$ for all $\mu$. Indeed, since UII implies $\phi(g, P, C^\mu) = \phi(g, P, C^\nu)$ for all $\mu$ and $\nu$, LI implies $\phi(g, P, C^\mu) = \phi(g, P, C)$ for all $\mu$.

Using the idea of routing-proofness from Moulin (2013) every agent $i$ could misrepresent her connection demand by splitting into $m_i$ aliases who each reports a connection demand. The characterization in Theorem 2 of allocation rules that truthfully implements MCCNs remains valid provided the condition in Theorem 2 is modified to $\Lambda_i((a_i, b_i), P_{-i}) \leq \sum_{k=1}^{m_i} \Lambda_k((\alpha_k, \beta_k), P_{-i})$.

The connection demand of every agent $i$ could consist of $m_i$ pairs of locations $(a_k^i, b_k^i)_{k=1}^{m_i}$ rather than a single pair $(a_i, b_i)$. The characterizations in Theorem 1 and Corollary 1 of simple and super simple allocation rules as well as the characterization in Theorem 2 of allocation rules that truthfully implements MCCNs remain valid. Indeed our proofs of Theorems 1 and 2 and Corollary 1 remain valid without any changes.

6 Final remarks

For the CN model with undemande locations: we have characterized allocation rules satisfying UII and NI as well as UII, NI and SI; we have shown that if an allocation rule implement MCCNs truthfully, then it satisfies UII and NI; and, we have characterized allocation rules satisfying UII, NI and SI truthfully implementing MCCNs. To our surprise it turned out an allocation rule satisfies UII, NI and SI if and only if relative cost shares depend on connection demands and nothing else. It was surprising for us because for the MCST model the folk rule and the rest of the family of oblighations rules satisfy UII, NI and SI. Consequently voluntarily participation is at odds with Individual Rationality in the CN model in contrast to the MCST model.
References


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