Gaussian Markov Random Field models for capturing localized residual spatio-temporal confounding in air pollution and health studies

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Acknowledgements

Work in this talk part of a larger collaboration:

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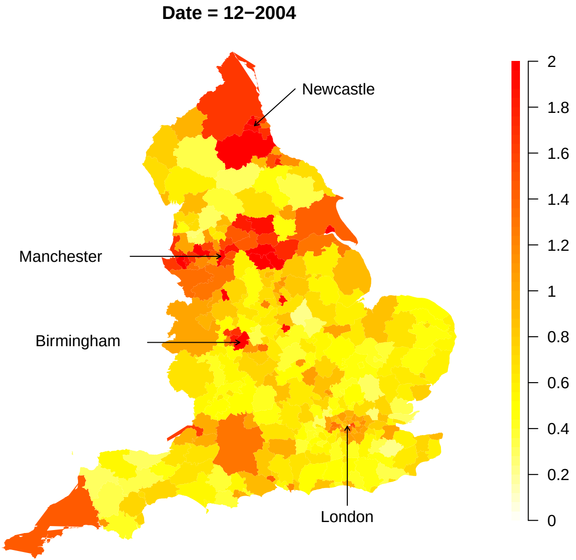
Context: England LHAs - Respiratory admission SIR



- 123 months
- 323 Local Health Authorities (LHAs)
- January 2001 to December 2011
- 42636 data points in total

Modelling disease counts in space

Typical spatial model might look like

$$
\begin{aligned}
Y_i | E_i, R_i &\sim \mathrm{Poisson}(E_i R_i) \\
\ln(R_i) &= \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j + \phi_i, \qquad i = 1, \ldots, N
\end{aligned}
$$

Where

Y_i	disease counts	R_i	disease risk
E_i	expected cases	$\sum_{j=1}^{p} x_{ij}\beta_j$	covariates

$$
\pi(\phi) \propto \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i\sim j} w_{ij}(\phi_i - \phi_j)^2\right)\right]
$$

Localised smoothing

Different strategies achieving this:

Treat ϕ's prior variance, σ, as spatially varying

- Brewer and Nolan (2007); Reich and Hodges (2008)

Treat the weights w_{ij} as random variables

- Ma, Carlin and Banerjee (2010); Lee and Mitchell (2013)

Use a clustering or grouping prior for the random effects

- Richardson and Green (2002); Knorr-Held and Raßer (2000)

CAR2 model

Ma, Carlin and Banerjee (2010)

Starting with Poisson model and Intrinsic CAR for random effects:

$$
\begin{aligned}
Y_i|E_i, R_i &\sim Poisson(E_i R_i) \\
\log(R_i) &= \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j + \phi_i, \qquad i = 1, \ldots, N \\
p(\phi|\sigma, \{w_{ij}\}) &= C(\sigma, \{w_{ij}\}) \exp\left(-\frac{1}{2\sigma^2}\sum_{i\sim j} w_{ij}(\phi_i - \phi_j)^2\right)
\end{aligned}
$$

Normalising term $C(\sigma, \{w_{ij}\})$ included to emphasise that w_{ij} is now being treated as unknown.

CAR2 model

Ma, Carlin and Banerjee (2010)

Idea: Call the non-zero w_{ij}, w_{ij}^{+}, and treat as Bernoulli random variables with unknown probabilities, p_{ij}.

Transform and smooth p_{ij} using a further CAR prior.

$$
\begin{aligned}
w_{ij}^{+}|p_{ij} &\sim \mathit{Bernoulli}(p_{ij}) \text{ and } \mathrm{logit}(p_{ij}) = \mathbf{z}_{ij}^{'}\gamma + \theta_{ij} \\
p(\boldsymbol{\theta}|\zeta) &\propto \exp\left(-\frac{1}{2\zeta^2}\sum_{ij\sim kl}(\theta_{ij}-\theta_{kl})^2\right)
\end{aligned}
$$

Pros: Maintains binary nature of w_{ij}

Cons: Assumes relatively strong smoothness over the w_{ij}; a lot of parameters to estimate $\phi, w_{ij}, \boldsymbol{\theta}$ that can be hard to identify.

