Imagine a transparent window that does not offset light rays, but that rotates them by an angle $\alpha = 90^\circ$ around the local window normal. A wave-optical consideration [1] reveals that such ray-rotating windows, if they existed, could create wave-optically impossible light-ray fields: light-ray direction is given by the gradient of the eikonal, $\nabla \phi$; for the example of an incident cylindrical wave front with a quadratic phase dependence of the form $\phi(x, y) \propto x^2/2$ in the plane of the window, the field of light-ray directions there is given by $p(x, y) \propto (x, 0, p_z)$ (we do not have enough information to calculate the $z$ component, but this is not necessary). After transmission through the window, the light-ray-direction field is $p'(x, y) \propto (0, x, p_z)$. The $z$ component of the curl of this light-ray field is $\nabla \times p'(x, y) \propto (\partial/\partial x)x - (\partial/\partial y)y \neq 0$, but as $\nabla \times \nabla \phi' = 0$ for all functions $\phi'$ almost everywhere, no eikonal $\phi'(x, y)$ can be constructed for the resulting light-ray field such that $p' = \nabla \phi'$.

An offset in the ray position that accompanies the light-ray-direction change can overcome this problem. For example, it is clearly possible to rotate the light-ray field in the above example as a whole. It is also possible to make the ray-position offset small, for example by rotating the incident wave front in small pieces. This approach of piecewise wave-front transformation is taken by so-called “metamaterials for rays” (METATOYs) [1] (but note that METATOYs are not metamaterials in the standard sense [2]). If each of these pieces is much larger than the wavelength but much smaller than the scale or variations for the incident ray-direction field, they can be as unnoticeable as a good monitor’s pixels. In this way, METATOYs can mimic wave-optically forbidden windows.

A number of METATOYs performing various forbidden light-ray-direction changes have been investigated theoretically [3–10]. So far, METATOYs performing three different light-ray-direction changes have been demonstrated experimentally: namely, flipping of one transverse ray-direction component [11], ray-optical refraction with a variation of Snell’s law in which the sines are replaced by tangents [12], and ray rotation around the window normal (through $90^\circ$, as already described above, and through other, in principle arbitrary, angles) [13].

Ray rotation around the local normal is particularly interesting, as it can be described formally as refraction at an interface between materials with a complex ray-optical refractive-index ratio [14]. Note that we call the refractive-index ratio ray-optical, as its effect is quite different from that of a material with a complex (standard) refractive index, which is associated with absorption [15]. The experimental demonstration of windows

Fig. 1. (Color online) Transformation of a ray cone originating from a point light source at $P$ into a ray hyperboloid by passage through a ray-rotating window, for different ray-rotation angles, $\alpha$. Light-ray trajectories are shown as cylinders. After transmission through the window, the rays lie on the surface of hyperboloids. For $\alpha = 90^\circ$, the hyperboloid’s waist plane coincides with the window plane. For $\alpha = 180^\circ$, the hyperboloid is degenerate: a cone. The images were created using the ray-tracing software TIM (The Interactive METATOY) [16].
representing a complex ray-optical refractive-index interface \(^{19}\) has prompted us to study the consequences of such unusual refraction for ray optics.

Here we describe an extension of geometrical imaging to complex object and image distances. We confirm that our description works for imaging between real object and image distances using parallel ray-rotation windows \(^{6}\). We believe that this work has the potential to lead to new optical design paradigms.

We are considering a planar window that rotates light rays by an angle \(\alpha\) around the window normal. Specifically, we are interested in its effect on rays from a point light source. A point light source emits light rays in all directions; any such light ray lies on a cone with its apex at the light-source position and its axis perpendicular to the ray-rotating window. Figure 1 shows ray-tracing simulations of light rays on one such cone before and after transmission through such windows. In the case \(\alpha = 180^\circ\), the rays behind the window form another cone with its apex at the image of the point light source. For all other nontrivial cases, the rays lie on the surface of a hyperboloid (of one sheet). (Trivial cases include \(\alpha = 0^\circ\) and the case of a point light source in the plane of the window, and in all of these, the rays remain on the original cone.)

Now consider a more general case, in which the ray-rotating window converts one ray hyperboloid into another ray hyperboloid (Fig. 2(a)). Figure 2(b) shows the trajectory of one light ray for this case, orthographically projected into the window plane. The projection of other rays on the same hyperboloids can be obtained by rotating the projection of the shown trajectory around the point \(A\), which is the projection of the common axis of both hyperboloids. The incident light ray intersects the window at the point \(I\), where it gets rotated through an angle \(\alpha\) around the window normal. As the (local) window normal is the axis around which the light ray is rotated, the angle \(\theta\) between the light ray and the window normal does not change in this process.

We define the point \(P\) as the center of the hyperboloid in front of the window. As it lies on the hyperboloid’s axis, its projection is the same as that of the axis. Similarly, we define the point \(P’\) as the center of the hyperboloid behind the window. We call the distance of \(P\) in front of the window \(z\) (if \(P\) is in front of the window, \(z > 0\), otherwise \(z < 0\) and that of \(P’\) behind the window \(z’\) (\(z’ > 0\) when \(P’\) is behind the window). The distance of the rays from \(P\) and \(P’\) (and also of the projections of the rays from the projections of \(P\) and \(P’\)) is, respectively, \(d\) and \(d’\).

