Fractality of light's darkness

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Natural light fields are threaded by lines of darkness. For monochromatic light, the phenomenon is familiar in laser speckle, i.e. the black points that appear in the scattered light. These black points are optical vortices that extend as lines throughout the volume of the field. We establish by numerical simulations, supported by experiments, that these vortex lines have the fractal properties of a Brownian random walk. Approximately 75% of the lines percolate through the optical beam, the remainder forming closed loops. Our statistical results are similar to those of vortices in random discrete lattice models of cosmic strings, implying that the statistics of singularities in random optical fields exhibit universal behaviour.

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Random physical processes often lead to complicated volume fields occurring in electromagnetism[1], quantum mechanics[2] and fluid mechanics. Turbulence in the latter is characterised by tangles of vortex lines, particularly in superfluids[3]. Random optical fields, especially speckle — familiar to all users of lasers — are also intertwined by optical vortices: optical phase singularities which are lines in three dimensions and places of complete destructive interference[4]. Around any of these lines, the optical phase changes by 2π and the local phasefronts are helical, inducing an optical vortex in the azimuthal energy flow[5], which can be understood as a localized orbital angular momentum[6]. The lines are easily observed in laser speckle, where the familiar black specks are intersections of the lines with the image plane. Vortices in speckle have been well-studied [7–10], but with emphasis on transverse 2D properties [1].

As laser speckle is the interference pattern generated by the random scattering of a coherent source, it can be described as a superposition of a large number of random plane waves. For three and four plane waves, the possible vortex topologies are arrays of infinite straight or helical lines, or planar loops[11]. When more plane waves are added at random, the three-dimensional topology becomes more complicated, and its statistics has hitherto lacked any systematic study. Nevertheless, this statistical topology is universal to random linear wave superpositions regardless of the physical system that they describe[12]. Superficially similar tangles of quantized vortex lines have been much studied in fields as diverse as BECs[13], superfluid turbulence[3], liquid crystals[14] and cosmic strings[15, 16]. In particular, the vortex structure studied in Ref. 16 has fractal properties very close to our findings for light.

The interference pattern formed by the superposition of plane waves is a deterministic problem and can therefore be calculated over any finite volume. In order to resolve topological ambiguity at the edges, we choose wavevectors lying on a rectangular grid in k-space, making the interference pattern periodic in x, y and z[17]. By tiling space with such Talbot cells, every vortex line eventually returns to its starting point within a cell. If this occurs within the same cell (i.e. after a zero net number of cell crossings), then the vortex is a closed loop, otherwise the vortex feature is an infinite periodic line.

To model such superpositions we generate sets of plane waves of wavelength $\lambda = 2\pi/k_0$, on a k-space grid of spacing δk. The real and imaginary components of the amplitude of each wave are Gaussian distributed, and the whole grid is subject to a Gaussian angular spectrum, characterised by a numerical aperture of K_σ/k_0, where K_σ is the standard deviation of the transverse component of the wavevectors. These Gaussian distributions are typical of those created by an expanded laser beam scattered from a rough surface[1, 18]. A concern is that the periodic interference patterns resulting from a discrete and finite k-space may not be representative of those generated from a continuous distribution. We therefore have calculated patterns based on different sizes of grid and tested the stability of the vortex statistics. In our case, for k-space grids larger than 23×23, the transverse vortex point density converges to the theoretical values predicted for a continuous spectrum[10]. In addition the 3D statistics also converge. We assume therefore that the interference patterns created from larger k-space grids are representative of the continuum.

Consequently we use a 27×27 k-space grid and calculate over a Talbot cell of $500 \times 500 \times 4000$ voxels. The Talbot cell's lateral and axial periodicities are $2\pi/\delta k$ and $4\pi k_0/\delta k^2$ respectively. We use $K_\sigma/k_0 = 1/330$, which is typical of a light field with low numerical aperture and corresponds to cells millimetres wide and metres in length. The vortex density is made isotropic by defining natural lateral and axial lengthscales[10]:

$$\Lambda_{x,y} = \lambda \left(\frac{k_0}{K_\sigma}\right) \text{ and } \Lambda_z = \lambda \left(\frac{k_0}{K_\sigma}\right)^2, \qquad (1)$$