A localised model in space and time

Poisson model for counts in space and time (indexed by i and j, respectively)

$$
\begin{aligned}
Y_{ij}|E_{ij}, R_{ij} &\sim \text{Poisson}(E_{ij}R_{ij}) \\
\log(R_{ij}) &= \beta_0 + \phi_{ij}, \quad i = 1, \ldots, N \text{ and } j = 1, \ldots, T
\end{aligned}
$$

Letting $\tilde{\phi}_t = (\phi_{1t}, \ldots, \phi_{Nt})$, where $t = 1, \ldots, T$.

$$
f(\tilde{\phi}_1, \ldots, \tilde{\phi}_T) = f(\tilde{\phi}_1) \prod_{t=2}^{T} f(\tilde{\phi}_t|\tilde{\phi}_{t-1}),
$$

where

$$
\begin{aligned}
\tilde{\phi}_1 &\sim \text{N}\left(\mathbf{0}, \sigma^2 Q(W, \epsilon)\right)^{-1} \\
\tilde{\phi}_t|\tilde{\phi}_{t-1} &\sim \text{N}\left(\alpha\tilde{\phi}_{t-1}, \sigma^2 Q(W, \epsilon)^{-1}\right).
\end{aligned}
$$

A localised model in space and time

Idea: Treat the non-zero elements of w_{ij}, w_{ij}^{+}, as random variables on $[0, 1]$.

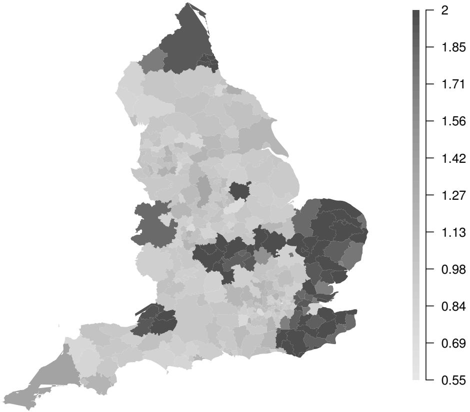
Then transform using $logit(w_{ij}^{+}) = z_{ij}$ and smooth these using a Leroux prior.

$$p(z_{ij}) \propto \exp\left[-\frac{1}{2\tau^2}\left(\rho\sum_{ij\sim kl}(z_{ij}-z_{kl})^2 + (1-\rho)\sum_{ij}(z_{ij})^2\right)\right]$$

Pros: Treating w_{ij} as non-binary makes things slightly easier; Leroux prior is a bit more flexible than the Intrinsic CAR model.

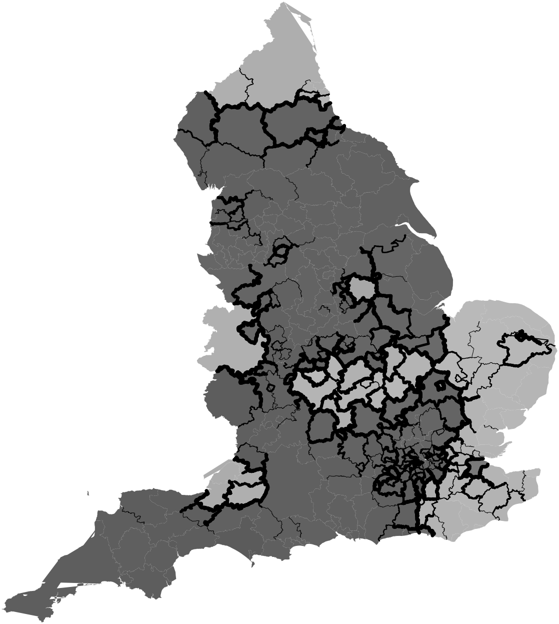
Cons: Still a lot of parameters to estimate