We calculate the dependence of \(z’\) and \(d’\) on \(z\), \(d\), \(\theta\), and \(\alpha\) as follows. We place a complex coordinate system into the projection plane with its origin at the projection of \(I\) and its real axis pointing in the opposite direction of the projection of the incident ray. Following \(^{9}\), we can then represent the projection of \(P\) and \(P’\) by the complex numbers

\[
Z_P = z_r \tan \theta - id, \quad Z_{P’} = -\exp(i\alpha)(z’_r \tan \theta + id’).
\]

(1)

As the projections of \(P\) and of \(P’\) are the same, \(Z_P = Z_{P’}\).

After division by \(\tan \theta\), this becomes

\[
z_r - i \frac{d}{\tan \theta} = -\exp(i\alpha)\left(z’_r + i \frac{d’}{\tan \theta}\right).
\]

(2)

Now we define the complex object and image distances

\[
z = z_r + i \frac{d}{\tan \theta}, \quad z’ = z’_r + i \frac{d’}{\tan \theta}.
\]

(3)

A positive imaginary part of a complex object or image distance implies rays that twist counterclockwise; a negative imaginary part implies clockwise twisting. Eq. (2) can then be written as

\[
z’ = -\exp(-i\alpha)z^*.
\]

(4)

This equation, which describes imaging between “point” light sources at complex positions, is the first key result of this paper.

Note that Eq. (4) does not contain the angle \(\theta\) between the light rays and the window normal. This means that it holds not only for light rays on one particular ray hyperboloid, but for all light rays that correspond to the point light sources at the complex object and image distances the window images into each other.

Which are those light rays that correspond to a particular point light source at a complex distance with imaginary part \(z_i\)? It can be seen from Eq. (3) that \(z_i = d/\tan \theta\). In the form \(d = z_i \tan \theta\), this allows calculation of the distance \(d\) from the hyperboloid center at which light rays that travel at an angle \(\theta\) with respect to the \(z\) direction travel. Special cases include light rays that travel in the \(z\) direction (\(\theta = 0^\circ\)), which pass through the hyperboloid center (\(d = 0\)), and light rays that travel at an angle 45° with respect to the \(z\) direction, which have a distance \(d = z_i\) from the hyperboloid center.

We now test our complex imaging equation, Eq. (4), in a number of special cases, each involving a point light
source at a real object distance \( z \) in front of a ray-rotating window. In the trivial case of a window that does not actually change the light-ray direction \( (\alpha = 0^\circ) \), the image distance correctly comes out as \( z' = -z \). In the case \( \alpha = 90^\circ \), \( z' = iz \), which means that the ray hyperboloid’s waist plane is in the window plane, which is confirmed visually by Fig. 1(b). Finally, in the case \( \alpha = 180^\circ \) (Fig. 1(c)), which describes the light-ray-direction change in negative refraction \([4, 9]\), the image should be at the same distance as the object but on the other side of the window, and indeed \( z' = z \).

We also test whether our formulation correctly describes imaging with two parallel ray-rotating windows (ray-rotation angles \( \alpha \) and \( \beta \), separation \( s \)). Only point light sources at a real distance

\[
z = \frac{-s \sin \beta}{\sin(\alpha + \beta)}
\]

in front of the first window are imaged to a real distance by such a combination of ray-rotating windows \([6]\). According to our formalism, the first window images the point light source to a complex distance

\[
z_1' = \exp(-ia) \frac{s \sin \beta}{\sin(\alpha + \beta)}.
\]

This intermediate image then acts as a point light source at a distance

\[
z_2 = s - z_1' = \exp(-i\beta) \frac{s \sin \alpha}{\sin(\alpha + \beta)}
\]

in front of the second window. Note that the complex conjugate of \( z_1' \) is used such that the imaginary part of the object distance \( z_2 \) is the same as that of the image distance \( z_1' \), which means that the handedness of the ray hyperboloid is unchanged. The intermediate object then gets imaged to a real distance

\[
z' = \frac{-s \sin \alpha}{\sin(\alpha + \beta)}
\]

behind the second window, which agrees with the result in \([6]\).

Finally, we use the fact that a ray-rotating window can be described as the interface between complex refractive indices \( n_1 \) and \( n_2 \), where \([14]\)

\[
\exp(i\alpha) = \frac{n_1}{n_2}.
\]

We can therefore write Eq. (4) in the form

\[
z' = -\frac{n_2}{n_1} z^*.
\]

Eq. (10) is the second key result of this paper. What makes this form significant is that this equation is also true if the refractive-index ratio is not in the form of Eq. (9), but only paraxially. It is therefore a generalization of the standard equation describing paraxial imaging by a planar refractive-index interface, which differs from Eqn. (10) only in the absence of the complex conjugation of \( z \). Remarkably, this generalization holds exactly in the nontrivial case of ray-rotating windows.

In summary, we have extended the laws of imaging at planar refractive-index interfaces to complex values of the refractive-index ratio and of object and image distance. Experimental tests should be possible using ray-rotating windows, which were recently demonstrated experimentally \([13]\).

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References

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