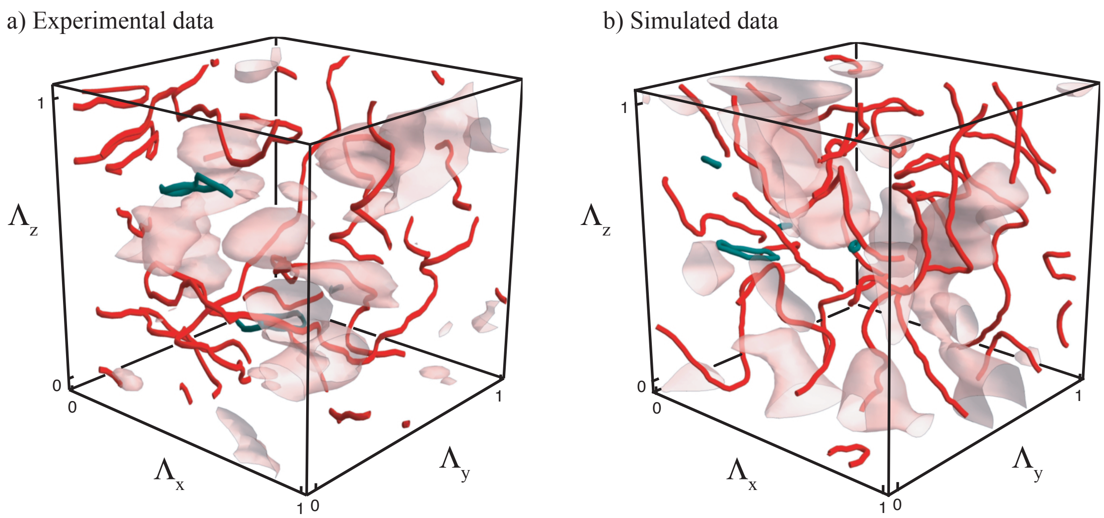

FIG. 1: Vortex structure in speckle: (a) experimental vortex structure obtained through interferometric measurements of laser speckle created by the scattering of a 10mm diameter HeNe laser beam through a ground glass screen and (b) numerical simulation of the vortex structure from Gaussian random wave superposition. The open vortex lines are plotted in red and the closed loops in dark green. Surfaces of 50% maximum intensity are also shown. Both are plotted over one natural volume of the speckle, $\Lambda_x \times \Lambda_y \times \Lambda_z$. Although different in detail, the two patterns have similar characteristics. (In color online).

giving a natural length $\Lambda = \sqrt[3]{\Lambda_{x,y}^2 \Lambda_z}$. By expressing all our observations in this natural unit, the results are independent of λ and K_σ, provided $K_\sigma \ll k_0$.

Using a high-end desktop computer, a random k-space distribution, the associated Talbot cell and subsequent vortex analysis can be completed in around 30 hours.

To complement our numerical simulations, we measured the vortex line topology in volumes of experimentally created laser speckle, synthesized by inserting a ground glass screen into a collimated HeNe ($\lambda = 633$nm) laser beam. Interferometric measurements of intensity and phase in successive cross sections in an experimental arrangement similar to that in Ref. 19 allowed us to map the three-dimensional vortex structure. Figure 1 illustrates examples of a numerically generated and experimentally observed Gaussian speckle pattern, showing both the bright optical speckles and the associated network of vortex lines.

Since the field of view of our experimental data is limited by the size of detector array, vortex lines that penetrate the sides of the volume cannot be distinguished from being segments of larger loop structures. Consequently, the possible statistical comparisons between the experimental and numerical data are limited. Inspection of figure 1 and other sets of data suggests that the simulated vortex structure is indeed similar to that of experimental observations. In both simulations and experiments, the transverse autocorrelation is Gaussian, giving the same vortex density ($2\pi/\Lambda^2$[10, 18]). We observe that the average number of closed vortex loops wholly enclosed within a natural unit volume for calculated data is 2.6 ± 1.6 (averaged over 50 natural volumes) and for experimental data, 2.0 ± 1.5 (averaged over 40 natural volumes). However, the limited field of view of our experiment data means that more detailed analysis relies on numerical simulation.

Figure 2 shows a single intensity transverse cross-section through the entrance face of a numerical Talbot cell and the full vortex line structure within the cell projected onto the yz-plane. The natural volume of Fig. 1 is only a small fraction of the volume of the associated Talbot cell, and the latter tiles three-dimensional space — allowing vortex loops to be unambiguously distinguished from infinite periodic vortex lines.