Illustrative simulations

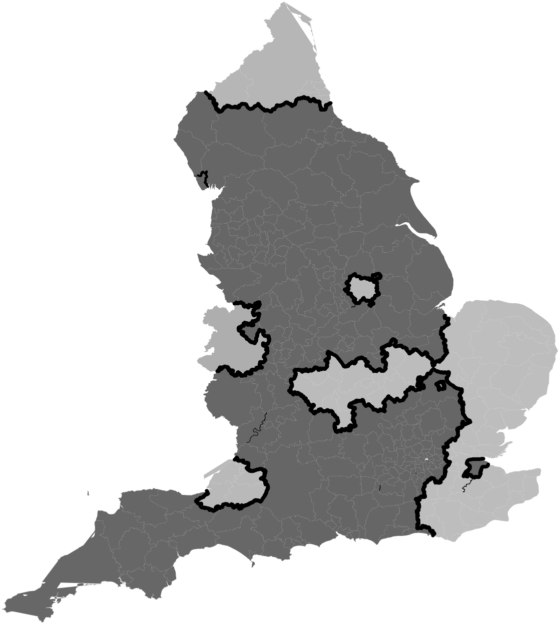


- English local authorities used as a spatial template
- What happens to boundary identification with increased temporal replication?
- What effect does ‘smoothing’ boundaries have? Can ρ be fixed?

Correlated boundaries: T = 10; $\rho = 0.993$



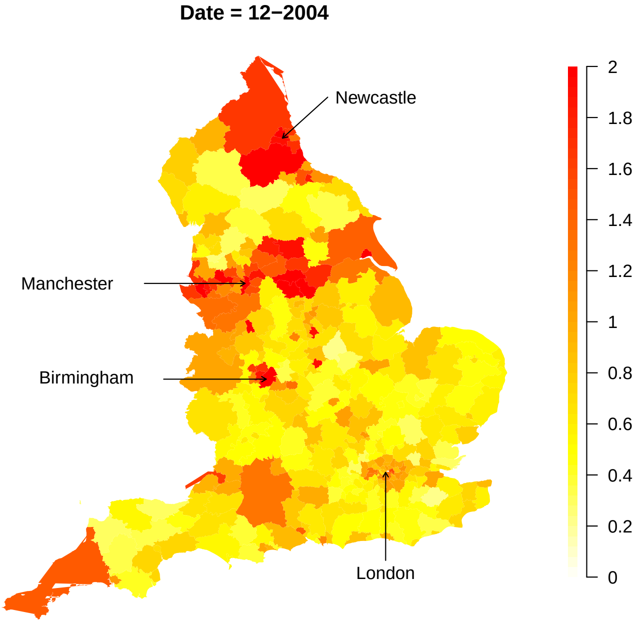
Uncorrelated boundaries: T $= 10$; $\rho = 0$



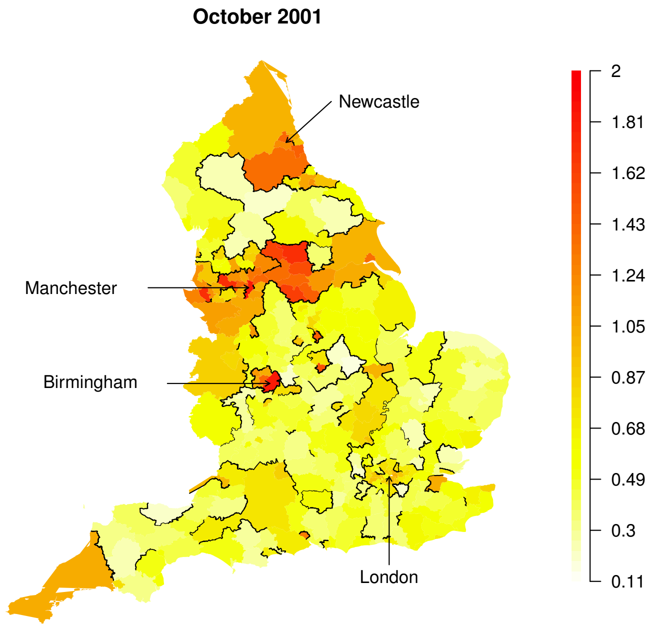
Validation of adaptive model by simulation

