Fourier-transforming a trapped Bose-Einstein Condensate by waiting a quarter of the trap period: simulation and applications

published in New J. Phys. ***8***, *196 (2006)*

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Abstract. We investigate the property of isotropic harmonic traps to Fourier transform a weakly interacting Bose-Einstein Condensate (BEC) every quarter of a trap period. We solve the Gross-Pitaevskii equation numerically to investigate the time evolution of interacting BECs in the context of the Fourier transform, and we suggest potential applications.

1. Introduction

The Fourier transform is important in optics, particularly in linear optics. For example, it is useful in understanding the behaviour of linear optical systems: every light beam can be described as a superposition of plane waves – the beam's Fourier components – and once a system is understood for the Fourier components the behaviour of general beams in the system can be constructed. Often it is important to physically Fourier-transform a light field experimentally; this is used in areas such as optical information processing and holography [1].

Within optical holography, Fourier holography gives the greatest freedom of shaping; for example, the intensity distribution of a light field's Fourier transform can be completely different from that of the original field. For this reason Fourier holography, which shapes a light beam in a plane where it is the Fourier transform of the original beam, can alter the beam most profoundly. The same argument also holds for Fourier holography of non-interacting, i.e. linear, Bose-Einstein Condensates (BECs), and an analogous argument – based on the evolution time it takes for an interacting BEC to lose all similarity to the original state – can be made for interacting BECs.

BECs are not usually linear, but they can be: Feshbach resonances [2, 3] can be used to tune the non-linearity of a BEC to almost any arbitrary value, and this technique is becoming increasingly commonplace in BEC experiments. This means that Fourier-transform-based experimental techniques for manipulating BECs could potentially be very important.

BECs are already routinely Fourier transformed by switching off the trap and allowing the BEC to expand for a relatively long time (e.g. [4]). While this leads to the BEC to sort itself into its Fourier components, the BEC ends up outside the trapping volume, so this technique is usually destructive.

A BEC at zero temperature is accurately described by the Gross-Pitaevskii equation. Starting with the Gross-Pitaevskii equation for a BEC containing N atoms,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(r) + g|\psi(r)|^2\right]\psi \tag{1}$$

(where the wave function is normalized as $\int \rho(r)\mathrm{d}\mathbf{r} = N\int|\psi(r)|^2\mathrm{d}\mathbf{r} = N$, N is the number of atoms, $V_{\mathrm{ext}} = \frac{1}{2}m\omega^2 r^2$ is the harmonic external trap potential and $g = N4\pi\hbar^2 a/m$ describes the two-body collisions between the atoms with the s-wave scattering length a), we show in this paper how, by exploring the dynamics and waiting for a quarter of a trap period, a BEC in an isotropic harmonic trap can be described as a Fourier transform.

The above-mentioned Fourier-transform mechanism is strictly speaking valid only for non-interacting BECs. As this does not apply to most BEC experiments, we investigate the evolution of interacting ($g \neq 0$) BECs in terms of the Fourier transform, specifically how far into the non-linear regime our Fourier-transform mechanism can be applied.

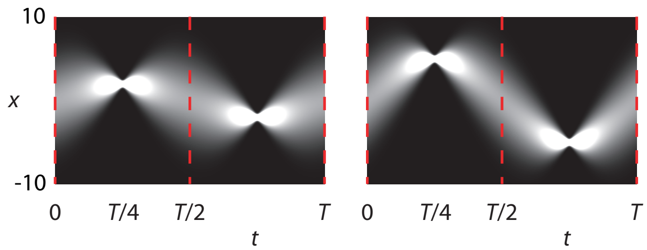

Figure 1. Dynamics of 1-dimensional breathing dipole modes – approximations to a BEC's Fourier components. The graphs show a grayscale representation of the density distribution along the x axis as a function of time, t, calculated using equation (3). At $t = 0$, the wave function is a Gaussian of width 4 (in units of the dipole-mode width) with a transverse wave number $k_x = 3$ (left) and $k_x = 5$ (right). The time evolution of the probability density is plotted over one trap period, T. At $t = T/4$, the width or the BEC has shrunk to one quarter of the dipole-mode width (generally it is inversely proportional to the width at $t = 0$) and its position is proportional to the transverse wave number at $t = 0$, k_x. Every further quarter of a trap period corresponds to an additional Fourier transform.