Analysis of several hundred Talbot cells generated using Gaussian distributions of wavevectors shows that periodic vortex lines account for about 75% of the total vortex line length, leaving 25% of the total vortex length as closed loops. This ratio is similar to that numerically found for the random lattice model of cosmic strings of Ref. 16. In that work, each point of a cubic lattice (also with periodic boundary conditions) is independently assigned a random discrete phase $0, 2\pi/3$, or $4\pi/3$ and interpolation through the volume gives a 3D quantized vortex tangle.

Since vortex lines are continuous we conjecture that many vortex lines completely traverse, i.e. percolate, real speckle fields of finite extent. In regions of low intensity, the field is strongly affected by background vacuum fluctuations[20], so the vortices are not stationary there. Consequently, although the vortex structure and topology are fixed within the bright regions of the beam, they are physically undetermined outside it.

Experiment[21] has confirmed predictions[22] that spe-

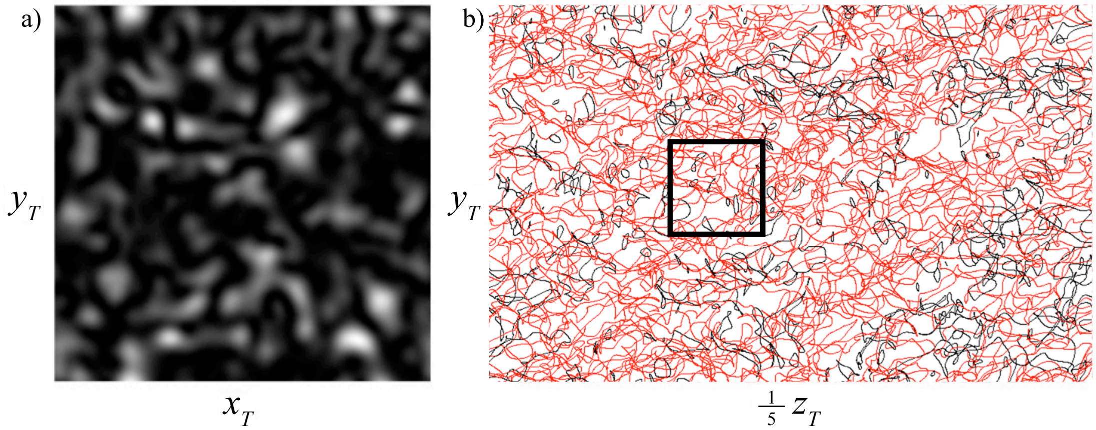

FIG. 2: Tangled vortices in a Talbot cell. An intensity xy-cross-section at the entrance face of a Talbot cell (a) and the associated vortex structure of a portion of a full cell (b). The subscript T denotes the Talbot distance in that direction. Closed vortex loops and open vortex lines are shown in black and red respectively. For comparison, an outline square (in thick black) shows the cross section of one natural volume $\Lambda_x \times \Lambda_y \times \Lambda_z$ (in color online).

cific superpositions of physical beams can contain linked and knotted vortex loops. We find that such topological configurations are extremely rare in our random interference patterns, with fewer than 1 in 100 loops threaded by a vortex line, and no knotted or linked loops have been found. Even in the controlled creation of loops and knots, the configuration dissolves on perturbation[23], and it is likely that such topologies occupy similarly small regions of probability space, hence are rare in random fields.

We investigate the self-similarity — that is, fractality — of the simulated vortex lines. The average arc length R between any two points on such a line, separated by a Pythagorean length L, is given by

$$< R >= P^{1-n} L^n, \qquad (2)$$

where P is a characteristic lengthscale below which the line is approximately straight. The reciprocal of n is the fractal dimension of the line[24]: for a straight line, $n = 1$, and for a Brownian random walk, $n = 1/2$.

Figure 3 is a log-log plot of the Pythagorean distance between points on the infinite vortex lines, sampled over many lines from different Talbot cells. It is linear over two decades, empirically implying significant self-similarity. The gradient 0.52 ± 0.01 suggests that over the marked range, the smoothly curved vortex lines are a close approximation to a Brownian random walk, with fractal dimension approximately 2. The y-intercept indicates a persistence length $P = 0.5\Lambda$, comparable to the coherence length of the optical field. As expected, at lengthscales below P, the vortex lines are straight (the curve gradient is 1). At longer lengthscales, the vortex lines inherit the periodicity of the Talbot cell array, again giving a gradient tending to unity.