		RMSE		
		non adaptive	$\rho = 0$	ρ varies
Temporal replication	**Time = 1**	0.1143	0.0971	0.0728
	Time = 5	0.0865	0.0499	0.0525
	Time = 20	0.0646	0.0390	0.0405
Relative risk	**a = 1**	0.0363	0.0386	0.0397
	a = 2	0.0868	0.0497	0.0524
	a = 3	0.1085	0.0587	0.0611
Expected cases	**E = 25**	0.1253	0.0686	0.0726
	E = 75	0.0866	0.0495	0.0521
	E = 200	0.0613	0.0375	0.0391

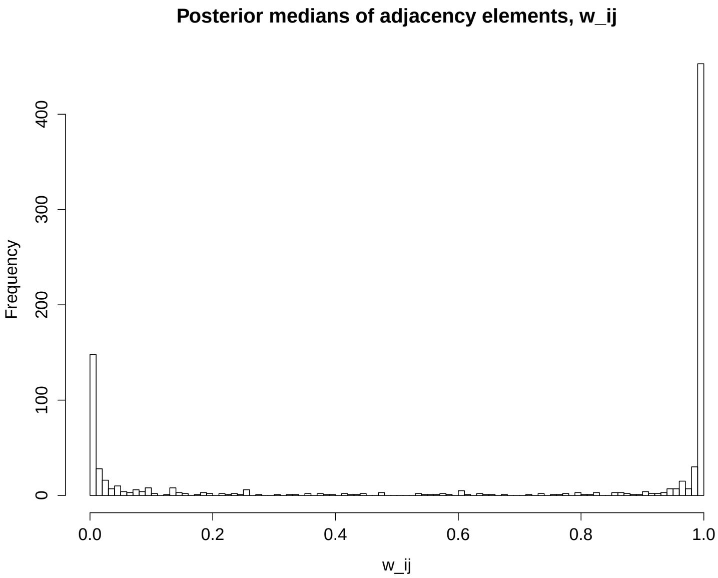
Application to England LHAs - Data



Application to England LHAs - Results



Application to England LHAs - Results



Comments

Adaptive model is succesful at identifying boundaries

- Simulation shows that performance is generally much better than for a standard model
- The prior distribution for boundaries is attractive as it encourages $w_{ij} = 1$ or $w_{ij} = 0$.
- Computation is important, and this model exploits matrix sparsity for speed.

In general, difficult to incorporate information about likely discontinuity structure into prior - compromise sometimes is required due to computational constraints:

- This is why CAR priors are nice: (matrix) sparsity; well established tricks for model-fitting; easy to write very efficient code.
- Example of alternative: Knorr-Held and Raßer (2000) use clustering prior that can induce smoothness; but requires RJMCMC.

Thanks for listening!

Some references

Besag, J., J. York, and A. Mollie (1991). Bayesian image restoration with two applications in spatial statistics. *Annals of the Institute of Statistics and Mathematics*

Lee, D. and R. Mitchell (2013). Locally adaptive spatial smoothing using conditional auto-regressive models. *Journal of the Royal Statistical Society: Series C*

Leroux, B., X. Lei, and N. Breslow. (1999). Estimation of disease rates in small areas: A new mixed model for spatial dependence, *Chapter Statistical Models in Epidemiology, the Environment and Clinical Trials, Springer-Verlag, New York*.

Ma, H. and Carlin, B. (2010) Hierarchical and Joint Site-Edge Methods for Medicare Hospice Service Region Boundary Analysis. *Biometrics*