2. Fourier-transforming by waiting half a trap period

Most BEC traps are harmonic, at least locally. The corresponding density and shape of the ground state depends on the interactions. For a weakly interacting case it takes the form of a Gaussian, whereas if the chemical potential $g\rho$ is larger than the ground-state energy of the harmonic trap, $\hbar\omega$, the density takes a parabolic shape, often referred to as the Thomas-Fermi solution. If the ground state is perturbed, interesting non-stationary modes can form [5, 6, 7]. We study here modes of a harmonic trap which behave like Fourier components.

If a trapped BEC is initially in the shape of the ground state but displaced sideways from the trap centre it will keep its shape and size but its centre of mass will oscillate about the trap centre with the trap period, T (the inverse of the trap frequency). Such a mode is called a dipole mode. If the width of the probability density is initially narrower than that of the ground state, the BEC expands and then contracts again – a so-called breathing mode. This oscillation happens over half a trap period for a weakly interacting gas, but the oscillation time changes with increasing interaction strength.

We consider here the dynamics of states that start off much wider than the ground state and which are positioned off-centre in the trap. To illustrate the main mechanisms we first show how a non-interacting BEC can be Fourier transformed in a harmonic trap. Figure 1 shows the simulated dynamics of such states in a 1-dimensional (1D) BEC. The dynamics of the states contain the characteristics of both the dipole mode and breathing mode, so we call them *breathing dipole modes.*

Breathing dipole modes can be seen as approximations to Fourier components. An analytical solution of the 1D Gross-Pitaevskii equation with starting conditions (at

$t = 0$)

$$\psi_{\sigma,k_x}(x,0) = \exp\left(-\frac{x^2}{2\sigma^2} + \mathrm{i}k_x x\right), \tag{2}$$

which is a centred Gaussian of width σ with a linear phase gradient proportional to the transverse wave number k_x, gives for time t

$$\psi_{\sigma,k_x}(x,t) = \left(\frac{\mathrm{i}\sin t}{\sigma^2} + \cos t\right)^{-1/2} \times \\ \exp\left(-\frac{x^2 \cot t + \mathrm{i}\sigma^2(x^2 + k_x^2 - 2k_x x \csc t)}{2\mathrm{i} + 2\sigma^2 \cot t}\right), \tag{3}$$

where length and time are respectively in units of $\sqrt{\hbar/m\omega}$ and $1/\omega$ of the harmonic oscillator. In these units, the trap period is $T = 2\pi$. For time $t = T/4$ this simplifies to

$$\psi_{\sigma,k_x}(x,T/4) = A \exp\left(-\frac{(x-k_x)^2}{2\tau^2}\right) \tag{4}$$

(where $A = \frac{\sigma(1-\mathrm{i})}{\sqrt{2}}$ and $\tau = 1/\sigma$), which is a flat-phased Gaussian whose x-position is proportional to k_x, the phase gradient at time $t = 0$. If the width of the Gaussian at $t = 0$, σ, is large, i.e. $\sigma \gg 1$, the width of the Gaussian at $t = T/4$, τ, becomes small, i.e. $\tau \ll 1$. This time evolution is approximately a Fourier transform, in which a uniform, inclined (with transverse wave number k_x), plane wave transforms into a δ function at a displacement proportional to k_x. In fact, in the limit $\sigma \to \infty$ the breathing dipole mode behaves exactly like a Fourier component. This is the key observation of this paper.

Another quarter of a trap period later, at $t = T/2$, ψ is back to a Gaussian peak of the original width σ and with a uniform phase gradient across it that is equal and opposite to that of the wave function at $t = 0$:

$$\psi_{\sigma,k_x}(x,T/2) = \mathrm{i} \exp\left(-\frac{x^2}{2\sigma^2} + \mathrm{i}(-k_x)x\right). \tag{5}$$

The time evolution between $t = T/4$ and $t = T/2$ corresponds to another Fourier transform. As a Fourier transform is equivalent to an inverse Fourier transform combined with a flipping of the original distribution, two successive Fourier transforms, and therefore time evolution through $T/2$, correspond to a flipping of the original distribution.

So the time evolution through $T/4$ of breathing dipole modes corresponds to the Fourier transform of individual Fourier components. For the time evolution through $T/4$ of arbitrary superpositions of breathing dipole modes to correspond to a Fourier transform, the individual Fourier components have to keep their relative phase at the trap centre ($x = 0$). According to equation (4), this is the case (with a constant phase offset of $-\pi/4$).