Following Ref. 16, we consider the implication of scale invariance — for lengthscales significantly greater than P — on the size distribution of vortex loops. If the loop

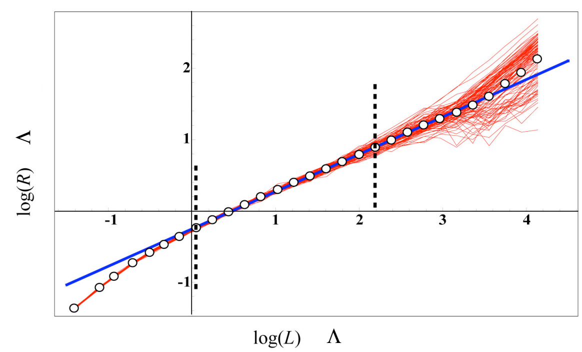

FIG. 3: The Pythagorean distance between two points on an open vortex line as a function of vortex line length, averaged over many pairs of points. Data is obtained from various speckle superpositions calculated using a k-space grid size of 27×27 in size. Circles mark the mean from 100 lines from different speckle superpositions and the straight line is the least square fit through the mean values. The gradient of 0.52 ± 0.01, fitted over the marked range, suggests a scale invariance over which the vortex lines have random walk, i.e. Brownian characteristics. The upper limit of this range is solely a result of the periodicity of the Talbot cell.

distribution is the same at all scales, then the number dN of closed loops per unit volume with sizes between R and $R + dR$, from equation 2 with Brownian exponent $n = 1/2$, is[16]

$$dN = CP^{-\frac{3}{2}} L^{-\frac{5}{2}} dL, \qquad (3)$$

where C is a dimensionless numerical constant.

Figure 4 shows the loop length distribution from the simulations. The distribution has a peak in loop size of approximately 0.3Λ, and suggests 3.9 loops per natural volume, both of which are consistent with figure 1 and other superpositions. For larger loops, the gradient

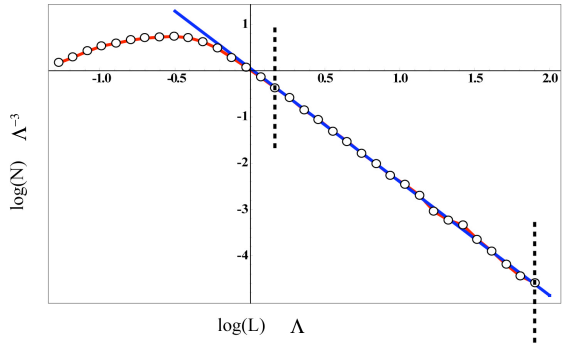

FIG. 4: The observed length distribution of vortex loops within numerically calculated random speckle patterns. Data is obtained from various speckle superpositions calculated using a k-space grid size of 27×27 in size. Approximately 80,000 loops are included in this distribution. The average number of loops contained within the natural speckle volume is 3.9 and the most common loop length is readily contained within it. The number density of large loops decreases, over the marked range, with a gradient of ≈ -2.5, consistent with Brownian scale invariance.

-2.46 ± 0.02 is consistent with equation 3, again supporting our observation that the vortex structures have a fractal self-similarity of Brownian character.

The results for the large-scale structure of the vortex lines and the size distribution of loops support the hypothesis that vortex lines in speckle fields display some degree of scale invariance. The lines have fractal dimension approximately 2 (Brownian), and the loop length distribution is consistent with this and scale invariance. These results are very similar to those found in the lattice model of Refs. 16 and 25, determined by independent random variables at discrete lattice sites, whereas our vortices are the nodes of independent random wave superpositions. This similarity may therefore be compared to that found [26] between nodal domain distributions in 2D real random waves, and 2-dimensional percolation. The interpretation of the vortices in the model of Ref. 16 was as the configuration of cosmic strings in the early universe; however, cosmic strings, as with other quantised vortices in physics such as those in superfluids, evolve according to nonlinear dynamics, which affects the overall fractal dimension [27] and loop length distribution [28]. We emphasize that the optical fields considered here are both linear and monochromatic, so there are no energetic considerations, and the statistics depend only on the probability distribution of the superposed random waves. We anticipate a closer analogy with other vortex systems for optical interference in a nonlinear medium, which would give an energy cost associated with vortex line curvature. The results presented here have been for a Gaussian angular spectrum, how many of these results are universal for all optical fields remains a point for further investigation.

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