To demonstrate the Fourier-transform property of a BEC's time evolution, we have numerically simulated [10] the time evolution of 2-dimensional (2D) and 3-dimensional (3D) non-interacting ($g = 0$) BECs in isotropic harmonic traps by integration of the Gross-Pitaveskii equation. (The dynamics for anisotropic traps are considerably more

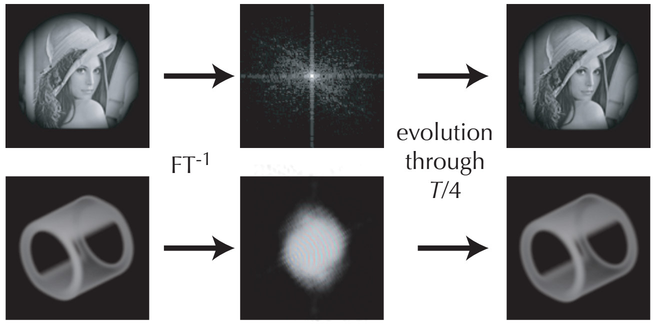

Figure 2. Simulated Fourier transform of non-interacting ($g = 0$) trapped 2-dimensional (top) and 3-dimensional (bottom) BECs by waiting a quarter of the trap period, that is $T/4$. In both cases we set the initial wave function (centre; log plot over four orders of magnitude) to the (discrete) inverse Fourier transform of distinctive shapes (left) – a well-known photo in the 2D case and a shape that looks circular from one direction and square from another in the 3D case – and simulated their time evolution over $T/4$, using a split-step technique. In the 2D case, the simulations were run on a grid of 96×96, representing a square of side length 34.7 dimensionless units. Density is represented as brightness (black represents zero density, white maximum density). In the 3D case, simulations were run on a $64 \times 64 \times 64$ grid, representing a cube of side length 35 dimensionless units. Density is represented as white opacity in front of a black background; the graphs were created using the *Volpack* rendering software [8, 9].

complex and outwith the scope of this paper.) We started with an initial wave function that is the (discrete) inverse Fourier transform of distinctive shapes: in the 2D case a photo frequently used in image processing, in the 3D case a shape that looks like a circle from one direction and like a square from another. Time evolution through $T/4$ should then return the BEC into its original shape. Figure 2 clearly shows that this is the case in our simulation.

Generally, through analogy with the propagation of light in quadratic graded-index (GRIN) wave guides [11, 12, 13], the time evolution of a non-interacting BEC in a harmonic trap can be described as a fractional Fourier transform.

3. Fourier filtering

As an example for an application of the Fourier transform we consider here a Fourier-filtering technique borrowed from optics which allows arbitrary shaping of both a wave function's amplitude and phase when only the amplitude can be shaped directly. (There are also optical techniques for shaping phase and amplitude when only the phase can be shaped directly [14], which, by analogy, would also work for BECs and make direct use of the various techniques for "phase imprinting" [15, 16].)

If a wave function with a uniform phase and arbitrary density distribution can be created, then part of it can be turned into an arbitrary state ψ_a as follows. The state

$$\psi_1 = \tilde{\psi}_\mathrm{a} \exp(i k_x x) + \tilde{\psi}_\mathrm{a}^* \exp(-i k_x x) + r \tag{6}$$

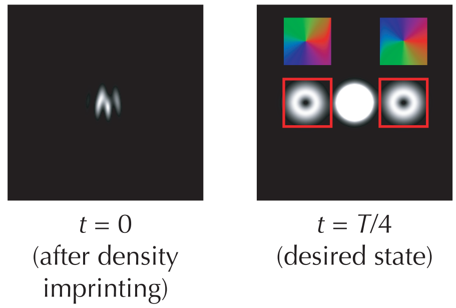

Figure 3. Shaping of a BEC in the Fourier domain. In this example the phase and density of part of a pancake BEC was shaped into a vortex at $t = T/4$ after only the density of the BEC was manipulated a quarter of a trap period earlier ($t = 0$). The state at $t = 0$ was calculated according to equation (6). At $t = T/4$, about 85% of the remaining BEC is in the central peak, the rest is distributed equally between the doughnut-shaped vortices with circulation +1 and −1 on either side (about 7.5% each). A colour representation of the phase in the areas around both vortices, marked by red squares, is shown above the squares. The simulations for this figure were performed on a 256×256 grid representing an area of 28×28 dimensionless units.

is a superposition of the state $\tilde{\psi}_{\mathrm{a}}$, the inverse Fourier transform of ψ_{a}, travelling in the positive x direction (with additional wave number k_x), its complex conjugate travelling in the negative x direction, and a real number r. As the sum of a complex number and its complex conjugate is a real number, this wave function is real everywhere, and if r is chosen to be sufficiently large it is also positive everywhere. Strictly speaking, Eq. (6) is difficult to justify in general because it is only true in the non-intercating case. If, however, the expansion is sufficiently fast so that the cloud separates before collisions have time to destroy any coherence, then Eq. (6) is a good starting point. The state ψ_1 can therefore be created from a BEC ground state, which has uniform phase, by manipulating the density of the BEC to be $|\psi_1|^2$, provided this does not influence the phase distribution. This can be achieved by adding to the harmonic trap a light-induced dipole trap, whose shape can be engineered using holograms. A quarter of a trap period later, after the light-induced dipole trap has been removed but with the harmonic trap still present, the three terms in equation (6) have – as a whole and individually – Fourier transformed and separated spatially (provided k_x was chosen to be sufficiently large). The parts of the BEC that correspond to the last two terms can be filtered out using the same technique for removing parts of a BEC by using a shaped resonant laser beam to selectively transfer atoms into an untrapped state. The remaining part is the desired state ψ_{a}, positioned off-centre in the trap. Figure 3 shows an example in a 2D BEC in which vortices are created from a trap ground state by manipulating the density of the ground state and waiting for a quarter of a trap period.

This is an example of the more general idea of shaping only part of a BEC into a fully specified state and accepting that there is a remainder which might have to be removed from the trap in order not to interfere with the desired part at a later stage.

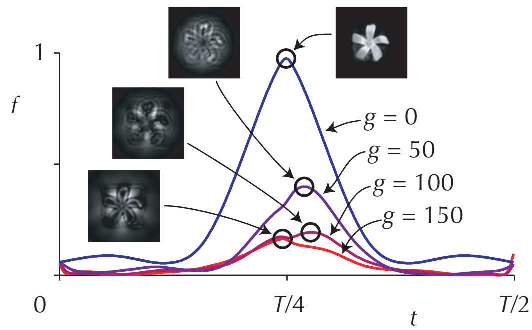

Figure 4. Time evolution of a BEC in a harmonic trap in terms of the Fourier transform of the initial wave function. The initial wave function ($t = 0$) was set to the inverse Fourier transform of wave function P with a distinctive density distribution. The graph shows the fraction, f, of the wave function $\psi(t)$ in the state P, calculated according to equation (7). The figure was calculated for various interaction strengths ($g = 0, 50, 100$ and 150). In the interaction-free case ($g = 0$), close to 100% of the BEC is in the Fourier-transformed state, i.e. P, at $T/4$. Insets show the density of the wave function corresponding to the peaks of the different curves, where for $g = 0, 50, 100$ and 150 respectively 97.5%, 39.8%, 19.3% and 16.3% of atoms are in the state P. The central squares of side length 20 dimensionless units are shown. Like those for figures 5 and 6, the simulations for this figure were performed on a 128×128 grid representing a square of side length 40.1 dimensionless units.

4. The role of interactions

For the case of strongly interacting BECs the situation inevitably becomes more complicated. Firstly, the dynamics of each individual Fourier component (breathing dipole mode) are altered as the period of the breathing mode is dependent on g, which means that at $t = T/4$ it no longer reaches its minimum size. Secondly, the Fourier components interact with each other. Here we investigate numerically the time evolution of 2D interacting BECs in the context of the Fourier transform. Throughout the paper and in all the figures we use the dimensionless parameter $g = 8\pi aN/d$ to describe the role of the interactions. Here the scattering length a is in units of the harmonic trap size $\sqrt{\frac{\hbar}{m\omega}}$ and d is the effective thickness of the cloud in the 2D simulations. As an illustration, a $g \approx 100$ would correspond to a strongly interaction gas well within the Thomas-Fermi regime.

Again we select distinctive states $P(x, y)$, set the initial condensate wave function to the inverse Fourier transform of $P(x, y)$ and simulate the time evolution in a harmonic trap. At each time step we calculate the fraction of the current state $\psi(t)$ in the state $P(x, y)$ by projecting $\psi(t)$ onto P, i.e.

$$f(t) = |\langle P|\psi(t)\rangle|^2 = \left|\int \mathrm{d}\mathbf{r} P(x, y)^* \psi(x, y, t)\right|^2 . \tag{7}$$

This assumes that both P and ψ are normalized. Figure 4 shows $f(t)$ over half a trap cycle for a distinctive pattern P for different values of g. In the case $g = 0$ the fraction of the condensate in the desired mode reaches almost 1 at $t = T/4$, confirming that in this case time evolution over a quarter of a trap period corresponds to a Fourier

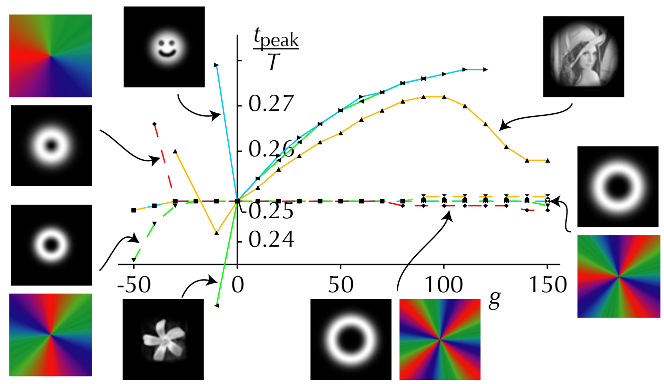

Figure 5. Time after which most of the BEC is in the Fourier-transformed state for various wave functions (insets) as a function of the interaction strength g. Only the data points for which more than 20% of the BEC are in the Fourier-transformed state are shown. Insets show a grayscale representation of the central 20×20 dimensionless units of the density of the state P for each curve, and a colour representation (the colour range red $\rightarrow$ green $\rightarrow$ blue represents phases between 0 to 2π) of the phase, unless the phase is uniform.

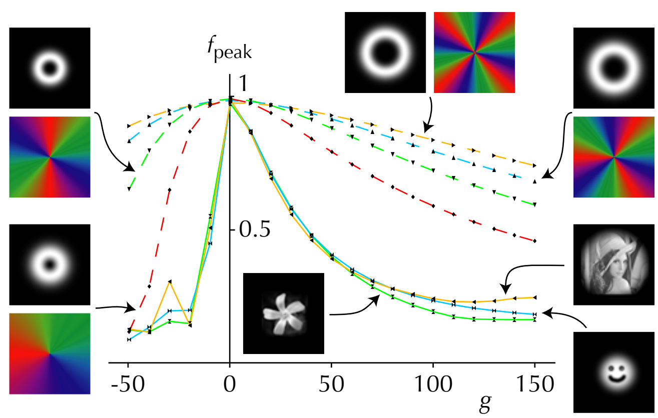

Figure 6. Maximum fraction in the state P of a BEC's wave function during simulated time evolution through $T/2$, starting with the Fourier transform of P, as a function of g, for various choices of P. The insets show a grayscale representation of the central 20×20 dimensionless units of the density of the state P for each curve, and a colour representation of the phase (unless the phase is uniform).

transform. For $g \neq 0$ there is still a sharp peak around $t = T/4$ (shifting to the right, in the direction of half the breathing mode period), but its height decreases with increasing value of g.

Figure 5 plots the time of the peak, t_{peak}, as a function of g for a variety of states P. The states P include various doughnut-shaped vortex states and states with density cross sections in the shape of a picture and flat phase. We arbitrarily chose the density distribution of our vortex states to be that of 2D Laguerre-Gaussian (LG) functions, given by

$$\Psi(r, \phi) \propto (r/r_0)^m e^{-(r/r_0)^2/2} e^{im\phi}, \tag{8}$$

where (r, ϕ) are circular coordinates around the trap centre, r_0 determines the size of the density doughnut, and m is the topological charge of the vortex. Figure 6 shows the peak fraction in state P for various patterns as a function of g.

These results show that waiting for $T/4$ does not always Fourier-transform the wave function of an interacting BEC perfectly, and that the detailed behaviour depends on the wave function. General trends that can be seen in our limited-size sample are that $f(t)$ has a maximum near $T/4$, that $t_{\text{peak}}(g)$ is longer than $T/4$ for $g > 0$, and that f_{peak} generally falls off as $|g|$ increases. However, our sample clearly does not represent all possible behaviours. Had we, for example, chosen the initial state to be stationary, then the graph of $f(t)$ would be a flat line and therefore not have a maximum near $T/4$. The detailed behaviour also depends on the size of the states: for example, if the soliton states are stretched they are no longer solitons and their behaviour will be different. Again, we chose the size of our states arbitrarily.

5. Conclusions

We have demonstrated that Fourier transforming a BEC non-destructively can be both easy and useful. However, for a reliable Fourier transform the BEC has to be only very weakly interacting.

For reasons of computational complexity most of our simulations were for 2D ('pancake') BECs. Corresponding results for 3D BECs will be an important extension of our work.

Acknowledgments

GW is supported by a studentship from the UK Engineering and Physical Sciences Research Council (EPSRC), PÖ is a Royal Society of Edinburgh Research Fellow, JC is a Royal Society University Research Fellow.

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